Delaunay Surfaces in $\mathbb{S}^3(\rho)$

J. Arroyo[a], O.J. Garay[a], A. Pámpano[a]

[a]Department of Mathematics, Faculty of Science and Technology, University of the Basque Country, Spain

Abstract. Recently, invariant constant mean curvature (CMC) surfaces in real space forms have been characterized locally by using extremal curves of a Blaschke type energy functional [5]. Here, we use this characterization to offer a new approach to some global results for CMC rotational surfaces in the 3-sphere.

1. Introduction

In [5] we studied CMC surfaces in Riemannian and Lorentzian 3-space forms, $M_r^3(\rho)$, which are invariant under the flow of a Killing vector field of the ambient space. We described any CMC invariant surface locally as a binormal evolution surface [4], [10]. As a consequence, they are *warped product surfaces* whose *warping functions* are solutions of an *Ermakov-Milne-Pinney* equation with constant coefficients.

On the other hand, although catenaries are known to be solutions of a classical variational problem (*they have the shape of a rope when fixing the extremes of it and letting gravity acts on the other part*) in 1930 Blaschke proved that they are also solutions of another variational problem ([6], pp. 38-39). To be more precise, Blaschke studied smooth immersed curves in $\mathbb{R}^3$ which are extremal for the curvature energy functional $\Theta(\gamma) = \int_\gamma \sqrt{\kappa}\, ds$, κ being the curvature of the curve, and he showed that *catenaries are critical for Θ when acting on planar curves*. Then, an extension of Blaschke's variational problem was considered in [5]. Namely, we introduced the functional $\Theta_\mu(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\, ds$, for a fixed $\mu \in \mathbb{R}$, and consider the associated variational problem when Θ_μ is acting on a certain space of smooth curves immersed in a Riemannian or Lorentzian 3-space form. The corresponding Euler-Lagrange equations, which are expressed in terms of the curvature and the torsion of the critical curves, were integrated. Furthermore, we also proved that extremals of Θ_μ evolving under an associated Killing field, ξ, produce invariant CMC surfaces in $M_r^3(\rho)$. And conversely, a CMC surface of $M_r^3(\rho)$ which is invariant by a one-parameter group of rigid motions was shown to be, locally, spanned by an extremal curve of $\Theta_\mu(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\, ds$ while evolving by ξ. In particular, we showed that CMC surfaces of revolution in Riemannian 3-space forms are locally spanned by an extremal curve of Θ_μ with zero torsion (planar extremals).

CMC surfaces immersed in the 3-sphere, $\mathbb{S}^3(\rho)$, have played a major role in Mathematics in last decades. In 1966, Almgren [1] proved that *any immersed minimal 2-sphere in $\mathbb{S}^3(\rho)$ must be totally geodesic* and, therefore,

congruent to the equator. Moreover, in 1970, Lawson [13] proved that, *given any positive integer m, there exist at least one compact embedded minimal surface in* $\mathbb{S}^3(\rho)$ *with genus m*. In fact, if the genus of the CMC surface is 1, he conjectured that *the only embedded minimal tori, up to rigid motions in* $\mathbb{S}^3(\rho)$, *is the Clifford torus*, [14]. Lawson's conjecture was recently proved by Brendle in [7]. Furthermore, adapting the technique of this proof, Andrews and Li [2] proved the Pinkall-Sterling conjecture: *any CMC tori embedded in* $\mathbb{S}^3(\rho)$ *must be rotationally symmetric*, [18]. In fact, these rotational CMC surfaces were completely classified by Perdomo, [16] and [17], and Andrews and Li [2].

In this note, we use the variational characterization of profile curves of invariant CMC surfaces to study global properties of these surfaces. More precisely, after describing the local characterization of CMC rotational surfaces of the 3-sphere in terms of the extremals of the Blaschke type energy, we study the existence of compact and embedded CMC rotational tori in $\mathbb{S}^3(\rho)$, by analyzing the simplicity and closedness of such critical curves. Then, some consequences are derived what provides a new approach to some of the results mentioned in the previous paragraph.

2. Extremals of a Blaschke's Type Energy and Delaunay Surfaces in $\mathbb{S}^3(\rho)$

Given a curve γ in a Riemannian 3-space form with sectional curvature ρ, $M^3(\rho)$, and a fixed constant $\mu \in \mathbb{R}$ we consider the following *curvature energy functional*

$$\Theta_\mu(\gamma) := \int_\gamma \sqrt{\kappa - \mu} = \int_0^L \sqrt{\kappa(s) - \mu}\, ds\,, \tag{1}$$

where, as usual, the arc-length or natural parameter is represented by $s \in [0, L]$, L being the length of γ, and the curvature of γ, $\kappa(s)$, is assumed to be greater than μ. This problem in the Euclidean 3-space $\mathbb{R}^3$ with $\mu = 0$ was studied by Blaschke in [6]. In [5], this variational problem was studied in any Riemannian and Lorentzian 3-space form. For the sake of simplicity, from now on we are just going to consider planar critical curves with non-constant curvature verifying $\kappa > \mu$. By "planar" we mean curves with constant zero torsion in the 3-sphere $\mathbb{S}^3(\rho)$, i.e., $\tau = 0$, which implies that they lie in a totally geodesic $\mathbb{S}^2(\rho) \subset \mathbb{S}^3(\rho)$. We define the following vector fields along γ

$$\mathcal{I} = \frac{1}{2\sqrt{\kappa - \mu}}B\,, \quad \mathcal{J} = \frac{2\mu - \kappa}{2\sqrt{\kappa - \mu}}T + \frac{1}{2}\frac{d}{ds}\Big(\frac{1}{\sqrt{\kappa - \mu}}\Big)N\,, \tag{2}$$

where $\{T, N, B\}$ represents the Frenet frame on γ. Now, a vector field W along a curve γ, which infinitesimally preserves unit speed parametrization is said to be a *Killing vector field along* γ (in the sense of [12]) if it evolves in the direction of W without changing shape, only position. In other words, the following equations must hold

$$W(v)(s, 0) = W(\kappa)(s, 0) = W(\tau)(s, 0) = 0\,,$$

($v = |\dot{\gamma}|$ and τ being the speed and torsion of γ, respectively) for any variation $\gamma(s, t)$ of γ having W as variation field. Then, it can be proved (for details, see [5]) that if γ is an extremal of (1) with zero torsion, then the vector fields $\mathcal{I}, \mathcal{J}$ given in (2) are Killing vector fields along γ and that the equation

$$\langle \mathcal{J}, \mathcal{J}\rangle + \rho\langle \mathcal{I}, \mathcal{I}\rangle = d\,, \tag{3}$$

for d a real constant, represents a first integral of the Euler-Lagrange equations of (1) for planar curves. It is easy to check that $\langle \mathcal{I}, \mathcal{J}\rangle = 0$ (since γ is planar, $\tau = 0$) and, in addition, an argument similar to that of [12] can be used to see that $\mathcal{I}$ and $\mathcal{J}$ can be uniquely extended to commuting Killing vector fields on the whole $\mathbb{S}^3(\rho)$, denoted again by $\mathcal{I}$ and $\mathcal{J}$, respectively. Hence, using spherical coordinates, $x(\theta, \sigma, \psi) = \frac{1}{\sqrt{\rho}}(\cos\theta\cos\sigma, \cos\theta\sin\sigma, \sin\theta\sin\psi, \sin\theta\cos\psi)$ in $\mathbb{S}^3(\rho)$ and denoting by $\partial_\sigma = x_\sigma$ and $\partial_\psi = x_\psi$, it can be assumed that [15]

$$\mathcal{I} = \sqrt{d}\,\partial_\sigma\,, \quad \mathcal{J} = \sqrt{\rho d}\,\partial_\psi\,. \tag{4}$$

We parametrize γ using these spherical coordinates as $\gamma(s) = x(\theta(s), \sigma(s), \psi(s))$. Then, computing the tangent vector of γ and combining with (2), we have that without loss of generality $\sigma(s) = 0$ and

$$\psi(s) = -2\sqrt{\rho\, d} \int \frac{(\kappa(s) - 2\mu)\sqrt{\kappa(s) - \mu}}{4d(\kappa(s) - \mu) - \rho}\, ds. \tag{5}$$

Therefore, taking into account that γ is arc-length parametrized we have that the planar critical curve of Θ_μ, (1), in $\mathbb{S}^3(\rho)$ can be parametrized as,

$$\gamma(s) = \frac{1}{2\sqrt{\rho\, d}\,\sqrt{\kappa(s) - \mu}}\left(\sqrt{\rho},\, 0,\, \sqrt{4d(\kappa(s) - \mu) - \rho}\,\sin\psi(s),\, \sqrt{4d(\kappa(s) - \mu) - \rho}\,\cos\psi(s)\right), \tag{6}$$

where $\psi(s)$ is given by (5) and $\kappa(s)$ is the curvature of γ. Observe that the critical curve $\gamma(s)$ crosses the pole of the parametrization, $\left(1/\sqrt{\rho}, 0, 0, 0\right)$, if and only if, $4d(\kappa - \mu) = \rho$.

Furthermore, equation (3) was completely solved in [5], and the curvatures of the profile curves of CMC rotational surfaces were explicitly determined. Indeed, we have

Proposition 2.1. *Let γ be a critical curve of* (1) *in* $\mathbb{S}^2(\rho)$ *with* $\kappa(s) > \mu$. *If γ has constant curvature, κ_o, then* $\kappa_o = \mu + \sqrt{\mu^2 + \rho}$, *with* $\mu^2 + \rho > 0$. *If $\kappa(s)$ is not constant, then, $\kappa(s) = \kappa_d(s), d \in \mathbb{R}$ where*

$$\kappa_d(s) = \frac{\rho + \mu^2}{2d + \mu - \sqrt{4d^2 + 4\mu d - \rho}\,\sin\left(2\sqrt{\rho + \mu^2}\,s\right)} + \mu. \tag{7}$$

We recall that the constant case in the above proposition gives rise to flat isoparametric surfaces, wich are well-known. On the other hand, if the curvature is not constant, the constant of integration of (7), d, is not entirely arbitrary since $d > \frac{-\mu + \sqrt{\mu^2 + \rho}}{2}$.

In [5] we have shown that invariant surfaces in Riemannian 3-space forms with constant mean curvature can be described locally as binormal evolution surfaces with velocity $G(s) = \frac{1}{2\sqrt{\kappa(s) - \mu}}$ whose initial vortex filament is critical for (1) and conversely (for details see [5]). In particular, in the case of the 3-sphere we have

Theorem 2.2. *Locally, a rotational surface of CMC H in $\mathbb{S}^3(\rho)$ can be described as a rotational surface S_γ shaped on a planar profile curve γ with curvature $\kappa(s)$ and is locally congruent to a piece of one of the following*

1. *The equator $\mathbb{S}^2(\rho)$; if $\kappa(s) = H = 0$.*
2. *A totally umbilical sphere; if $\kappa(s) = |H| \neq 0$.*
3. *A Hopf Torus*

$$\mathbb{S}^1\left(\sqrt{\rho + \kappa^2}\right) \times \mathbb{S}^1\left(\frac{\sqrt{\rho}}{\kappa}\sqrt{\rho + \kappa^2}\right),$$

if $\kappa(s) = -|H| + \sqrt{H^2 + \rho}$.

4. *A binormal evolution surface parametrized by*

$$x(s, t) = \frac{1}{2\sqrt{\rho\, d}\,\sqrt{\kappa(s) - \mu}}\left(\sqrt{\rho}\cos t,\, \sqrt{\rho}\sin t,\, \sqrt{4d(\kappa(s) - \mu) - \rho}\,\sin\psi(s),\, \sqrt{4d(\kappa(s) - \mu) - \rho}\,\cos\psi(s)\right), \tag{8}$$

where the profile curve γ is critical for (1) *with curvature* (7), *and where $|\mu| = |H|$.*

3. Embedded CMC Tori in the Round 3-Sphere

Now, we want to understand which among the rotational CMC surfaces of $\mathbb{S}^3(\rho)$ given in (iv) of Theorem 2.2, are closed. There are two possibilities to be considered. The first one corresponds to planar critical curves that meet the axis of rotation, while the second one comes from planar closed critical curves not touching that axis. However, it can be checked that the Euclidean radius of curvature of the orbits is positive $4d(\kappa - \mu) > 0$ [15] what means that planar critical curves of the extended Blaschke's energy (1) never cut the axis of rotation. By this reason, we study closure conditions for profile curves of Θ_μ (1), when acting on curves immersed in $\mathbb{S}^2(\rho)$, since closed planar critical curves with curvature given by (7) will generate CMC rotational tori, and these are the only possible non-isoparametric closed CMC rotational surfaces in $\mathbb{S}^3(\rho)$. Moreover, by the recently proved Pinkall-Sterling's conjecture, these are the only possible non-isoparametric CMC tori embedded in $\mathbb{S}^3(\rho)$. Now, planar critical curves of Θ_μ (1), are completely
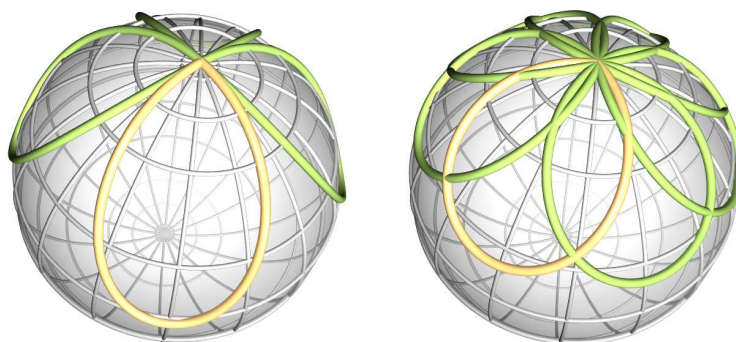

Figure 1: Closed planar extremal curves in $\mathbb{S}^2(1)$ passing through the pole $(4\mu d = 1)$ for $\mu \simeq 0.312$ (Left) and $\mu \simeq 0.634$ (Right).

determined by the curvature $\kappa(s)$, (7) which is periodic of period $\varrho = \frac{\pi}{\sqrt{\rho + \mu^2}}$. Then, using (7), it is easy to check that whenever $4\mu d \neq \rho$, at points of γ where the curvature attains maxima and minima the vector field $\mathcal{J}$, (2), has only component in T, which means that the critical curve is bounded between the two parallels of $\mathbb{S}^2(\rho)$ corresponding to the integral curves of $\mathcal{J}$ at the maximum and minimum of $\kappa(s)$, respectively. What is more, the length of $\mathcal{J}$ vanishes, if and only if, $\kappa(s)$ reaches its minimum and $4\mu d = \rho$ (observe that since $d > 0$, this equality can only occur for positive values of μ). In this particular case, the critical curve crosses the pole of the parametrization (see the parametrization given in (8) and Figure 1). However, notice that it is possible to find a reparametrization of the critical curve in order to avoid this singularity. For instance, in Lemma 3.1 of [17] it has been done when $\rho = 1$.

In Figure 1 and 2, we have plot a few instances of closed critical curves of Θ_μ, (1) in $\mathbb{S}^2(1)$. These curves have periodic curvature, $\kappa(s)$, given by (7). The yellow part of these pictures corresponds with that piece of the curve covered in one period of the curvature. Notice that, as the curvature is the same for each period of it, our critical curve is nothing but congruent copies of the yellow part, that is, the whole curve can be constructed by gluing smoothly as many copies of the trace covered in one period of the curvature as needed to close up the curve.

However, not every critical curve with periodic curvature (7) is closed and a closure condition needs to be satisfied. In $\mathbb{S}^3(\rho)$, equation (3) implies that the constant of integration d must be positive and from (5) we obtain that the condition for a critical $\gamma(s)$ to close up is

$$I := -\frac{1}{2}\sqrt{\rho\,d}\,\Lambda(d) := \sqrt{\rho\,d} \int_0^\varrho \frac{(\kappa(s) - 2\mu)\,\sqrt{\kappa(s) - \mu}}{4d(\kappa(s) - \mu) - \rho}\,ds = \frac{n\,\pi}{m}, \tag{9}$$

for some integers n and m. Observe that n denotes the number of times the critical curve goes around the pole of the parametrization in order to close up, and that m is the number of lobes the critical curve has.
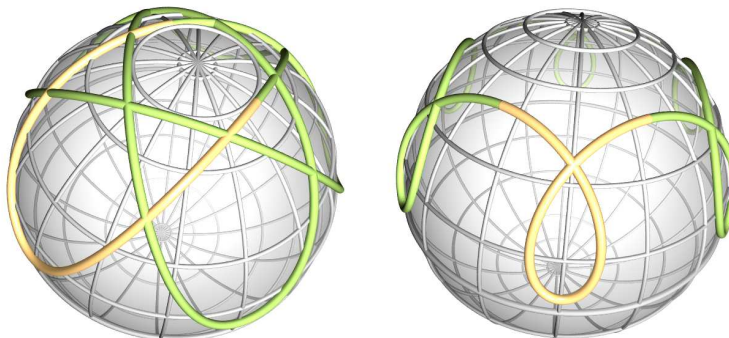
Figure 2: Closed planar extremal curves in $\mathbb{S}^2(1)$ for: $\mu = -0.1$ and $d \simeq 1.27$ (Left); and, $\mu = 1$ and $d \simeq 1.81$ (Right). They close up for the values $m = 5$ and $n = 3$ (resp., $m = 6$ and $n = 1$).

Then, we can prove the existence of closed planar critical curves with non-constant curvature for any value of μ in the following way.

Let γ be a planar ($\tau = 0$) critical curve of $\mathbf{\Theta}_\mu$ (1), completely determined by the curvature, $\kappa(s)$, given by (7). As these curvatures are periodic functions with period $\varrho = \frac{\pi}{\sqrt{\rho + \mu^2}}$, if we call α to the maximum curvature of $\kappa(s)$ and β to the minimum curvature, we obtain

$$\alpha = \kappa\left(\frac{\pi}{4\sqrt{\rho + \mu^2}}\right) \geq \kappa\left(-\frac{\pi}{4\sqrt{\rho + \mu^2}}\right) = \beta.$$

And the function on the left hand side of (9) can be rewritten as

$$I = 2\sqrt{\rho\, d} \int_{-\frac{\pi}{4\sqrt{\rho + \mu^2}}}^{\frac{\pi}{4\sqrt{\rho + \mu^2}}} \frac{(\kappa(s) - 2\mu)\sqrt{\kappa(s) - \mu}}{4d(\kappa(s) - \mu) - \rho}\, ds. \tag{10}$$

Now, differentiating equation (7), we get that $\kappa_s = 2(\kappa - \mu)\sqrt{4d(\kappa - \mu) - (\kappa - 2\mu)^2 - \rho}$, which can be written in terms of the maximum and minimum values of the curvature, α and β, as

$$\kappa_s = 2(\kappa - \mu)\sqrt{(\alpha - \kappa)(\kappa - \beta)}. \tag{11}$$

Using (11) to make a change of variable in (10), we get

$$I = \sqrt{\rho\, d} \int_\beta^\alpha \frac{\kappa - 2\mu}{(4d(\kappa - \mu) - \rho)\sqrt{(\kappa - \mu)(\alpha - \kappa)(\kappa - \beta)}}\, d\kappa.$$

This integral can be written as a linear combination of complete elliptic integrals of first, second and third kind (for details see [11]). In fact, since $d > 0$, we have the following relation, $4d(\kappa - 2\mu) = (4d(\kappa - \mu) - \rho) + \rho - 4\mu d$, and, therefore, we conclude that

$$I = \frac{\sqrt{\rho}}{4\sqrt{d}}\left(\int_\beta^\alpha \frac{d\kappa}{\sqrt{(\kappa - \mu)(\alpha - \kappa)(\kappa - \beta)}}\right.$$
$$\left. + (\rho - 4\mu d)\int_\beta^\alpha \frac{d\kappa}{(4d(\kappa - \mu) - \rho)\sqrt{(\kappa - \mu)(\alpha - \kappa)(\kappa - \beta)}}\right). \tag{12}$$

Take into account that $\rho = 4\mu d$ is a special case where the second integral does not appear because if $\rho = 4\mu d$ then the critical curve passes through the pole, as we have explained in the beginning of the section.

Now, following the notation of [11], we define

$$p = \sqrt{\frac{\alpha - \beta}{\alpha - \mu}}, \quad q = \sqrt{1 - p^2}, \quad \nu = \frac{(\alpha - \beta)(\alpha + \beta - 4\mu)}{(\alpha - \mu)(\alpha + \beta - 4\mu) - \rho}.$$ (13)

Then, from (11), we know that $4d(\kappa - \mu) - (\kappa - 2\mu)^2 - \rho = (\alpha - \kappa)(\kappa - \beta)$ which gives us $\alpha + \beta = 4(d + \mu)$ and $\alpha\beta = 4\mu^2 + 4\mu d + \rho$. Thus, finally, we obtain that

$$\alpha = \frac{\sqrt{\rho + \mu^2}}{q} + \mu, \quad \beta = q\sqrt{\rho + \mu^2} + \mu.$$ (14)

Therefore, using relations (14) we see that (12) can be reduced

$$I = \frac{2\sqrt{\rho}}{\sqrt{\alpha + \beta - 4\mu}\,\sqrt{\alpha - \mu}}\left(K(p) + \frac{\rho - \mu(\alpha + \beta - 4\mu)}{(\alpha - \mu)(\alpha + \beta - 4\mu) - \rho}\Pi(\nu, p)\right),$$ (15)

where $K(p)$ and $\Pi(\nu, p)$ denote the complete elliptic integrals of first and third kind with modulo p and argument ν, respectively. Thus, we obtain

Theorem 3.1. *For any value of μ, there exist closed planar critical curves in $\mathbb{S}^2(\rho)$ of the energy Θ_μ defined in* (1) *with non-constant curvature, $\kappa(s)$, given by* (7).

Proof. Consider a planar critical curve γ of Θ_μ (1), with non-constant periodic curvature $\kappa(s)$, (7). We need to check that the clousure condition (9) is verified. Remember that we have written the function I in terms of elliptic integrals, (15). We first begin by translating the different values of the parameter d into the new parameter q introduced in (13). The value $d = \frac{-\mu + \sqrt{\rho + \mu^2}}{2}$ corresponds to $q = 1$, while, the limit $d \to \infty$, now reads $q \to 0$. Moreover, $d = \frac{\rho}{4\mu}$, that is, when the critical curve passes through the pole, is represented by $q = \frac{\mu}{\sqrt{\rho + \mu^2}}$. Then, with the notation introduced in (13), it can be checked that $p^2 < \nu < 1$. Hence,

$$\Pi(\nu, p) = \frac{\pi}{2}\sqrt{\frac{\nu}{(1 - \nu)(\nu - p^2)}}\,\Lambda_o\left(\arcsin\sqrt{\frac{\nu - p^2}{\nu(1 - p^2)}}, p\right),$$ (16)

where Λ_o represents the *Heuman's lambda*, (for instance, see Appendix B of [3]). Moreover, $d > \frac{-\mu + \sqrt{\rho + \mu^2}}{2} > 0$ and using (15) we get

$$I = q\phi K(p) + \frac{\pi}{2}\epsilon\Lambda_o\left(\arcsin\phi, p\right),$$ (17)

where ϵ represents the sign of $\rho - 4\mu d$ and ϕ is given by

$$\phi = \sqrt{\frac{\nu - p^2}{\nu(1 - p^2)}} = \frac{1}{q}\sqrt{\frac{\rho}{4d(\alpha - \mu)}}.$$

If $\mu \leq 0$, then necessarily $\epsilon = 1$ and then I defined in (9) is a monotonically decreasing function of d, when $d \in \left(\frac{-\mu + \sqrt{\rho + \mu^2}}{2}, \infty\right)$, bounded by (see Appendix A of [15])

$$\arcsin\sqrt{\frac{\rho}{\rho + \mu^2}} < I < \frac{\pi}{(\rho + \mu^2)^{\frac{1}{4}}}\sqrt{\frac{\rho}{2\left(-\mu + \sqrt{\rho + \mu^2}\right)}}.$$ (18)

On the other hand, if $\mu > 0$, we need to take out the case $d = \frac{\rho}{4\mu}$ and consider d moving in $d \in \left(\frac{-\mu + \sqrt{\rho + \mu^2}}{2}, \frac{\rho}{4\mu} \right) \cup \left(\frac{\rho}{4\mu}, \infty \right)$. Then, taking into account the sign of ϵ, we obtain that I is a monotonically decreasing function of d which is bounded by (see Appendix A of [15])

$$- \arcsin \sqrt{\frac{\rho}{\rho + \mu^2}} \; < \; I \; < \; \frac{\mu}{\sqrt{\rho + \mu^2}} K\left(\sqrt{\frac{\rho}{\rho + \mu^2}} \right) - \frac{\pi}{2}, \tag{19}$$

if $\epsilon = -1$, or, in the case $\epsilon = 1$ we have the following upper and lower bounds

$$\frac{\mu}{\sqrt{\rho + \mu^2}} K\left(\sqrt{\frac{\rho}{\rho + \mu^2}} \right) + \frac{\pi}{2} \; < \; I \; < \; \frac{\pi}{(\rho + \mu^2)^{\frac{1}{4}}} \sqrt{\frac{\rho}{2\left(-\mu + \sqrt{\rho + \mu^2}\right)}}. \tag{20}$$

Thus, in all the three cases (18)-(20), we can always find some integers m and n, such that, $m I = n \pi$, that is, there are closed critical curves. To finish the proof, we are going to consider now the case where the critical curve passes through the pole. That is, when $\rho = 4\mu d$. As mentioned before, for this case we only have the first integral in (12). Moreover, in this case, $q = \frac{\mu}{\sqrt{\rho + \mu^2}}$ and μ can take values in $(0, \infty)$, since $\rho > 0$. In this case, from (15) we have

$$I = \frac{\mu}{\sqrt{\rho + \mu^2}} K\left(\sqrt{\frac{\rho}{\rho + \mu^2}} \right). \tag{21}$$

Then, I is increasing in μ and $0 < I < \pi/2$ (see Appendix A of [15]). Therefore, there are also closed critical curves passing through the pole. □

Notice that planar critical curves of Θ_μ (1) are all immersed in the totally geodesic sphere $\mathbb{S}^2(\rho)$ (see Figures 1 and 2). However, in order to study embeddedness of the associated rotational CMC surfaces, we need to determine under what conditions planar critical curves are embedded in $\mathbb{S}^2(\rho)$, that is, when these critical curves are simple.

Theorem 3.2. *Assume that γ is a planar critical curve of Θ_μ (1) with non-constant curvature, $\kappa(s)$, (7), immersed in $\mathbb{S}^2(\rho)$. Then, if $\mu > 0$, γ is not simple. Moreover, if $\mu \leq 0$, γ will be simple, if and only if, it is closed and it closes up after one trip around the pole.*

Proof. Let γ be a planar critical curve of (1), with curvature, $\kappa(s)$ (which is given by (7)). Then, the function

$$\widetilde{I}(s) = \frac{(\kappa(s) - 2\mu) \sqrt{\kappa(s) - \mu}}{4 d (\kappa(s) - \mu) - \rho}, \tag{22}$$

verifies that;

1. If $\mu \leq 0$ or, $\mu > 0$ and $4 \mu d \leq \rho$, then it never changes sign.
2. If $\mu > 0$ and $4 \mu d > \rho$, it changes sign.

Recall that $2d > -\mu + \sqrt{\rho + \mu^2}$. Moreover, notice that if $\mu \leq 0$, then $\widetilde{I}(s)$, (22), does not change sign and, therefore, the function $\psi(s)$, (5), is monotone. Furthermore, the planar critical curve γ will be simple, unless it closes up in more than one round. Thus, $\gamma \subset \mathbb{S}^2(\rho)$ is simple, if and only if, it closes up in one round, that is, by checking the image of I, (18), if there exists an integer m, such that,

$$\arcsin \sqrt{\frac{\rho}{\rho + \mu^2}} \; < \; \frac{\pi}{m} \; < \; \frac{\pi}{(\rho + \mu^2)^{\frac{1}{4}}} \sqrt{\frac{\rho}{2\left(-\mu + \sqrt{\rho + \mu^2}\right)}},$$

and verifying $mI = \pi$. Now, if $\mu > 0$ and $4\mu d < \rho$, the function $\widetilde{I}(s)$, (22), has always the same sign. That is, by monotonicity of $\psi(s)$, (5), γ will be simple, if and only if, it closes up in one round. Using Appendix A of [15], the absolute value of the total angular variation, $|2I|$, of γ in $\mathbb{S}^2(\rho)$, (20), is always bigger than π, but smaller than 2π. Thus, γ cannot close up in one period, and it travels more than one round from the second period, so it cuts itself. That is, γ is not simple.

Let's study now the case $\mu > 0$ and $4\mu d = \rho$. In this case, γ passes through the pole exactly once in each period of its curvature, therefore the only option for γ to be simple is that it closes up in just one period. But, if we look at the absolute value of the total angular variation, $|2I|$, for this case, $(0, \pi)$, we realize that $\gamma \subset \mathbb{S}^2(\rho)$ does not travel one whole round in each period, since $|2I|$ is always smaller than π.

Finally, if $\mu > 0$ and $4\mu d > \rho$, we have that $\widetilde{I}(s)$, (22), has changes of sign, what it means that γ goes back and it is clear that γ is not simple. □

Notice that what Theorem 3.2 tells us is that there exist closed critical curves embedded in $\mathbb{S}^2(\rho)$. In fact, the condition in this case reduces to γ closing up in one round and not having self-intersections in one period of its curvature.
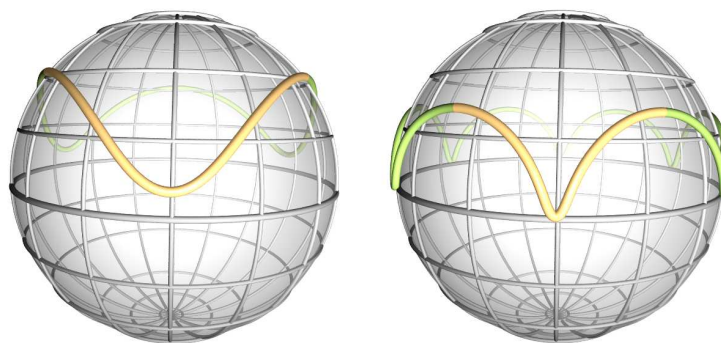

Figure 3: Closed and simple planar extremal curves in $\mathbb{S}^2(1)$ for: $\mu = -1$ and $d \simeq 2.48$ (Left); and, $\mu = -2$ and $d \simeq 16.19$ (Right).

The condition "closing up in just one round" means that the angular variation, I, must be equal π/m, for an integer m. This function has been proved to be bijective in Appendix A of [15], where the analysis made with elliptic integrals lead to the monoticity of I. A different proof when $\rho = 1$ can be found in [2]. This bijection means that for each m, there exists just one d such that $mI = \pi$. However, the choice of the integer m is not totally free, since it is constrained by μ. Indeed, if we combine Theorem 3.1 and Theorem 3.2 to obtain closed and simple planar critical curves we get that $\mu < 0$ and that for any $m > 1$

$$\mu \in \left(-\sqrt{\rho}\, \frac{m^2 - 2}{2\sqrt{m^2 - 1}}, -\sqrt{\rho} \cot \frac{\pi}{m} \right). \tag{23}$$

Finally, we sum up all this information in the following corollary,

Corollary 3.3. *Let γ be a planar closed critical curve of Θ_μ (1), with non-constant curvature, $\kappa(s)$, (7), embedded in $\mathbb{S}^2(\rho)$. Then, $\mu \neq -\sqrt{\frac{\rho}{3}}$ is negative.*

Proof. From Theorem 3.2, we know that necessarily $\mu \leq 0$. Moreover, as explained above, it must verify (23), for any $m > 1$. Then, we can check that for any strictly negative $\mu \neq -\sqrt{\frac{\rho}{3}}$, there exists such $m > 1$. □

Of course, if the profile curve γ is closed, then the corresponding rotational surface S_γ would also be closed (see Figure 4), and, what is more, if the planar critical curve is simple and closed, then γ sweeps out a closed surface S_γ embedded in $\mathbb{S}^3(\rho)$. Hence, Theorem 2.2 and the bijection of I imply that once we fix the
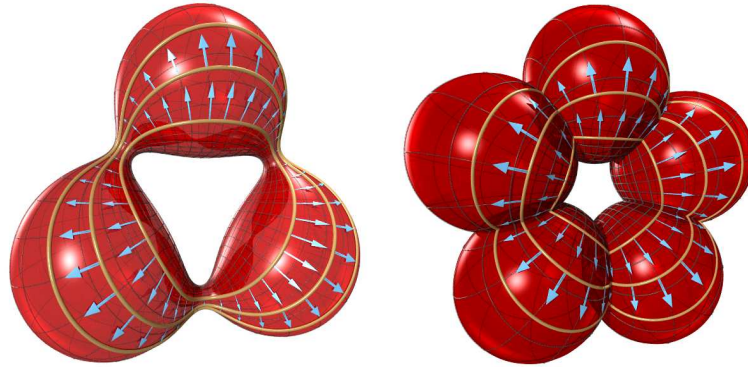
Figure 4: Stereographic projections of two closed CMC rotational surfaces in $\mathbb{S}^3(\rho)$ showing the binormal evolution (in blue) of the filaments (in yellow).

CMC H, for each $m > 1$, there exist at most one compact embedded non-isoparametric rotational surface of CMC H in $\mathbb{S}^3(\rho)$.

Observe that since critical curves with non-constant curvature do not meet the axis of rotation, the binormal evolution surfaces of point (iv) of Theorem 2.2 are all local descriptions of topological torus, therefore, both the Hopf tori and the binormal evolution surfaces have genus one. Moreover, notice that the interval (23), is precisely the interval given by Perdomo in [16]. In fact, for each possible value H and any $m > 1$ such that

$$|H| \in \left(\sqrt{\rho} \cot \frac{\pi}{m}, \ \sqrt{\rho} \, \frac{m^2 - 2}{2\sqrt{m^2 - 1}} \right),$$

there exists a compact embedded non-isoparametric surface of genus one given by point (iv) of Theorem 2.2 (see some of them in Figures 4 and 5). Moreover, our construction here gives a way of proving Ripoll's Theorem [19], which states that *for any $H \neq 0, \pm \sqrt{\frac{\rho}{3}}$, there exists a non-isoparametric torus of CMC H*. In Figure 5, we can see the stereographic projection of three of these surfaces for $m = 3$, 4 and 5, respectively.
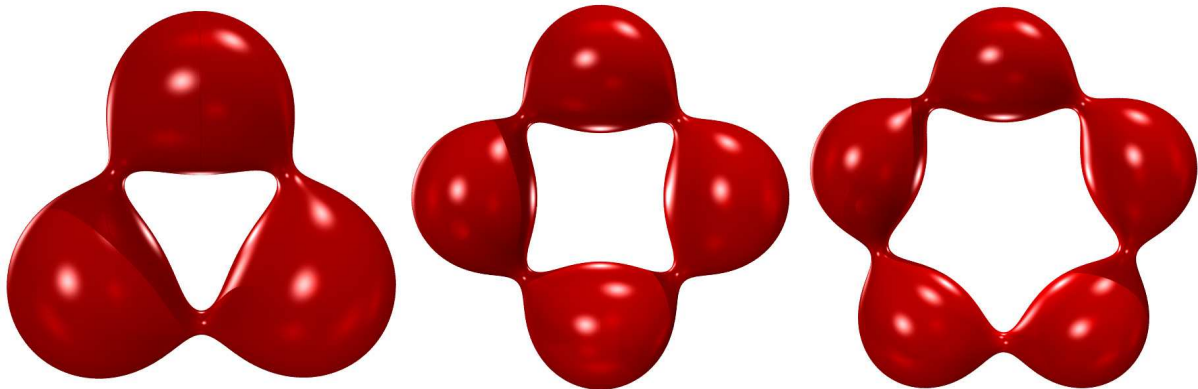

Figure 5: Stereographic projections of embedded CMC rotational surfaces in $\mathbb{S}^3(\rho)$.

Furthermore, the Pinkall-Sterling's conjecture [18] (recently proved in [2]) asserts that *any CMC tori embedded in $\mathbb{S}^3(\rho)$ must be rotationally symmetric*. Therefore, points (iii) and (iv) in Theorem 2.2 (together with the restriction from Corollary 3.3) give rise to a complete classification of CMC tori embedded in $\mathbb{S}^3(\rho)$. In particular, the Lawson's conjecture (see [7] and [14]) is verified. Indeed, if a rotational torus is

minimal, then μ must be zero and there are no closed simple planar critical curves of Θ_μ (1) (see Corollary 3.3), therefore, the only minimal torus embedded in $\mathbb{S}^3(\rho)$ is locally given in point (iii) of Theorem 2.2, i.e, $\mathbb{S}^1\left(\sqrt{2\rho}\right) \times \mathbb{S}^1\left(\sqrt{2\rho}\right)$, which is a Hopf torus, usually called the *Clifford torus*.

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