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Second Type Almost Geodesic Mappings of Special Class and Their Invariants

Nenad O. Vesić[a], Mića S. Stanković[a]

[a]*Department of Mathematics, Faculty of Sciences and Mathematics, Niš*

Abstract. Invariants of almost geodesic mappings of a generalized Riemannian space are discussed in this paper. As a special case, invariants of equitorsion almost geodesic mappings of this type are discussed in here.

1. Introduction and preliminaries

Geodesic lines and their generalizations are important for applications of differential geometry in physics. A diffeomorphism $f : \mathbb{R}_N \to \overline{\mathbb{R}}_N$ of Riemannian spaces $\mathbb{R}_N$ and $\overline{\mathbb{R}}_N$ endowed with symmetric metric tensor g_{ij} is called the *almost geodesic mapping* if it maps any geodesic line of the space $\mathbb{R}_N$ into an almost geodesic line of the space $\overline{\mathbb{R}}_N$. Sinyukov involved this concept of research for the mappings between affine connected spaces without torsion (see [16]). J. Mikeš [1, 7–9] significantly contributed to the study of geodesic and almost geodesic mappings of affine connected, Riemannian and Einstein spaces. Invariants of almost geodesic mappings of a generalized Riemannian space will be searched in this paper. The almost geodesic mappings of generalized Riemannian spaces and of spaces with non-symmetric affine connection as well are discussed in [17–20, 25, 26].

An N-dimensional manifold endowed with metric tensor g_{ij} non-symmetric in indices i and j is *the generalized Riemannian space* $\mathbb{GR}_N$ in the sense of Eisenhart definition [3–5].

Because of the non-symmetry $g_{ij} \neq g_{ji}$, the symmetric and anti-symmetric part of metric tensor g are defined as

$$g_{\underline{ij}} = \frac{1}{2}(g_{ij} + g_{ji}) \quad \text{and} \quad g_{\underset{\vee}{ij}} = \frac{1}{2}(g_{ij} - g_{ji}). \tag{1}$$

We assume that is $\det[g_{\underline{ij}}] \neq 0$. Tensor $g^{\underline{ij}}$ is determined by the condition $g_{\underline{i\alpha}} g^{\underline{\alpha j}} = \delta^j_i$ where δ^j_i is a Cronecker's symbol. Affine connection coefficients of the space $\mathbb{GR}_N$ are generalized Christoffel symbols Γ^i_{jk} of this space defined as

$$\Gamma^i_{jk} = \frac{1}{2} g^{\underline{i\alpha}}(g_{j\alpha,k} - g_{jk,\alpha} + g_{\alpha k,j}), \tag{2}$$

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Email addresses: vesko1985@pmf.ni.ac.rs (Nenad O. Vesić), stmica@mts.rs (Mića S. Stanković)

for partial derivation $\partial/\partial x^k$ denoted by comma. These coefficients are non-symmetric by indices j and k. For this reason, their symmetric and anti-symmetric parts are:

$$\Gamma^i_{\underline{jk}} = \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj}) \quad \text{and} \quad \Gamma^i_{\underset{\vee}{jk}} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}). \tag{3}$$

The symmetric parts $\Gamma^i_{\underline{jk}}$ are the affine connection coefficients of *the associated Riemannian space* $\mathbb{R}_N$ [10, 11]. The anti-symmetric part $\Gamma^i_{\underset{\vee}{jk}}$ of the Christoffel symbol Γ^i_{jk} is *the torsion tensor* of the space $\mathbb{GR}_N$. It also holds

$$\Gamma^\alpha_{\underline{i\alpha}} = \frac{1}{2}\Big(\ln|g|\Big)_{,i} \quad \text{and} \quad \Gamma^\alpha_{\underset{\vee}{i\alpha}} = 0. \tag{4}$$

One kind of covariant derivation with regard to the affine connection of associated space $\mathbb{R}_N$ is defined as (see [7–9, 16]). For example, for the tensor a^i_j, we have

$$a^i_{j|k} = a^i_{j,k} + \Gamma^i_{\underline{\alpha k}} a^\alpha_j - \Gamma^\alpha_{\underline{jk}} a^i_\alpha. \tag{5}$$

Unlike the affine connection of a non-symmetric affine connection spaces, one may discover four kinds of affine connection of a generalized Riemannian space $\mathbb{GR}_N$. With regard to these kinds of affine connection, S. M. Minčić obtained twelve curvature tensors [10, 11]

$$K^i_{jmn} = R^i_{jmn} + u\Gamma^i_{\underset{\vee}{jm}|n} + u'\Gamma^i_{\underset{\vee}{jn}|m} + v\Gamma^\alpha_{\underset{\vee}{jm}}\Gamma^i_{\underset{\vee}{\alpha n}} + v'\Gamma^\alpha_{\underset{\vee}{jn}}\Gamma^i_{\underset{\vee}{\alpha m}} + w\Gamma^\alpha_{\underset{\vee}{mn}}\Gamma^i_{\underset{\vee}{\alpha j}}, \tag{6}$$

for real constants u, u', v, v', w and

$$R^i_{jmn} = \Gamma^i_{\underline{jm},n} - \Gamma^i_{\underline{jn},m} + \Gamma^\alpha_{\underline{jm}}\Gamma^i_{\underline{\alpha n}} - \Gamma^\alpha_{\underline{jn}}\Gamma^i_{\underline{jm}}. \tag{7}$$

Many books and research papers are dedicated to the study of spaces with torsion, generalized Riemannian spaces and mappings between them [2–6, 10–15, 17–25, 27–30]. The aim of this paper is to obtain invariants of special almost geodesic mappings of the second type.

2. Invariants of second type almost geodesic mappings

A mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ determined with the equations

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \psi_j\delta^i_k + \psi_k\delta^i_j + 2F^i_j\sigma_k + 2F^i_k\sigma_j + \xi^i_{jk} \tag{8}$$

$$F^i_{\underset{p}{j|k}} + F^i_{\underset{p}{k|j}} + 2F^i_\alpha F^\alpha_j\sigma_k + 2F^i_\alpha F^\alpha_k\sigma_j + 2\xi^i_{\alpha j}\sigma_k + 2\xi^i_{\alpha k}\sigma_j = \mu_j F^i_k + \mu_k F^i_j + \nu_j\delta^i_k + \nu_k\delta^i_j, \tag{9}$$

$p = 1, 2$, for 1-forms $\psi_j, \sigma_j, \mu_j, \nu_j$, affinor structure F^i_j and the tensor ξ^i_{jk} anti-symmetric in indices j and k is called the almost geodesic mapping of the second type and the p-th kind.

Second type almost geodesic mapping satisfies the property of reciprocity if it preserves the affinor structure F^i_j and the corresponding inverse mapping f^{-1} is the second type almost geodesic mapping of the p-th kind. This mapping satisfies the property of reciprocity if and only if $F^i_\alpha F^\alpha_j = e\delta^i_j, e = \pm 1, 0$. These mappings are elements of the class $\underset{p}{\pi_2}(e)$.

2.1. Generalized Thomas projective parameter

Let us consider an almost geodesic mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ of a type $\underset{p}{\pi}2(e)$ determined by the affinor $F^i_j = \frac{1}{2} g^{\underline{i\alpha}} g_{\underset{\vee}{j\alpha}}$.

We have that is

$$\Gamma^i_{\underset{\vee}{jk}} = F^i_{j|k} - F^i_{k|j} - \frac{1}{2} g^{\underline{i\alpha}} g_{\underset{\vee}{jk}|\alpha}. \tag{10}$$

From this equation, one obtains that is

$$\overline{\Gamma}^i_{\underset{\vee}{jk}} - \Gamma^i_{\underset{\vee}{jk}} = \underset{(p)}{\overline{\tau}}{}^i_{jk} - \underset{(p)}{\tau}{}^i_{jk}, \tag{11}$$

$p = 1, \ldots, 4$, for

$$\underset{(1)}{\tau}{}^i_{jk} = \Gamma^i_{\underline{\alpha k}} F^\alpha_j - \Gamma^i_{\underline{\alpha j}} F^\alpha_k - \frac{1}{2} g^{\underline{i\alpha}} g_{\underset{\vee}{jk}|\alpha}, \tag{12}$$

$$\begin{aligned}\underset{(2)}{\tau}{}^i_{jk} &= \frac{1}{N+1}\Big(\delta^i_k(\Gamma^\beta_{\underline{\alpha\beta}} F^\alpha_j + (N+1)e\sigma_j) - \delta^i_j(\Gamma^\beta_{\underline{\alpha\beta}} F^\alpha_k + (N+1)e\sigma_k)\Big)\\ &+ (F^i_k F^\alpha_j - F^i_j F^\alpha_k)\sigma_\alpha - \frac{1}{2} g^{\underline{i\alpha}} g_{\underset{\vee}{jk}|\alpha} - \frac{1}{N+1}\Big(F^i_j(\Gamma^\alpha_{\underline{k\alpha}} + \sigma_\alpha F^\alpha_k) - F^i_k(\Gamma^\alpha_{\underline{j\alpha}} + \sigma_\alpha F^\alpha_j)\Big),\end{aligned} \tag{13}$$

$$\begin{aligned}\underset{(3)}{\tau}{}^i_{jk} &= e\delta^i_k \sigma_j - \frac{1}{N+1}\delta^i_j\Big(\Gamma^\beta_{\underline{\alpha\beta}} F^\alpha_k + e\sigma_k\Big)\\ &+ \Gamma^i_{\underline{\alpha k}} F^\alpha_j + F^i_j F^\alpha_k \sigma_\alpha - \frac{1}{2} g^{\underline{i\alpha}} g_{\underset{\vee}{jk}|\alpha} - \frac{1}{N+1} F^i_k\Big(\Gamma^\alpha_{\underline{j\alpha}} + \sigma_\alpha F^\alpha_j\Big),\end{aligned} \tag{14}$$

$$\begin{aligned}\underset{(4)}{\tau}{}^i_{jk} &= -e\delta^i_j \sigma_k + \frac{1}{N+1}\delta^i_k\Big(\Gamma^\beta_{\underline{\alpha\beta}} F^\alpha_j + e\sigma_j\Big)\\ &- \Gamma^i_{\underline{\alpha j}} F^\alpha_k - F^i_k F^\alpha_j \sigma_\alpha - \frac{1}{2} g^{\underline{i\alpha}} g_{\underset{\vee}{jk}|\alpha} + \frac{1}{N+1} F^i_j\Big(\Gamma^\alpha_{\underline{k\alpha}} + \sigma_\alpha F^\alpha_k\Big).\end{aligned} \tag{15}$$

Moreover, it holds

$$\overline{\Gamma}^i_{\underline{jk}} - \Gamma^i_{\underline{jk}} = \underset{(q)}{\overline{\omega}}{}^i_{jk} - \underset{(q)}{\omega}{}^i_{jk}, \tag{16}$$

$q = 1, 2$, for

$$\underset{(1)}{\omega}{}^i_{jk} = \Gamma^i_{\underline{jk}}, \tag{17}$$

$$\underset{(2)}{\omega}{}^i_{jk} = -F^i_j \sigma_k - F^i_k \sigma_j + \frac{1}{N+1}\delta^i_j\Big(\Gamma^\alpha_{\underline{k\alpha}} + F^\alpha_k \sigma_\alpha\Big) + \frac{1}{N+1}\delta^i_k\Big(\Gamma^\alpha_{\underline{j\alpha}} + F^\alpha_j \sigma_\alpha\Big). \tag{18}$$

Lemma 2.1. [27] *Let $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ be an almost geodesic mapping of a type $\underset{t}{\pi}(e)$, $t = 1, 2$ determined with $F^i_j = \frac{1}{2} g^{\underline{i\alpha}} g_{\underset{\vee}{j\alpha}}$. The geometrical objects*

$$\mathcal{T}^i_{jk} = \Gamma^i_{\underline{jk}} - \underset{(2)}{\omega}{}^i_{jk}, \quad \underset{(p)}{\mathcal{T}}{}^i_{jk} = \Gamma^i_{jk} - \underset{(2)}{\omega}{}^i_{jk} - \underset{(p)}{\tau}{}^i_{jk}, \quad \underset{(p)}{\hat{\mathcal{T}}}{}^i_{jk} = \Gamma^i_{\underset{\vee}{jk}} - \underset{(p)}{\tau}{}^i_{jk}, \tag{19}$$

are invariants of the mapping f. □

2.2. Generalized Weyl projective tensor

We have that is

$$\begin{aligned}\underset{(p)}{\overline{\hat{\mathcal{T}}}}{}^{i}_{jm\|n} - \underset{(p)}{\hat{\mathcal{T}}}{}^{i}_{jm|n} &= \overline{\Gamma}{}^{i}_{\underset{\vee}{jm}\|n} - \underset{(p)}{\overline{\tau}}{}^{i}_{jm\|n} - \Gamma^{i}_{\underset{\vee}{jm}|n} + \underset{(p)}{\tau}{}^{i}_{jm|n}\\ &= \underset{(q_1)}{\overline{\omega}}{}^{i}_{\alpha n}\underset{(p)}{\overline{\hat{\mathcal{T}}}}{}^{\alpha}_{jm} - \underset{(q_2)}{\overline{\omega}}{}^{\alpha}_{jn}\underset{(p)}{\overline{\hat{\mathcal{T}}}}{}^{i}_{\alpha m} - \underset{(q_3)}{\overline{\omega}}{}^{\alpha}_{mn}\underset{(p)}{\overline{\hat{\mathcal{T}}}}{}^{\alpha}_{j\alpha} - \underset{(q_1)}{\omega}{}^{i}_{\alpha n}\underset{(p)}{\hat{\mathcal{T}}}{}^{\alpha}_{jm} + \underset{(q_2)}{\omega}{}^{\alpha}_{jn}\underset{(p)}{\hat{\mathcal{T}}}{}^{i}_{\alpha m} + \underset{(q_3)}{\omega}{}^{\alpha}_{mn}\underset{(p)}{\hat{\mathcal{T}}}{}^{\alpha}_{j\alpha}\\ &= \underset{\substack{(q)\\(p)}}{\overline{\zeta}}{}^{i}_{jmn} - \underset{\substack{(q)\\(p)}}{\zeta}{}^{i}_{jmn},\end{aligned}\tag{20}$$

for $q=(q_1,q_2,q_3), q_1,q_2,q_3\in\{1,2\}$ and

$$\underset{\substack{(q)\\(p)}}{\zeta}{}^{i}_{jmn} = \underset{(q_1)}{\omega}{}^{i}_{\alpha n}\underset{(p)}{\hat{\mathcal{T}}}{}^{\alpha}_{jm} - \underset{(q_2)}{\omega}{}^{\alpha}_{jn}\underset{(p)}{\hat{\mathcal{T}}}{}^{i}_{\alpha m} - \underset{(q_3)}{\omega}{}^{\alpha}_{mn}\underset{(p)}{\hat{\mathcal{T}}}{}^{\alpha}_{j\alpha}.\tag{21}$$

From this equation, we get it holds the following equation

$$\overline{\Gamma}{}^{i}_{\underset{\vee}{jm}\|n} = \Gamma^{i}_{\underset{\vee}{jm}|n} + \underset{(p)}{\overline{\tau}}{}^{i}_{jm\|n} + \underset{\substack{(q)\\(p)}}{\overline{\zeta}}{}^{i}_{jmn} - \underset{(p)}{\tau}{}^{i}_{jm\|n} - \underset{\substack{(q)\\(p)}}{\zeta}{}^{i}_{jmn}.\tag{22}$$

From the equality $\underset{(p^1)}{\overline{\hat{\mathcal{T}}}}{}^{\alpha}_{jm}\underset{(p^2)}{\overline{\hat{\mathcal{T}}}}{}^{i}_{\alpha n} = \underset{(p^1)}{\hat{\mathcal{T}}}{}^{\alpha}_{jm}\underset{(p^2)}{\hat{\mathcal{T}}}{}^{i}_{\alpha n}$, we obtain that is

$$\overline{\Gamma}{}^{\alpha}_{\underset{\vee}{jm}}\overline{\Gamma}{}^{i}_{\underset{\vee}{\alpha n}} = \Gamma^{\alpha}_{\underset{\vee}{jm}}\Gamma^{i}_{\underset{\vee}{\alpha n}} + \underset{\substack{(q^2)\\(q^1)}}{\overline{\Theta}}{}^{i}_{jmn} - \underset{\substack{(q^2)\\(q^1)}}{\Theta}{}^{i}_{jmn},\tag{23}$$

for

$$\underset{\substack{(q^2)\\(q^1)}}{\Theta}{}^{i}_{jmn} = \Gamma^{\alpha}_{\underset{\vee}{jm}}\underset{(q^2)}{\tau}{}^{i}_{\alpha n} + \Gamma^{i}_{\underset{\vee}{\alpha n}}\underset{(q^1)}{\tau}{}^{\alpha}_{jm} - \underset{(q^1)}{\tau}{}^{\alpha}_{jm}\underset{(q^2)}{\tau}{}^{i}_{\alpha n}.\tag{24}$$

It is obtained [27] that the geometrical objects

$$\underset{(2)}{\mathcal{W}}{}^{i}_{jmn} = R^{i}_{jmn} + \underset{(2)}{\widetilde{\mathcal{W}}}{}^{i}_{jmn} \quad\text{and}\quad \underset{(2)}{W}{}^{i}_{jmn} = R^{i}_{jmn} + \underset{(2)}{\widetilde{W}}{}^{i}_{jmn},\tag{25}$$

for

$$\begin{aligned}\underset{(2)}{\widetilde{\mathcal{W}}}{}^{i}_{jmn} &= \big(\sigma_j F^i_m + \sigma_m F^i_j\big)_{|n} - \big(\sigma_j F^i_n + \sigma_n F^i_j\big)_{|m}\\ &+ \frac{1}{(N+1)^2}\delta^i_n\Big((N+1)\Big(\Gamma^{\alpha}_{\underline{j\alpha}|m} + \sigma_\alpha F^{\alpha}_{j|m} - \Gamma^{\beta}_{\underline{\alpha\beta}}(\sigma_j F^{\alpha}_m + \sigma_m F^{\alpha}_j)\Big) + \Gamma^{\alpha}_{\underline{j\alpha}}\Gamma^{\beta}_{\underline{m\beta}} + \Gamma^{\alpha}_{\underline{j\alpha}}F^{\beta}_m\sigma_\beta + \Gamma^{\alpha}_{\underline{m\alpha}}F^{\beta}_j\sigma_\beta\Big)\\ &- \frac{1}{(N+1)^2}\delta^i_m\Big((N+1)\Big(\Gamma^{\alpha}_{\underline{j\alpha}|n} + \sigma_\alpha F^{\alpha}_{j|n} - \Gamma^{\beta}_{\underline{\alpha\beta}}(\sigma_j F^{\alpha}_n + \sigma_n F^{\alpha}_j)\Big) + \Gamma^{\alpha}_{\underline{j\alpha}}\Gamma^{\beta}_{\underline{n\beta}} + \Gamma^{\alpha}_{\underline{j\alpha}}F^{\beta}_n\sigma_\beta + \Gamma^{\alpha}_{\underline{n\alpha}}F^{\beta}_j\sigma_\beta\Big)\\ &- \frac{1}{N+1}\delta^i_j\Big(\Gamma^{\alpha}_{\underline{m\alpha}|n} - \Gamma^{\alpha}_{\underline{n\alpha}|m} + \sigma_\alpha(F^{\alpha}_{m|n} - F^{\alpha}_{n|m})\Big),\end{aligned}\tag{26}$$

$$\begin{aligned}\underset{(2)}{\widetilde{W}}{}^{i}_{jmn} &= \big(\sigma_j F^i_m + \sigma_m F^i_j\big)_{|n} - \big(\sigma_j F^i_n + \sigma_n F^i_j\big)_{|m}\\ &+ \frac{1}{N-1}\delta^i_n\Big(R_{jm} + \big(\sigma_j F^{\alpha}_m + \sigma_m F^{\alpha}_j\big)_{|\alpha} - \sigma_{\alpha|m}F^{\alpha}_j - \sigma_\alpha F^{\alpha}_{j|m}\Big) - \frac{1}{N^2-1}\delta^i_n(F^{\alpha}_{j|m} - F^{\alpha}_{m|j})\sigma_\alpha\\ &- \frac{1}{N-1}\delta^i_m\Big(R_{jn} + \big(\sigma_j F^{\alpha}_n + \sigma_n F^{\alpha}_j\big)_{|\alpha} - \sigma_{\alpha|n}F^{\alpha}_j - \sigma_\alpha F^{\alpha}_{j|n}\Big) + \frac{1}{N^2-1}\delta^i_m(F^{\alpha}_{j|n} - F^{\alpha}_{n|j})\sigma_\alpha\\ &+ \frac{1}{N+1}\delta^i_j\Big(\sigma_{\alpha|n}F^{\alpha}_m + \sigma_\alpha F^{\alpha}_{m|n} - \sigma_{\alpha|m}F^{\alpha}_n - \sigma_\alpha F^{\alpha}_{n|m}\Big),\end{aligned}\tag{27}$$

are invariants of an almost geodesic mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ of a type $\underset{t}{\pi}2(e), t = 1, 2$ determined by afinor $2F^i_j = g^{\underline{i\alpha}} g_{\underset{\vee}{j\alpha}}$.

Theorem 2.2. *Let* $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ *be an almost geodesic mapping of a type* $\underset{t}{\pi}2(e), t = 1, 2$ *determined by affinor* $2F^i_j = g^{\underline{i\alpha}} g_{\underset{\vee}{j\alpha}}$. *The families of geometrical objects*

$$\begin{aligned}\underset{(2)}{\mathcal{W}}{}^i_{(p).(q).jmn} = K^i_{jmn} + \underset{(2)}{\widetilde{\mathcal{W}}}{}^i_{jmn} - u\Big(\underset{(p^1)}{\tau}{}^i_{jm|n} + \underset{\substack{(q^1)\\(p^1)}}{\zeta}{}^i_{jmn}\Big) - u'\Big(\underset{(p^2)}{\tau}{}^i_{jn|m} + \underset{\substack{(q^2)\\(p^2)}}{\zeta}{}^i_{jmn}\Big)\\ - v\underset{\substack{(q^4)\\(q^3)}}{\Theta}{}^i_{jmn} - v'\underset{\substack{(q^6)\\(q^5)}}{\Theta}{}^i_{jnm} - w\underset{\substack{(q^8)\\(q^7)}}{\Theta}{}^i_{mnj},\end{aligned} \tag{28}$$

$$\begin{aligned}\underset{(2)}{W}{}^i_{(p).(q).jmn} = K^i_{jmn} + \underset{(2)}{\widetilde{W}}{}^i_{jmn} - u\Big(\underset{(p^1)}{\tau}{}^i_{jm|n} + \underset{\substack{(q^1)\\(p^1)}}{\zeta}{}^i_{jmn}\Big) - u'\Big(\underset{(p^2)}{\tau}{}^i_{jn|m} + \underset{\substack{(q^2)\\(p^2)}}{\zeta}{}^i_{jmn}\Big)\\ - v\underset{\substack{(q^4)\\(q^3)}}{\Theta}{}^i_{jmn} - v'\underset{\substack{(q^6)\\(q^5)}}{\Theta}{}^i_{jnm} - w\underset{\substack{(q^8)\\(q^7)}}{\Theta}{}^i_{mnj},\end{aligned} \tag{29}$$

are families of invariants of the mapping f.

Proof. We have that is

$$\begin{aligned}\overline{K}^i_{jmn} = K^i_{jmn} + \big(\overline{R} - R\big)^i_{jmn} + u\Big(\overline{\Gamma}^i_{\underset{\vee}{jm}\|n} - \Gamma^i_{\underset{\vee}{jm}|n}\Big) + u'\Big(\overline{\Gamma}^i_{\underset{\vee}{jn}\|m} - \Gamma^i_{\underset{\vee}{jn}|m}\Big)\\ + v\Big(\overline{\Gamma}^\alpha_{\underset{\vee}{jm}}\overline{\Gamma}^i_{\underset{\vee}{\alpha n}} - \Gamma^\alpha_{\underset{\vee}{jm}}\Gamma^i_{\underset{\vee}{\alpha n}}\Big) + v'\Big(\overline{\Gamma}^\alpha_{\underset{\vee}{jn}}\overline{\Gamma}^i_{\underset{\vee}{\alpha m}} - \Gamma^\alpha_{\underset{\vee}{jn}}\Gamma^i_{\underset{\vee}{\alpha m}}\Big) + w\Big(\overline{\Gamma}^\alpha_{\underset{\vee}{mn}}\overline{\Gamma}^i_{\underset{\vee}{\alpha j}} - \Gamma^\alpha_{\underset{\vee}{mn}}\Gamma^i_{\underset{\vee}{\alpha j}}\Big).\end{aligned}$$

From the equalities $\overline{\underset{(2)}{\mathcal{W}}}{}^i_{jmn} = \underset{(2)}{\mathcal{W}}{}^i_{jmn}$ and $\underset{(2)}{\overline{W}}{}^i_{jmn} = \underset{(2)}{W}{}^i_{jmn}$ such as the equations (22, 23) as well, we obtain that is

$$\overline{\underset{(2)}{\mathcal{W}}}{}^i_{(p).(q).jmn} = \underset{(2)}{\mathcal{W}}{}^i_{(p).(q).jmn} \quad \text{and} \quad \underset{(2)}{\overline{W}}{}^i_{(p).(q).jmn} = \underset{(2)}{W}{}^i_{(p).(q).jmn},$$

which proves this theorem. □

Corollary 2.3. *The invariants* (26, 27, 28, 29) *satisfy the following equations*

$$\begin{aligned}\underset{(2)}{\mathcal{W}}{}^i_{(p).(q).jmn} ={}& \underset{(2)}{\mathcal{W}}{}^i_{jmn} + u\Big(\underset{(p^1)}{\mathcal{T}}{}^i_{jm|n} - \underset{\substack{(q^1)\\(p^1)}}{\zeta}{}^i_{jmn}\Big) - u'\Big(\underset{(p^2)}{\mathcal{T}}{}^i_{jn|m} - \underset{\substack{(q^2)\\(p^2)}}{\zeta}{}^i_{jmn}\Big)\\ &+ v\Big(\Gamma^\alpha_{\underset{\vee}{jm}}\Gamma^i_{\underset{\vee}{\alpha n}} - \underset{\substack{(q^4)\\(q^3)}}{\Theta}{}^i_{jmn}\Big) + v'\Big(\Gamma^\alpha_{\underset{\vee}{jn}}\Gamma^i_{\underset{\vee}{\alpha m}} - \underset{\substack{(q^6)\\(q^5)}}{\Theta}{}^i_{jnm}\Big) + w\Big(\Gamma^\alpha_{\underset{\vee}{mn}}\Gamma^i_{\underset{\vee}{\alpha j}} - \underset{\substack{(q^8)\\(q^7)}}{\Theta}{}^i_{mnj}\Big),\end{aligned}\tag{30}$$

$$\begin{aligned}\underset{(2)}{\mathcal{W}}{}^i_{(p).(q).jmn} ={}& \underset{(2)}{W}{}^i_{jmn} + \underset{(2)}{\mathcal{W}}{}^i_{jmn} - \underset{(2)}{\widetilde{W}}{}^i_{jmn} + u\Big(\underset{(p^1)}{\mathcal{T}}{}^i_{jm|n} - \underset{\substack{(q^1)\\(p^1)}}{\zeta}{}^i_{jmn}\Big) - u'\Big(\underset{(p^2)}{\mathcal{T}}{}^i_{jn|m} - \underset{\substack{(q^2)\\(p^2)}}{\zeta}{}^i_{jmn}\Big)\\ &+ v\Big(\Gamma^\alpha_{\underset{\vee}{jm}}\Gamma^i_{\underset{\vee}{\alpha n}} - \underset{\substack{(q^4)\\(q^3)}}{\Theta}{}^i_{jmn}\Big) + v'\Big(\Gamma^\alpha_{\underset{\vee}{jn}}\Gamma^i_{\underset{\vee}{\alpha m}} - \underset{\substack{(q^6)\\(q^5)}}{\Theta}{}^i_{jnm}\Big) + w\Big(\Gamma^\alpha_{\underset{\vee}{mn}}\Gamma^i_{\underset{\vee}{\alpha j}} - \underset{\substack{(q^8)\\(q^7)}}{\Theta}{}^i_{mnj}\Big),\end{aligned}\tag{31}$$

$$\begin{aligned}\underset{(2)}{W}{}^i_{(p).(q).jmn} ={}& \underset{(2)}{\mathcal{W}}{}^i_{jmn} + \underset{(2)}{\widetilde{W}}{}^i_{jmn} - \underset{(2)}{\widetilde{\mathcal{W}}}{}^i_{jmn} + u\Big(\underset{(p^1)}{\mathcal{T}}{}^i_{jm|n} - \underset{\substack{(q^1)\\(p^1)}}{\zeta}{}^i_{jmn}\Big) - u'\Big(\underset{(p^2)}{\mathcal{T}}{}^i_{jn|m} - \underset{\substack{(q^2)\\(p^2)}}{\zeta}{}^i_{jmn}\Big)\\ &+ v\Big(\Gamma^\alpha_{\underset{\vee}{jm}}\Gamma^i_{\underset{\vee}{\alpha n}} - \underset{\substack{(q^4)\\(q^3)}}{\Theta}{}^i_{jmn}\Big) + v'\Big(\Gamma^\alpha_{\underset{\vee}{jn}}\Gamma^i_{\underset{\vee}{\alpha m}} - \underset{\substack{(q^6)\\(q^5)}}{\Theta}{}^i_{jnm}\Big) + w\Big(\Gamma^\alpha_{\underset{\vee}{mn}}\Gamma^i_{\underset{\vee}{\alpha j}} - \underset{\substack{(q^8)\\(q^7)}}{\Theta}{}^i_{mnj}\Big),\end{aligned}\tag{32}$$

$$\begin{aligned}\underset{(2)}{W}{}^i_{(p).(q).jmn} ={}& \underset{(2)}{W}{}^i_{jmn} + u\Big(\underset{(p^1)}{\mathcal{T}}{}^i_{jm|n} - \underset{\substack{(q^1)\\(p^1)}}{\zeta}{}^i_{jmn}\Big) - u'\Big(\underset{(p^2)}{\mathcal{T}}{}^i_{jn|m} - \underset{\substack{(q^2)\\(p^2)}}{\zeta}{}^i_{jmn}\Big)\\ &+ v\Big(\Gamma^\alpha_{\underset{\vee}{jm}}\Gamma^i_{\underset{\vee}{\alpha n}} - \underset{\substack{(q^4)\\(q^3)}}{\Theta}{}^i_{jmn}\Big) + v'\Big(\Gamma^\alpha_{\underset{\vee}{jn}}\Gamma^i_{\underset{\vee}{\alpha m}} - \underset{\substack{(q^6)\\(q^5)}}{\Theta}{}^i_{jnm}\Big) + w\Big(\Gamma^\alpha_{\underset{\vee}{mn}}\Gamma^i_{\underset{\vee}{\alpha j}} - \underset{\substack{(q^8)\\(q^7)}}{\Theta}{}^i_{mnj}\Big),\end{aligned}\tag{33}$$

for the above defined $\underset{(2)}{\widetilde{\mathcal{W}}}{}^i_{jmn}, \underset{(2)}{\mathcal{W}}{}^i_{jmn}$. □

Corollary 2.4. *Let* $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ *be an equitorsion almost geodesic mapping of a type* $\underset{t}{\pi}_2(e)$, $t = 1, 2$, *determined with the affinor* $2F^i_j = g^{\underline{i\alpha}} g_{\underset{\vee}{j\alpha}}$. *The families of geometrical objects*

$$\underset{(2)}{\mathcal{E}}{}^i_{(q).(1).jmn} = K^i_{jmn} + \underset{(2)}{\widetilde{\mathcal{W}}}{}^i_{jmn} - u\, \underset{\substack{(q^1)\\(0)}}{\zeta}{}^i_{jmn} - u'\, \underset{\substack{(q^2)\\(0)}}{\zeta}{}^i_{jnm},\tag{34}$$

$$\underset{(2)}{\mathcal{E}}{}^i_{(q).(2).jmn} = K^i_{jmn} + \underset{(2)}{\widetilde{W}}{}^i_{jmn} - u\, \underset{\substack{(q^1)\\(0)}}{\zeta}{}^i_{jmn} - u'\, \underset{\substack{(q^2)\\(0)}}{\zeta}{}^i_{jnm},\tag{35}$$

for $q = (q^1, q^2) = \big((q^1_1, q^1_2, q^1_3), (q^2_1, q^2_2, q^2_3)\big)$ *and*

$$\underset{\substack{(q)\\(0)}}{\zeta}{}^i_{jmn} = \underset{(q_1)}{\omega}{}^i_{\alpha n}\Gamma^\alpha_{\underset{\vee}{jm}} - \underset{(q_2)}{\omega}{}^\alpha_{jn}\Gamma^i_{\underset{\vee}{\alpha m}} - \underset{(q_3)}{\omega}{}^\alpha_{mn}\Gamma^i_{\underset{\vee}{j\alpha}},$$

are families of invariants of the mapping f. □

Corollary 2.5. *The invariants* (26, 27, 34, 35) *satisfy the following equations*

$$\begin{aligned}\underset{(2)}{\mathcal{E}}{}^{i}_{(q).(1).jmn} &= \underset{(2)}{\mathcal{W}}{}^{i}_{jmn} + u\Big(\Gamma^{i}_{\underset{\vee}{jm|n}} - \underset{\substack{(q^1)\\(0)}}{\zeta}{}^{i}_{jmn}\Big) + u'\Big(\Gamma^{i}_{\underset{\vee}{jn|m}} - \underset{\substack{(q^2)\\(0)}}{\zeta}{}^{i}_{jnm}\Big)\\ &\quad + v\Gamma^{\alpha}_{\underset{\vee}{jm}}\Gamma^{i}_{\underset{\vee}{\alpha n}} + v'\Gamma^{\alpha}_{\underset{\vee}{jn}}\Gamma^{i}_{\underset{\vee}{\alpha m}} + w\Gamma^{\alpha}_{\underset{\vee}{mn}}\Gamma^{i}_{\underset{\vee}{\alpha j}},\end{aligned} \tag{36}$$

$$\begin{aligned}\underset{(2)}{\mathcal{E}}{}^{i}_{(q).(1).jmn} &= \underset{(2)}{W}{}^{i}_{jmn} + \underset{(2)}{\widetilde{\mathcal{W}}}{}^{i}_{jmn} - \underset{(2)}{\widetilde{W}}{}^{i}_{jmn} + u\Big(\Gamma^{i}_{\underset{\vee}{jm|n}} - \underset{\substack{(q^1)\\(0)}}{\zeta}{}^{i}_{jmn}\Big) + u'\Big(\Gamma^{i}_{\underset{\vee}{jn|m}} - \underset{\substack{(q^2)\\(0)}}{\zeta}{}^{i}_{jnm}\Big)\\ &\quad + v\Gamma^{\alpha}_{\underset{\vee}{jm}}\Gamma^{i}_{\underset{\vee}{\alpha n}} + v'\Gamma^{\alpha}_{\underset{\vee}{jn}}\Gamma^{i}_{\underset{\vee}{\alpha m}} + w\Gamma^{\alpha}_{\underset{\vee}{mn}}\Gamma^{i}_{\underset{\vee}{\alpha j}},\end{aligned} \tag{37}$$

$$\begin{aligned}\underset{(2)}{\mathcal{E}}{}^{i}_{(q).(2).jmn} &= \underset{(2)}{\mathcal{W}}{}^{i}_{jmn} + \underset{(2)}{\widetilde{W}}{}^{i}_{jmn} - \underset{(2)}{\widetilde{\mathcal{W}}}{}^{i}_{jmn} + u\Big(\Gamma^{i}_{\underset{\vee}{jm|n}} - \underset{\substack{(q^1)\\(0)}}{\zeta}{}^{i}_{jmn}\Big) + u'\Big(\Gamma^{i}_{\underset{\vee}{jn|m}} - \underset{\substack{(q^2)\\(0)}}{\zeta}{}^{i}_{jnm}\Big)\\ &\quad + v\Gamma^{\alpha}_{\underset{\vee}{jm}}\Gamma^{i}_{\underset{\vee}{\alpha n}} + v'\Gamma^{\alpha}_{\underset{\vee}{jn}}\Gamma^{i}_{\underset{\vee}{\alpha m}} + w\Gamma^{\alpha}_{\underset{\vee}{mn}}\Gamma^{i}_{\underset{\vee}{\alpha j}},\end{aligned} \tag{38}$$

$$\begin{aligned}\underset{(2)}{\mathcal{E}}{}^{i}_{(q).(2).jmn} &= \underset{(2)}{W}{}^{i}_{jmn} + u\Big(\Gamma^{i}_{\underset{\vee}{jm|n}} - \underset{\substack{(q^1)\\(0)}}{\zeta}{}^{i}_{jmn}\Big) + u'\Big(\Gamma^{i}_{\underset{\vee}{jn|m}} - \underset{\substack{(q^2)\\(0)}}{\zeta}{}^{i}_{jnm}\Big)\\ &\quad + v\Gamma^{\alpha}_{\underset{\vee}{jm}}\Gamma^{i}_{\underset{\vee}{\alpha n}} + v'\Gamma^{\alpha}_{\underset{\vee}{jn}}\Gamma^{i}_{\underset{\vee}{\alpha m}} + w\Gamma^{\alpha}_{\underset{\vee}{mn}}\Gamma^{i}_{\underset{\vee}{\alpha j}},\end{aligned} \tag{39}$$

for the above defined $\underset{(2)}{\widetilde{\mathcal{W}}}{}^{i}_{jmn}, \underset{(2)}{\widetilde{W}}{}^{i}_{jmn}$. □

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