



Symmetries in 4-dimensional Manifolds

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Abstract. In this paper some algebraic and geometrical properties of symmetries (taken here as Lie algebras of smooth Killing vector fields) on a 4-dimensional manifold of arbitrary signature will be described. The discussion will include the theory of the distributions arising from such vector fields, their resulting orbit and isotropy structure and certain stability properties which these orbits may, or may not, possess. A link between the isotropies and the restrictions on the fundamental tensors of Ricci and Weyl (in terms of the subalgebras of the Lie algebras $o(4)$, $o(1, 3)$ and $o(2, 2)$) will be briefly discussed.

1. Introduction

Let M be a 4-dimensional, smooth, connected, paracompact, Hausdorff manifold with metric g of signature either $(+, +, +, +)$ (positive definite), $(+, +, +, -)$ (Lorentz) or $(+, +, -, -)$ (neutral), collectively referred to as (M, g) . (In the Lorentz case, paracompactness follows automatically from the other conditions [1]). In an attempt to deal with all signatures simultaneously, if T_mM denotes the tangent space to M at $m \in M$ and $u, v \in T_mM$, $u \cdot v$ denotes their inner product, $g(m)(u, v)$, at m . A non-zero member $u \in T_mM$ is called *spacelike* if $u \cdot u > 0$, *timelike* if $u \cdot u < 0$ and *null* if $u \cdot u = 0$ and the 1-dimensional subspaces (*directions*) that each span are called, respectively, *spacelike*, *timelike* and *null*. A 2-dimensional subspace V of T_mM is called *spacelike* if each non-zero member of V is spacelike, or each non-zero member of V is timelike, *timelike* if V contains exactly two, distinct, null directions, *null* if V contains exactly one null direction and *totally null* if each non-zero member of V is null. Thus a totally null 2-space, which can only arise for neutral signature, consists, apart from the zero vector, of null vectors any two of which are orthogonal. For neutral signature, a 3-dimensional subspace of T_mM is *spacelike* (respectively, *timelike*, *null*) if its normal is spacelike (respectively, timelike, null) whereas for Lorentz signature convention demands it is spacelike (respectively, timelike) if its normal is timelike (respectively, spacelike) and null if its normal is null. If g is of positive definite signature the term subspace, together with its dimension, are all that is required. The property (spacelike, timelike, null, etc) of a path or submanifold is sometimes referred to as its *nature*.

Let X be a global, smooth vector field on M each of whose local flows ϕ_t is an isometry, that is, its pullback ϕ_t^* satisfies $\phi_t^*(g) = g$. Then, equivalently, X satisfies $\mathcal{L}_X g = 0$ on M (using \mathcal{L} to denote a Lie derivative) and X is a *Killing vector field* on M . The collection of all such vector fields is a finite-dimensional Lie algebra under the Lie bracket operation called the *Killing algebra* and is denoted by $K(M)$. If $X \in K(M)$, then X satisfies Killing's equations

$$X_{a;b} + X_{b;a} = 0 \Leftrightarrow X_{a;b} = F_{ab} = -F_{ba}, \quad X^a{}_{;bc} = F^a{}_{bc} = R^a{}_{bcd}X^d \quad (1)$$

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where a semi-colon denotes a covariant derivative with respect to the (unique) Levi-Civita connection of g , R^a_{bcd} are the components of the associated curvature tensor and F is the (skew-symmetric) Killing bivector of X . The equations (1) reduce to first order differential equations for the components of X and F along any path in M and reveal that, since M is connected, a member $X \in K(M)$ is uniquely determined by its value and that of its Killing bivector, at any $m \in M$ and hence that $\dim K(M) \leq 10$, these results being independent of signature. Thus, if $X \in K(M)$ vanishes on some non-empty open subset of M , $X \equiv 0$ on M . [It is remarked that it is assumed that there are no local (non-extendible to M) Killing vector fields on M and that all the “symmetry” is contained in the global Killing algebra $K(M)$.]

The theory of Killing symmetry is interesting and important and has been used to significant effect in producing exact solutions of Einstein’s equations in his general relativity theory. It is also useful from a purely geometrical viewpoint. This paper extends the work of the author and Bahar Kırık [2].

2. Generalised Killing Distributions and Orbits

Define a *generalised* (or *singular*) distribution $D : m \rightarrow J_m$ on M where $m \in M$ and J_m is the subspace of $T_m M$ given by $J_m = \{X(m) : X \in K(M)\}$ (noting that $\dim J_m$ is not necessarily constant on M). A submanifold N of M with (smooth) inclusion $i : N \rightarrow M$ is then an *integral manifold* of D if for each $m \in N$ the differential of i at m , $i_*(m)$, satisfies $i_*(T_m N) = J_m$. The distribution D is called *integrable* if there is a maximal, connected integral manifold of D through each $m \in M$. Since $K(M)$ is finite-dimensional, D is necessarily integrable [3] and the integral manifolds of D are *leaves* of M and satisfy the condition (not necessarily satisfied by any submanifold) that if N is an integral manifold of D and $f : M \rightarrow M$ is smooth with range in N then $f : M \rightarrow N$ is smooth [4]. These integral manifolds have a geometrical interpretation in that they are precisely the equivalence classes corresponding to the equivalence relation $m \sim m'$ where $m, m' \in M$ and some finite sequence of local flows of members of $K(M)$ maps $m \rightarrow m'$ [5]. Such integral manifolds are called *orbits* (of $K(M)$) and each $X \in K(M)$ is tangent to the integral manifold through m at any $m \in M$. Now let $m \in M$ and define a subalgebra I_m of $K(M)$ by $I_m = \{X \in K(M) : X(m) = 0\}$. [In this case any local flow ϕ_t of X satisfies $\phi_t(m) = m$ and so m is a *zero* of X and a *fixed point* of ϕ_t .] The subalgebra I_m is called the *isotropy subalgebra* (of $K(M)$) at m and is easily checked to be Lie isomorphic to the Lie algebra of all $\{F^a_b(m)\}$ where $F(m)$ is the Killing bivector of some member of I_m (with matrix commutation as its Lie product) under the map which associates $X \in I_m$ with its Killing bivector at m . Finally, for any $m \in M$, the linear map $K(M) \rightarrow T_m M$ given by $X \rightarrow X(m)$ has range space J_m and kernel I_m and so $\dim K(M) = \dim I_m + \dim O_m$ where O_m is the orbit (of $K(M)$) through m . This equation links the (m -dependent) dimensions of I_m and O_m with the (fixed by (M, g)) dimension of $K(M)$.

It is easily checked that the nature of an orbit is the same at each of its points, as is the nature of the tangent to the integral curve of a Killing vector field. An orbit O of $K(M)$ is called *proper* if $1 \leq \dim O \leq 3$. A *proper* orbit O will be called *stable* (respectively, *dimensionally stable*) if, for each $m \in O$, there exists an open neighbourhood U of m in M such that each orbit intersecting U has the same nature and dimension as O (respectively, has the same dimension as O) [6, 7]. If a proper orbit O is spacelike or timelike and $i : O \rightarrow M$ is the usual inclusion map for the submanifold O , the pullback $h \equiv i^*g$ is a metric on O . Now each $X \in K(M)$ is tangent to O at any $m \in O$ and so there exists a unique smooth vector field \tilde{X} on O such that $i_*\tilde{X} = X$ (see, e.g. [8]) and \tilde{X} is a Killing vector field for h (see, e.g. [6, 7]). Thus one has a map (in fact a Lie algebra homomorphism) $K(M) \rightarrow K(O)$ where $K(O)$ is the Killing algebra for (O, h) . This map need not be either injective or surjective [2, 6, 7].

It is useful to note the following decomposition of M with respect to $K(M)$ and with $K(M)$ assumed non-trivial. For each $0 \leq i \leq 4$ let $V_i = \{m \in M : \dim J_m = i\}$ and *disjointly* decompose $M = \bigcup_{i=0}^4 V_i$ as

$$M = \bigcup_{i=0}^4 \text{int} V_i \cup Z = V_4 \cup \bigcup_{i=0}^3 \text{int} V_i \cup Z. \quad (2)$$

where int denotes the interior in the manifold topology on M and the closed subset Z is defined by the disjointness of the decomposition. It can be shown that $\text{int} Z = \emptyset$ [2, 6, 7]. Thus M is decomposed into

open subsets of constant orbit dimension apart from the closed set Z which has empty interior. On any component of V_4 , if non-empty, the symmetries act “homogeneously”. The subset $M \setminus Z$ is the union of V_4 with all (proper) dimensionally stable orbits of $K(M)$ and is open and dense in M whilst Z is the union of V_0 and all (proper) not dimensionally stable orbits of $K(M)$ (since $K(M)$ is not trivial and hence $\text{int}V_0 = \emptyset$.) Any proper orbit of *maximum dimension* (in an obvious sense) is dimensionally stable.

3. Isotropy Structure

If O_m is the Killing orbit through m and $\dim O_m = n' < n = \dim K(M)$ then I_m is non-trivial. Let $X \in I_m$ be non-trivial, so that $X(m) = 0$ (and its Killing bivector $F(m) \neq 0$.) If one chooses normal coordinates x^a about m with domain U then, since the local flows of X are affine and preserve geodesics *and* their affine parameters, X is “linearised” on U , that is, its components are linear functions of x^a and satisfy $X^a = F^a_b(m)x^b$ in U [6, 9]. Thus if $F(m)$, which must have matrix rank an even integer since it is skew-symmetric, has rank 4, the zero m of X is *isolated* (that is, it is the only zero of X in some open neighbourhood of m) whereas if it has rank 2, the coordinates of the zeros of X in U are found from the kernel of $F(m)$ and constitute a 2–dimensional submanifold of U and hence of M . In this latter case $F(m)$ is called *simple* and can be written in the form $F^{ab}(m) = r^a s^b - s^a r^b$ (or, more briefly, as $r \wedge s$ where \wedge is the usual wedge product) where the 2–dimensional subspace of $T_m M$ spanned by $r, s \in T_m M$ is uniquely determined by $F(m)$ and called the *blade* of $F(m)$. For the above orbit O_m (and reducing U if necessary) choose independent members $X_1, \dots, X_{n'} \in K(M)$ which span the tangent space to O_m at each point in $U \cap O_m$ and give rise to independent tangent vectors at each point of U and a smooth vector field k which is orthogonal to each of $X_1, \dots, X_{n'}$ on U . Then along any path c in O_m through m with tangent $p(t)$ any $X \in I_m$ satisfies $(X^a k_a)_;b p^b = 0$ for each (independent) choice of k and p (which depend on O_m). Evaluating this at m gives $F_{ab}(m)k^a p^b = 0$. If O_m is dimensionally stable the Killing vectors $X_1, \dots, X_{n'}$ may be taken to span the orbits on U and so the stronger result $F_{ab}(m)k^b = 0$ is achieved [6, 7]. When the type and dimension of the orbit is known these equations give restrictions on the dimension of (the Killing bivector representation of) I_m and, *in the dimensionally stable case, any solution $F(m)$ is simple*.

For example, suppose $g(m)$ is positive definite and that there exists a 1–dimensional orbit O with $m \in O$. Then there exists $X \in K(M)$ spanning O at m . Choosing an orthonormal basis x, y, z, w at m and normal coordinates based on it one can arrange that $X(m) = (\partial/\partial x)_m$ and, in an obvious abuse of notation, $F_{ab}x^a y^b = F_{ab}x^a z^b = F_{ab}x^a w^b = 0$ where $F = F(m)$ and $x^a = (\partial/\partial x)_m$, etc. Thus with $\langle \rangle$ denoting the span of its included members $I_m \subset \langle y \wedge z, y \wedge w, z \wedge w \rangle$ with each member of I_m satisfying $F^a_b x^b = 0$ and is thus simple. This example is easily modified, if O is spacelike or timelike, to the Lorentz and neutral signature cases. Further, for any signature and *any* (1–dimensional) orbit type, if O is dimensionally stable, I_m is trivial.

Next, if g has positive definite signature and O is 2–dimensional one may choose the above orthonormal basis where x^a and y^a are tangent to O at m and $F(m)$ satisfies $F_{ab}x^a z^b = F_{ab}x^a w^b = F_{ab}y^a z^b = F_{ab}y^a w^b = 0$. Thus $F(m)$ lies in $\langle x \wedge y, z \wedge w \rangle$ and $\dim I_m \leq 2$ whilst, if O dimensionally stable, $F(m)$ is a multiple of $x \wedge y$ and $\dim I_m = 1$. This technique is easily extended to Lorentz and neutral signature if O is spacelike or timelike and with a little more effort if O is null or totally null.

Next, if g is of neutral signature and O_m is null with $\dim O_m = 3$ then if $l, n, L, N \in T_m M$ constitute a basis each member of which is null (and with the only non-vanishing inner products $l.n = L.N = 1$) and with l, L, N spanning O_m (so that l is the null normal to O_m) $F(m)$ lies in $\langle l \wedge n, l \wedge L, l \wedge N, L \wedge N \rangle$ and $\dim I_m \leq 4$. Here, in the dimensionally stable case, $F(m)$ lies in $\langle l \wedge L, l \wedge N, L \wedge N \rangle$ and $\dim I_m \leq 3$. The Lorentz case is similar in an appropriately modified basis.

Finally, and with g of neutral signature, let O be a 3–dimensional timelike orbit. Choose an orthonormal basis x, y, s, t at m with $x.x = y.y = -s.s = -t.t = 1$ and with x, y, s spanning the tangent space to O at m . The above results show that $F(m)$ lies in $\langle x \wedge y, x \wedge s, y \wedge s \rangle$ and so, at m , $F_{ab}t^b = 0$. Thus $F(m)$ is always simple. Similar comments apply to the spacelike case, to both (3–dimensional) spacelike and timelike cases when g has Lorentz signature and in all (3–dimensional) cases if g is positive definite. Thus for each of these $\dim I_m \leq 3$. When all the above results, for each orbit type and dimension and for each signature, are combined they place restrictions on $\dim K(M)$, as remarked above, and lead to the following theorem which

covers all signatures. It is not claimed that all situations can occur but many can and some examples can be found in [2, 6, 7, 10].

Theorem 3.1. *Let M be a 4–dimensional manifold with metric of arbitrary signature. Then the following hold for the orbits of $K(M)$ where $K(M)$ is assumed non-trivial.*

(i) *If $K(M)$ admits a (proper) dimensionally stable orbit O with $m \in O$ and $I_m M$ non-trivial the non-zero Killing bivectors in I_m are simple and have a common annihilator (that is, there exists $k \in T_m M$ such that $F_{ab} k^b = 0$ for each $F \in I_m$). Each such vector k is orthogonal to the orbit at m and the transformations associated with I_m act as the identity map on the subspace of $T_m M$ orthogonal to O .*

(ii) *If there exists a 3-dimensional, null orbit (Lorentz or neutral) $3 \leq \dim K(M) \leq 7$ whilst if there exists a 3–dimensional, null, dimensionally stable orbit (Lorentz or neutral) or a 3–dimensional timelike or spacelike orbit (Lorentz or neutral and for positive definite, spacelike), $3 \leq \dim K(M) \leq 6$.*

(iii) *If there exists a 2–dimensional, totally null orbit (neutral), $2 \leq \dim K(M) \leq 7$, if there exists a 2–dimensional, null orbit (Lorentz or neutral), $2 \leq \dim K(M) \leq 5$ and if there exists a 2–dimensional, spacelike or timelike orbit (Lorentz and neutral and for positive definite, spacelike), $2 \leq \dim K(M) \leq 4$. If there exists a 2–dimensional, dimensionally stable orbit (all signatures), $2 \leq \dim K(M) \leq 3$.*

(iv) *If there exists a 1–dimensional, null orbit (Lorentz or neutral), $1 \leq \dim K(M) \leq 5$, if there exists a 1–dimensional, spacelike or timelike orbit (Lorentz and neutral and for positive definite, spacelike), $1 \leq \dim K(M) \leq 4$ and if there exists a 1–dimensional, dimensionally stable orbit (all signatures), $\dim K(M) = 1$.*

4. Orbit Restrictions

There are further restrictions on certain orbit dimensions other than those contained in Theorem 3.1 and which can be seen from the following argument.

For all signatures, suppose there exists a 1–dimensional, non-dimensionally stable, spacelike or timelike orbit O associated with $K(M)$ and that $m \in O$. Choose normal coordinates y^a (with domain V) about m generated by a basis for $T_m M$ one member of which is tangent to O and the others orthogonal to O . Thus one may arrange that $X \in K(M)$ and $X(m) = \partial/\partial y^1$ and the 3–dimensional coordinate plane H in V given by $y^1 = 0$ is then generated by geodesics through m orthogonal to O at m . Then use the “straightening out lemma” for X ([9] section 4.1.14) to get a local chart (U, ψ) of M with $m \in U$, and, with an abuse of notation, $\psi(U) = H' \times I$, $H' \subset H$ and $I = (-a, a)$ for some $a \in \mathbb{R}$. The curves of the form $q \times I$ are integral curves of X for each $q \in H'$. Now each local flow ϕ_t of X preserves geodesics and orthogonality and thus maps H' at m to a similar section of the above flow box of X generated by geodesics orthogonal at the corresponding point of O to the integral curve of X in O . Now any other $Y \in K(M)$ is tangent to O , hence orthogonal to these sections at each point of O and hence (by a classical result saying that the inner product of a Killing vector field and the tangent to an affinely parametrised geodesic is constant along the geodesic -see, e.g. [11]) is at any $m' \in U$ orthogonal to a geodesic from m' to some point of O . This restriction shows that the orbits in U are at most 3–dimensional and with at least one orbit of dimension 2 or 3 since O is not dimensionally stable.

Now suppose there exists a 3–dimensional spacelike or timelike orbit O , for any signature. Let $m \in O$, let X, Y, Z be independent members of $K(M)$ and let U an open neighbourhood of m such that X, Y, Z span the tangent space to O at each point of $U \cap O$ and give independent members of $T_m M$ for each $m \in U$. Then U may be chosen such that there exists a smooth nowhere-zero spacelike or timelike (depending on the nature of O) vector field T on U which is orthogonal to X, Y, Z on U and whose integral curves are geodesics. The last of these follows from Killing’s equations for X, Y, Z since $0 = (T^a X_a)_{;b} T^b = T^a_{;b} T^b X_a$ on U and similarly for Y and Z . Now using the straightening out lemma again, this time for T , one may build an open neighbourhood $V \subset U$ containing m . Since any $W \in K(M)$ is tangent to $O \cap V \subset O \cap U$ it is orthogonal to T on V by the classical result mentioned above and hence any orbit in V is at most 3–dimensional (and hence exactly 3–dimensional by an appeal to rank) and of the same nature as O . Thus any 3–dimensional non-null orbit is stable.

A similar argument shows, again for any signature, that if $V_0 \neq \emptyset$, using normal coordinates about $m \in V_0$, there exists a neighbourhood U of m in which the orbits have dimension ≤ 3 .

It is clear that any proper, spacelike or timelike, dimensionally stable orbit is stable.

Theorem 4.1. *Let M be a 4–dimensional manifold with metric of arbitrary signature. Then any 3–dimensional non-null orbit is stable and any m lying on a spacelike or timelike 1–dimensional orbit admits a neighbourhood in which all orbits are at most 3–dimensional. Any proper, spacelike or timelike, dimensionally stable orbit is stable.*

Some examples to illustrate this theorem (and theorem 3.1) are given in [2].

5. Restrictions on the Ricci and Weyl Tensors

The work in section 3 dealt with the situation when a non-trivial member $X \in K(M)$ vanished at $m \in M$ so that m was a zero of X and a fixed point of the associated local flows ϕ_t of X . In this case the pushforward ϕ_{t*} is a linear isometry on $T_m M$ with respect to $g(m)$. Since $X \in K(M)$, one has the relations $\phi_t^* Ricc = Ricc$ and $\phi_t^* C = C$ where $Ricc$ denotes the Ricci tensor arising from the curvature tensor on M and C is the associated Weyl conformal tensor. Thus for example, for $u, v \in T_m M$, $Ricc(u, v) = Ricc(\phi_{t*}(u), \phi_{t*}(v))$ and (algebraic) restrictions are placed on $Ricc(m)$ (and similarly on $C(m)$). Such restrictions depend on the structure of the Lie algebra I_m (now of dimension ≥ 1) expressed in bivector form (section 3). It turns out (sections 3 and 4) that if m lies on any proper, dimensionally stable orbit or any 1– or 3–dimensional spacelike or timelike orbit, I_m has a bivector representation consisting of simple bivectors which possess a common annihilator (that is, there exists $k \in T_m M$ such that any bivector F in this representation satisfies $F^a_b k^b = 0$) and hence can be shown to have dimension ≤ 3 . Such subalgebras are called *special*. To put this into practice one requires an algebraic classification of $Ricc$ and C for each signature and these can be found, for neutral signature respectively, in [12] and [13]. For Lorentz signature the algebraic classification of C is the Petrov classification [14] (as modified by Pirani [15]) and convenient forms for the algebraic types for $Ricc(m)$ and $C(m)$ can be found in [6]. The (Petrov) types are labelled **I**, **II**, **III**, **D**, **N** and **O**, where the last type means that $C(m) = 0$. The type of $Ricc(m)$ can be given by its Segre symbol. For positive definite signature the algebraic types for $Ricc(m)$ are just the diagonalisable (over \mathbb{R}) Segre types. For $C(m)$ one must consider how to algebraically classify C for this signature. This is rather similar to that given for neutral signature in [13] but much less complicated. Further details on the isotropy structure and the consequent algebraic types will be given by Dr B Kırık in her contribution to this volume [10]. It is remarked here, for later use and for all signatures, that if m lies on some *dimensionally stable* orbit and $\dim I_m \geq 3$ the Weyl tensor vanishes at m (and in the Lorentz case this result applies at any $m \in M$ where $\dim I_m \geq 3$ —see, e.g., [6, 16]) and again for any $m \in M$, if $\dim I_m \geq 4$, $Ricc(m)$ is proportional to $g(m)$ [2, 6].

6. Further Comments

The results of theorem 3.1 lead to further results in those cases when $K(M)$ is of high dimension. The next theorem summarises some of these results.

Theorem 6.1. (i) *Suppose $\dim K(M) = 7$. Then for Lorentz and neutral signatures V_4 is dense in M and if $M \neq V_4$ any orbit in $M \setminus V_4$ is not dimensionally stable. In the positive definite case $M = V_4$.*

(ii) *Suppose $\dim K(M) \geq 8$. For all signatures, $M = V_4$ and (M, g) is an Einstein space. If, in addition, $\dim K(M) \geq 9$ (M, g) is a conformally flat, Einstein space, hence of constant curvature, and there exists, locally, a 10–dimensional Killing algebra. If M is simply connected, $\dim K(M) = 10$.*

(iii) *Suppose $\dim K(M) = 6$. Then, in the positive definite case $M = V_4$ or $M = V_3$. Otherwise, either $M = V_4$, or $V_4 = \emptyset$ or $\emptyset \neq V_4 \neq M$ and in this last case $M \setminus V_4$ admits a non-dimensionally stable orbit which is either 2–dimensional and totally null (neutral signature) or 3–dimensional and null (Lorentz or neutral signature). Again, for Lorentz or neutral signatures, if $V_4 = \emptyset$ the subset U of points of M lying on proper dimensionally stable orbits is open and dense in M and so, on U , $\dim I_m \geq 3$ and $C(m)$ vanishes there and hence on M . Thus (M, g) is conformally flat.*

Proof

(i) (Lorentz and neutral signatures.) If V_4 (which is open in M) is not dense in M there exists a non-empty open subset $U \subset M \setminus V_4$. Then the orbit of maximum dimension (≤ 3) intersecting U non-trivially is dimensionally stable and theorem 3.1 gives the contradiction that $\dim K(M) \leq 6$. Thus V_4 is dense in M and it follows that if $M \neq V_4$ any orbit intersecting $M \setminus V_4$ is not dimensionally stable. In the positive definite case it follows immediately from theorem 3.1 that $M = V_4$.

(ii) (All signatures.) If $\dim K(M) \geq 8$ then $\dim I_m \geq 4$ for each $m \in M$ and so (M, g) is an Einstein space [2, 6, 13]. It follows from theorem 3.1 that $M = V_4$. For $\dim K(M) \geq 9$, $\dim I_m \geq 5$ for each $m \in M$ and so $C \equiv 0$ on M [2, 6, 13] and (M, g) is a conformally flat Einstein space and is hence of constant curvature. Thus a local Lie algebra of Killing vector fields of dimension 10 is admitted about each point and which may be extended globally to M if M is simply connected [17, 18].

(iii) For $\dim K(M) = 6$ (Lorentz or neutral signatures) if $\emptyset \neq V_4 \neq M$ (and since M is connected and V_4 is open in M) $M \setminus V_4$ admits a non-dimensionally stable orbit (otherwise it would be open and contradict the connectedness of M) which is either 2-dimensional and totally null or 3-dimensional and null. If $V_4 = \emptyset$ and if a non-empty open subset is contained in $M \setminus U$ the orbit of maximum dimension intersecting this subset non-trivially is a proper, dimensionally stable orbit and a contradiction to the definition of U is obtained. Thus U is (open and) dense in M . So for $m \in U$, $\dim I_m \geq 3$ and $C(m)$ vanishes there (since m lies on a dimensionally stable orbit) and hence on M . Thus (M, g) is conformally flat. For the positive definite case theorem 3.1 shows that $M = V_3 \cup V_4$ with V_4 open (and V_3 is open from theorem 4.1) and the result follows from the connectedness of M .

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