



A Study on Killing Vector Fields in Four-Dimensional Spaces

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Abstract. In the present study, some properties of Killing vector fields are investigated on 4–dimensional manifolds in case of the signature of the metric tensor g is either Lorentz or positive definite or neutral. First of all, the notation and the main object of the study are introduced on these manifolds. Later on, some special subalgebras are examined for the members of the Killing algebra when the Killing vector field vanishes at a point of the manifold admitting any of these metric signatures. The constraints of this examination to the Weyl conformal curvature tensor and the Ricci tensor are then studied and some results are obtained. Finally, some examples related to these results are given for all metric signatures.

1. Introduction

The concept of symmetry is an important and interesting subject not only in mathematics but also in physics. Referring to the literature, there are many applications of symmetries to Albert Einstein's general theory of relativity. These allow the use of geometrical techniques in modelling some problems in physics. For this reason, the role of geometry in physics has a significant place. It is also important to investigate the geometrical features of symmetries and the related structures in mathematics. Roughly speaking, a symmetry is defined by local transformations preserving some geometrical properties of the manifold. One of the useful items that helps defining such symmetries is the notion of vector field. Some well-known examples of the local transformations of some special symmetries, which can be represented by vector fields, are (local) diffeomorphisms preserving the metric (in this case the associated vector fields are so called *Killing vector fields*), (local) diffeomorphisms preserving the metric up to a conformal factor (the associated vector fields are called *conformal vector fields*), (local) diffeomorphisms preserving the metric up to a constant conformal factor (the associated vector fields are called *homothetic vector fields*), (local) diffeomorphisms preserving the geodesics (the associated vector fields are called *projective vector fields*), (local) diffeomorphisms preserving the geodesics and their affine parameters (the associated vector fields are called *affine vector fields*) and so on. Examining these symmetries on 4–dimensional manifolds admitting a Lorentz metric (known as *space-times*) is of interest to many researchers and it is a popular issue in the literature and so, it is not possible to include all these works because of being studied from different angles (however, in connection with this study see, e.g., [1–3, 8–10, 12, 25]).

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One of the most widely studied vector fields both in differential geometry and in physics is Killing vector fields which are named after the German Mathematician Wilhelm Karl Joseph Killing. Killing vector fields are defined to be smooth vector fields and they preserve the metric tensor of the manifold which are important types of symmetries as mentioned above. On the other hand, the investigation of the geometric structures of 4–dimensional manifolds admitting a metric attracts the attention of researchers, e.g., see [7, 8, 13, 14, 17–19, 22, 26]. Under a geometric point of view, it is also interesting to study special symmetries in 4–dimensional manifolds admitting a metric other than the Lorentz metric which can only be of signatures $(+, +, +, +)$ (positive definite signature) or $(+, +, -, -)$ (neutral signature). For instance, Killing symmetries on 4–dimensional manifolds admitting a neutral metric has recently been examined in [16] and various results on the Lie algebra of global Killing vector fields and the theory of Killing orbits are obtained (a similar study on space-times is given in [9]). This paper is an extension of [16] to all metric signatures in 4–dimensional manifolds.

The aim of this work is to study some properties of Killing vector fields on 4–dimensional manifolds admitting a metric. Section 2 is devoted to introduce the notation and basic concepts. In Section 3, the geometry of orbits, which is closely related to the Lie algebra of Killing vector fields, and the isotropy subalgebras are considered and special subalgebras of the orthogonal algebra of g (depending on the metric signature which is either $o(2, 2)$ or $o(1, 3)$ or $o(4)$) are discussed. After that, in Section 4, some applications of Killing symmetries to Ricci and Weyl conformal curvature tensors are investigated when there is an isotropy and possible algebraic types of these special tensors are listed. In Section 5, some metric examples are given to illustrate some results of the study.

2. Preliminaries

Let M be a smooth manifold of dimension $n = 4$ and let g be a (smooth) metric of arbitrary signature on M and this couple will be denoted by (M, g) . Throughout the following, it will be assumed that M is Hausdorff, connected and paracompact (the last necessarily holding if g has Lorentz signature, see [6]). The tangent space at $p \in M$ is written as T_pM and $0 \neq v \in T_pM$ is called *spacelike* if $g(v, v) > 0$, *timelike* if $g(v, v) < 0$ and *null* (or *lightlike*) if $g(v, v) = 0$ at p . For each signature, one can define a basis for T_pM , e.g., see [8, 19, 26] (and these notations will be adapted). A 2–dimensional subspace (which will be referred to as a 2–space) of T_pM can be spacelike (each non-zero member of it is spacelike or each non-zero member of it is timelike) or timelike (it has exactly two, distinct, null directions) or null (it has exactly one null direction) or totally null (each non-zero member of it is null and hence any two non-zero members of it are orthogonal). It is noted that there are only spacelike 2–spaces for the signature $(+, +, +, +)$ and totally null 2–spaces only occur in $(+, +, -, -)$. Similarly, a 3–dimensional subspace of T_pM can be classified according to its normal (for the details of the neutral signature case we refer to [16] and for the Lorentz case [8]). Let Λ_pM be space of all 2–forms where the members of it are also called *bivectors*. A bivector F is called *simple* (or *non-simple*) if its rank is 2 (or 4). In the former case, F can be written as $F^{ij} = v^i w^j - w^i v^j$ for $v, w \in T_pM$ where the 2–space spanned by v and w is called the *blade* of F and which is uniquely determined by it and denoted by $v \wedge w$. A simple bivector is called *spacelike* (respectively, *timelike*, *null* or *totally null*) if its blade is *spacelike* (respectively, *timelike*, *null* or *totally null*). It is useful to note that a bivector F is simple if and only if there exists $0 \neq v \in T_pM$ such that $F_{ij}v^j = 0$ if and only if $F_{i[jk]} = 0$ if and only if $F_{ij}F^{jk} = F_{ij}F^{ji} = 0$ where $*$ denotes the Hodge duality operator (for details see, page 175 of [8] where this result is proved for Lorentz signature but it is true for all signatures).

Now let X be a global vector field on (M, g) . Suppose that the following equations are satisfied:

$$\nabla_j X_i = \frac{1}{2} h_{ij} + F_{ij}, \quad F_{ij} = -F_{ji}, \quad h_{ij} = h_{ji} = \mathcal{L}_X g_{ij} \tag{1}$$

where ∇ denotes the covariant derivative with respect to the Levi-Civita connection of g , $F \in \Lambda_pM$ with components $F_{ij} = -F_{ji}$, the X^i 's are the components of the vector field X , \mathcal{L} is the Lie derivative and h is a symmetric tensor field. The equation (1) gives some special vector fields as follows. If h satisfies

$$\nabla_k h_{ij} = 2g_{ij}\phi_k + g_{ik}\phi_j + g_{jk}\phi_i \tag{2}$$

for some necessarily closed 1-form ϕ on (M, g) , then X is called a *projective vector field*. Moreover, if h given in (1) is *parallel*, that is, if $\nabla h \equiv 0$ (equivalently, $\phi \equiv 0$ in (2)), X is called an *affine vector field*. The algebra of affine vector fields has two important subalgebras known as the *homothetic algebra* and the *Killing algebra*. For an affine vector field X , if $h = 2cg$ for a constant $c \in \mathbb{R}$ (and hence, $\mathcal{L}_X g = 2cg$ in (1)), then X is named as *homothetic*. Furthermore, in case of $c = 0$ (that is, $h \equiv 0$ on M and so, $\mathcal{L}_X g = 0$), then X is called a *Killing vector field*. In addition to these definitions, if $\mathcal{L}_X g = 2\mu g$ for a smooth function μ on M , X is said to be a *conformal vector field*. For details of these vector fields given above, we refer to [8]. Besides all these, if $\mathcal{L}_X \text{Riem} = 0$, $\mathcal{L}_X \text{Ricc} = 0$ and $\mathcal{L}_X C = 0$ hold where *Riem*, *Ricc* and *C* denote the Riemann curvature tensor, the Ricci tensor and the Weyl conformal curvature tensor of (M, g) , respectively, then X is called a *curvature collineation (CC)*, *Ricci collineation (RC)* and *Weyl collineation (WC)*, respectively.

Let us now consider the Lie algebra of global Killing vector fields, that is, the Killing algebra denoted by $\mathcal{K}(M)$. As mentioned above, a Killing vector field $X \in \mathcal{K}(M)$ satisfies the condition $\mathcal{L}_X g = 0$. This equation is equivalent to $\varphi_t^*(g) = g$ for the pullback φ_t^* of each local diffeomorphism φ_t associated with X (that is, the *local flows* of X) for each $X \in \mathcal{K}(M)$. Thus, for $X \in \mathcal{K}(M)$, the following Killing equations are satisfied:

$$\nabla_j X_i + \nabla_i X_j = 0 \quad (\Leftrightarrow \nabla_j X_i = F_{ij} = -F_{ji}), \quad \nabla_k \nabla_j X^i = \nabla_k F^i_j = R^i_{jkl} X^l \tag{3}$$

where X^i and R^i_{jkl} are the components of the Killing vector field X and *Riem*, respectively. Moreover, $F \in \Lambda_p M$ in (3) is called the *Killing bivector* of X with components $F_{ij} = -F_{ji}$. It is well known that (3) yields a first order system of differential equations (in the 6 components of F and 4 components of X) along any curve in M and one can see that $\dim \mathcal{K}(M)$ is finite dimensional and, in particular, ≤ 10 . [It is noted that much more generally, $\dim \mathcal{K}(M) \leq \frac{1}{2}n(n+1)$ for a smooth manifold M of dimension n .]

3. The Geometry of Killing Orbits and Isotropies

Suppose that $\mathcal{K}(M)$ is non-trivial and for any $p \in M$, consider the linear map $f : \mathcal{K}(M) \rightarrow T_p M$ which assigns each $X \in \mathcal{K}(M)$ to its value $X(p)$ at p . Then there is a maximal connected submanifold of M through each $p \in M$ called the *Killing orbit* of $\mathcal{K}(M)$ (and which is a leaf of M , see e.g., [8, 9, 16, 20, 23, 24]). Moreover, the rank-nullity theorem shows that $\dim \mathcal{K}(M) = \dim \mathcal{O}_p + \dim \mathcal{I}_p$ where \mathcal{O}_p is the orbit through $p \in M$ and $\mathcal{I}_p = \{X \in \mathcal{K}(M) : X(p) = 0\}$ is the kernel of the map f called the *isotropy algebra* at p . It is true that the collection $\{F^i_j(p) : X \in \mathcal{I}_p\}$ under matrix commutation is a Lie algebra isomorphic to \mathcal{I}_p and it is either a Lie subalgebra of $o(2, 2)$ (for neutral signature) or $o(1, 3)$ (for Lorentz signature) or $o(4)$ (for positive definite signature). If $X \in \mathcal{I}_p$, then p is called a *zero* of X and p is a fixed point of the local flow φ_t associated with X . Besides, if \mathcal{I}_p is non-trivial, then $F(p) \neq 0$ where F is the Killing bivector of X . It is useful to note that if $X(p) = 0$ and $F(p) = 0$ ($p \in M$), then $X \equiv 0$ on M . Thus, if $X \in \mathcal{K}(M)$ vanishes on some non-empty open subset of M , it follows that X vanishes on M .

The mathematical structure of the Killing orbits is quite important because of their several properties which are based on the dimension and nature (timelike, spacelike, null and totally null depending on the metric signature) and which for $(+, +, -, -)$ are considered in [16], for $(+, +, +, -)$ in [9] and for the extension to all signatures in [15]. It is known that the nature of an orbit \mathcal{O} of $\mathcal{K}(M)$ is constant at all points of \mathcal{O} . If $\dim \mathcal{O}$ is either 1, 2 or 3, then we shall call \mathcal{O} a *proper orbit*. For each signature, the possibilities for such orbits have been found in [15] and this involves a study of the algebraic nature of \mathcal{I}_p for points p on the orbit and its relationship to the nature of the orbit. A proper orbit \mathcal{O} is said to be *stable* (respectively, *dimensionally stable*) if, for each $p \in \mathcal{O}$, there exists an open neighbourhood of p such that each orbit intersecting that neighbourhood non-trivially has the same nature *and* dimension as \mathcal{O} (respectively, has the same dimension as \mathcal{O}), [8, 9, 16].

One of the other important concepts in this study is to define special subalgebras of the orthogonal algebra of g . We shall be named a subalgebra as a *special subalgebra* if its members have a common annihilator, in other words, there exists $v \in T_p M$ such that $F_{ij} v^j = 0$ for each bivector F in this subalgebra. This yields that each of its bivectors is simple and its dimension is ≤ 3 , [16, 19]. It is useful to note that for $p \in \mathcal{O}$ where \mathcal{O} is a dimensionally stable orbit, \mathcal{I}_p is special, [9, 16]. According to these, one can now find all special subalgebras for each signature. The special subalgebras of $o(2, 2)$ are found in [16] and isomorphic

to types 1(a), 1(b), 1(c), 1(d), 2(g), 2(h)($\alpha\beta = 0$), 2(k), 3(c) and 3(d)($\alpha = 0$) where the subalgebras of $o(2,2)$ are tabulated in [26]. When g has Lorentz signature, the special subalgebras are those labelled $R_2, R_3, R_4, R_6, R_8, R_{10}, R_{11}$ and R_{13} where the subalgebras of $o(1,3)$ are tabulated in [22] (for details, see [8]). We can now find the special subalgebras of $o(4)$ from the labelling given in [19]. It is useful to note that sometimes only subalgebras which can be holonomy algebras were needed (see, e. g., [17, 19, 21, 26]). In particular, a necessary condition for a 1–dimensional subalgebra to be a holonomy algebra is that it is spanned by a simple bivector. However, there is no such a restriction here and all subalgebras should be examined. For positive definite signature, there are extra 1–dimensional subalgebras of $o(4)$ which can not represent a holonomy algebra for (M, g) and which are given in an orthonormal basis x, y, z, w by $S_a \equiv \langle x \wedge y + z \wedge w \rangle$ and $S_b \equiv \langle x \wedge y + \gamma(z \wedge w) \rangle$ where $\langle \rangle$ denotes a spanning set, $\gamma \in \mathbb{R}$ and $0 \neq \gamma \neq \pm 1$.

For each subalgebra of $o(4)$, the special subalgebras (up to isomorphism) are listed in the third column of Table 1. In the first column of Table 1, all subalgebras of $o(4)$ are labelled and in the second column a basis is given for these subalgebras where $\overset{+}{S} \equiv \{F : F = \overset{*}{F}\}$ is the Lie algebra $o(3)$ and $0 \neq G \in \bar{S} \equiv \{F : F = -\overset{*}{F}\}$ for $F \in \Lambda_p M$. The fourth and fifth columns will be achieved in Section 4.

Table 1: Subalgebras of $o(4)$

Type	Basis	Special Subalgebras	Ricci Type	Weyl Type
S_1	$x \wedge y$	S_1	$\{1(11)\}, \{1(111)\}, \{(11)(11)\}, \{(1111)\}$	$(D, D), (D, O)^*, (O, O)$
S_a	$x \wedge y + z \wedge w$	None	$\{(11)(11)\}, \{(1111)\}$	$(D, I), (D, D), (O, I), (D, O)^*, (O, O)$
S_b	$x \wedge y + \gamma(z \wedge w)$	None	$\{(11)(11)\}, \{(1111)\}$	$(D, D), (D, O)^*, (O, O)$
S_2	$x \wedge y, z \wedge w$	S_1	$\{(11)(11)\}, \{(1111)\}$	$(D, D), (D, O)^*, (O, O)$
S_3	$x \wedge y, x \wedge z, y \wedge z$	S_1, S_3	$\{1(111)\}, \{(1111)\}$	(O, O)
$\overset{+}{S}_3$	$\overset{+}{S}$	None	$\{(1111)\}$	$(O, I), (O, D), (O, O)$
$\overset{+}{S}_4$	$\overset{+}{S}, G$	S_1	$\{(1111)\}$	$(O, D), (O, O)$
S_6	$o(4)$	S_1, S_3	$\{(1111)\}$	(O, O)

Let us now construct the third column of Table 1. First of all, it is trivial from the definition of the special subalgebra that the only 1–dimensional special subalgebra is S_1 and as being 1–dimensional and non-simple, there is no special subalgebra for the types S_a and S_b . Similarly, the 2–dimensional type S_2 contains the special subalgebra S_1 . The 3–dimensional subalgebra S_3 is itself special since its spanning members have a common annihilator w and each of its bivectors is simple. Moreover, it contains special subalgebra of type S_1 . For the 3–dimensional subalgebra $\overset{+}{S}_3$ with spanning members $F' \equiv x \wedge y + z \wedge w$, $G' \equiv x \wedge z + w \wedge y$ and $H' \equiv x \wedge w + y \wedge z$, by taking any combination $F = \theta F' + \xi G' + \epsilon H'$ for a non-zero member $F \in \overset{+}{S}_3$ where $\theta, \xi, \epsilon \in \mathbb{R}$, one calculates $\overset{*}{F}_{ij}F^{ij} = 4(\theta^2 + \xi^2 + \epsilon^2)$ since $F = \overset{*}{F}$. If $\overset{*}{F}_{ij}F^{ij} = 0$, then $\theta = \xi = \epsilon = 0$ and so $F \equiv 0$. Hence, there is no simple member or special subalgebra in $\overset{+}{S}_3$. For subalgebra $\overset{+}{S}_4$ spanned by $\overset{+}{S}$ and $G \in \bar{S}$, one can take F', G', H' as above and $G = x \wedge y - z \wedge w$ (the choice of $G \in \bar{S}$ can be found in [19]). If $E = \theta F' + \xi G' + \epsilon H' + \lambda G$ where $E \in \Lambda_p M$ and $\theta, \xi, \epsilon, \lambda \in \mathbb{R}$, then $\overset{*}{E}_{ij}E^{ij} = 4(\theta^2 + \xi^2 + \epsilon^2 - \lambda^2)$. This computation shows that $\overset{*}{E}_{ij}E^{ij}$ can be zero and it gives the special subalgebra of type S_1 . Finally, the type $o(4)$ covers all special subalgebras which are types S_1 and S_3 . Therefore, we obtain the following theorem.

Theorem 3.1. *Let (M, g) be a structure with M being a smooth, connected, 4–dimensional manifold and g being a (smooth) positive definite metric on M . Then, the special subalgebras of $o(4)$ are isomorphic to types S_1 and S_3 as given in Table 1.*

It is remarked that the isotropy algebra \mathcal{I}_p does not have to be special when a proper orbit \mathcal{O} is not dimensionally stable. For neutral signature case, some examples have been discovered in [16] (for additional examples, see Section 5).

4. Special Tensor Fields and Applications of Symmetries

One must relate the properties of $\mathcal{K}(M)$ to the geometric structures of (M, g) . This leads us to study the theory of restrictions on the Weyl and Ricci tensors at a zero of some non-trivial Killing vector field X and it is based on the algebraic nature of these tensors at $p \in M$. For neutral signature case, the general algebraic study of these tensors has been recently completed in [13, 14]. In this signature, a second order symmetric tensor at p (and hence *Ricc* at p) is one of the following Segre types (or their possible degeneracies): $\{1111\}$, $\{11z\bar{z}\}$, $\{z\bar{z}w\bar{w}\}$, $\{211\}$, $\{2z\bar{z}\}$, $\{22\}$ (over \mathbb{C}), $\{22\}$ (over \mathbb{R}), $\{31\}$, $\{4\}$. Besides, the Weyl types are labelled as **I**, **II**, **III**, **D**₁, **D**₂, **N** and **O**, [14]. Here, type **O** denotes the case when the Weyl tensor vanishes. Moreover, in the case that the isotropy algebra \mathcal{I}_p is not trivial, the effects on *Ricc* and \bar{C} have been investigated in [16] and all possible Ricci and Weyl types are found for all subalgebras of $o(2, 2)$.

For Lorentz signature, the algebraic classification of the Weyl tensor is well known as the Petrov classification where the possible types are denoted by **I**, **II**, **III**, **D**, **N** and **O**. In addition, the possible Segre types for a second order symmetric tensor are $\{1111\}$, $\{11z\bar{z}\}$, $\{211\}$, $\{31\}$ or possible degeneracies of these types (for details see, e. g., [8]).

When (M, g) has positive definite signature, a similar classification of C can be given as follows:

Let us consider the (linear) Weyl map on $\Lambda_p M$ defined by $f : \Lambda_p M \rightarrow \Lambda_p M$ which assigns $F^{ij} \in \Lambda_p M$ to $C^{ij} F^{kl} \in \Lambda_p M$ where $C^{ij} F^{kl}$ denote the components of C . Moreover, one can define the self dual and anti-self dual parts of C denoted by $\overset{+}{W}$ and \bar{W} , respectively. In this case, C is decomposed uniquely as $C = \overset{+}{W} + \bar{W}$ where $\overset{+}{W} = \overset{+}{W}$ and $\bar{W} = -\bar{W}$. In component form, it can be written as

$$C_{ijkl} = \overset{+}{W}_{ijkl} + \bar{W}_{ijkl} \tag{4}$$

where $\overset{+}{W} \equiv \frac{1}{2}(C + {}^*C)$, $\bar{W} \equiv \frac{1}{2}(C - {}^*C)$ and ${}^*C = C^*$ (the left and right duals of C).

On the other hand, the Weyl map f can be written as $f = \overset{+}{f} + \bar{f}$ where $\overset{+}{f}$ and \bar{f} arise from $\overset{+}{W}(p)$ and $\bar{W}(p)$ given in (4). Then the 3-dimensional subalgebras $\overset{+}{S}$ (spanned by $F' \equiv x \wedge y + z \wedge w$, $G' \equiv x \wedge z + w \wedge y$ and $H' \equiv x \wedge w + y \wedge z$) and \bar{S} (spanned by $\bar{F} \equiv x \wedge y - z \wedge w$, $\bar{G} \equiv x \wedge z - w \wedge y$ and $\bar{H} \equiv x \wedge w - y \wedge z$) are invariant subspaces of $\overset{+}{f}$ and \bar{f} , respectively. Since each of $\overset{+}{S}$ and \bar{S} is isometric to \mathbb{R}^3 with the 3-dimensional Euclidean metric of signature $(+1, +1, +1)$ (together with the bivector metric defined by $P(F, G) = P_{ijkl} F^{ij} G^{kl} = F^{ij} G_{ij}$ for $F, G \in \Lambda_p M$ where $P_{ijkl} = \frac{1}{2}(g_{ik}g_{jl} - g_{il}g_{jk})$), each of $\overset{+}{W}(p)$ and $\bar{W}(p)$ can be algebraically classified according to the Segre type of the maps $\overset{+}{f}$ and \bar{f} . This yields that $C(p)$ is diagonalisable over \mathbb{R} and the types for $C(p)$ are just the pairs of possibilities for the Segre types of the maps $\overset{+}{f}$ and \bar{f} which can be either $\{111\}$ or $\{1(11)\}$ or $\{(111)\}$. Let us label these types by **I**, **D** and **O**, respectively. Therefore, the types for $C(p)$ are given by (up to isomorphism) **(I, I)**, **(I, D)**, **(I, O)**, **(D, D)**, **(D, O)** and **(O, O)** (cf [4]).

One of the important concepts in studying C at $p \in M$ is to consider the canonical tetrads in which C achieves its canonical form. Firstly, it will be appropriate to make the definition of the *fix group* for all signatures.

Definition 4.1. Let η be a tetrad transformation at p defined by $(a, b, c, d) \rightarrow (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ where (a, b, c, d) and $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ are bases at p . Let F be a bivector at p given in the basis (a, b, c, d) and also in the basis $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ and suppose that these two expressions are identical under these bases. The collection of all such transformations is a group called the *fix group* of F .

It is noted that depending on the metric signature, the fix group is a Lie subgroup of $O(2, 2)$ (for neutral signature) or $O(1, 3)$ (for Lorentz signature) or $O(4)$ (for positive definite signature) and, up to isomorphism, is independent of the original basis chosen. So, all of these bases defined above are connected by some members of $o(2, 2)$ (for neutral signature) or $o(1, 3)$ (for Lorentz signature) or $o(4)$ (for positive definite signature). For neutral signature, the tetrad changes which fix such bivectors can be found in [14, 21]. On

the other hand, the relationship between the canonical bases for each type for C has been shown in [14]. For Lorentz case, it is known that the Petrov types **I**, **II** and **III** determine their canonical tetrads uniquely up to orientation changes and signs. However, the types **D** and **N** determine these tetrads up to a 2–dimensional abelian subgroup of the Lorentz group acting on T_pM , [5]. For positive definite case, the fix group of any $F \in \Lambda_pM$ can be found by using similar techniques those performed in the other metric signatures. To find some of them, firstly, let the algebraic types for $C(p)$ be of the form (\mathbf{A}, \mathbf{B}) where \mathbf{A} and \mathbf{B} can be of the types **I**, **D** and **O**. It is noted that the types (\mathbf{A}, \mathbf{B}) and (\mathbf{B}, \mathbf{A}) are isomorphic and so there will be no distinction between them. Each type of $C(p)$ can be written as the sum of the canonical forms corresponding to $\overset{+}{W}$ and \bar{W} given in (4), respectively. For example, consider the type (\mathbf{D}, \mathbf{D}) . Then it can be shown that the bivectors $F' \equiv x \wedge y + z \wedge w \in \overset{+}{S}$ and $\bar{F} \equiv x \wedge y - z \wedge w \in \bar{S}$ are fixed by the **D** type of $\overset{+}{W}$ and the **D** type of \bar{W} . Therefore, the fix group arises from the subalgebra $\langle x \wedge y, z \wedge w \rangle$ which is isomorphic to 2–dimensional subalgebra S_2 in Table 1. For type (\mathbf{O}, \mathbf{I}) , the tetrad changes preserve \bar{F} , \bar{G} and \bar{H} . In this case, one gets the 3–dimensional subalgebra $\langle \bar{S} \rangle$ isomorphic to \bar{S}_3 in Table 1. Similarly, for type (\mathbf{O}, \mathbf{D}) , the 4–dimensional subalgebra of $o(4)$ spanned by $\overset{+}{S}$ and \bar{G} is obtained and it is isomorphic to $\overset{+}{S}_4$ in Table 1. For the other types of $C(p)$, similar steps can be done. For each $p \in M$, one can then calculate the possibilities of $C(p)$ for the associated isotropy subalgebra \mathcal{I}_p . Now if $X \in \mathcal{K}(M)$ and $X(p) = 0$, the map φ_{t^*} preserves C at p (this yields that $\mathcal{L}_X C = 0$). Therefore, finding of the canonical bases given above is required. For each subalgebra of $o(4)$, possible candidates for \mathcal{I}_p (when $C(p)$ has the types given above) are listed in the fifth column of Table 1. Here the type denoted by $(\mathbf{D}, \mathbf{O})^*$ means (\mathbf{D}, \mathbf{O}) and (\mathbf{O}, \mathbf{D}) . Recall that there is an obvious isomorphism between the Weyl types (\mathbf{A}, \mathbf{B}) and (\mathbf{B}, \mathbf{A}) for any types \mathbf{A} and \mathbf{B} and that the listing of subalgebras $o(4)$ in Table 1 is up to isomorphism. It should also be noted that when pairing off a Weyl type with its associated isotropy algebra, these isomorphisms care are needed in Table 1. For neutral signature, the required Weyl types have been recently found in [16] and for Lorentz signature, the results are known, see [16] and [5, 8].

Let us now examine the algebraic type of $Ricc(p)$ when the isotropy subalgebra \mathcal{I}_p is not trivial. If $X \in \mathcal{I}_p$, then $X(p) = 0$ and one has the condition $\mathcal{L}_X Ricc = 0$ at p . In fact, for $u, v \in T_pM$, $Ricc(p)$ is restricted as $Ricc(u, v) = \varphi_t^* Ricc(u, v) = Ricc(\varphi_{t^*}(u), \varphi_{t^*}(v))$ where φ_t is the corresponding local flow for $X \in \mathcal{K}(M)$. These conditions give the following equivalent equation which is more convenient for the calculations:

$$R_{ik}F^k_j + R_{kj}F^k_i = 0 \tag{5}$$

where F is the Killing bivector of X . From (5), it can be concluded that for each $X \in \mathcal{I}_p$, the Killing bivectors at p restrict $Ricc$ at p and the Segre types for $Ricc(p)$ can be obtained for each subalgebras of $o(4)$, $o(1, 3)$ and $o(2, 2)$ depending on the metric signature.

For neutral signature, all Segre types for $Ricc(p)$ are found in [16] which are $\{11(11)\}$, $\{1(111)\}$, $\{(11)(11)\}$, $\{(1111)\}$, $\{(11)z\bar{z}\}$, $\{(21)1\}$, $\{2(11)\}$, $\{(211)\}$, $\{(31)\}$, $\{22\}$ (over \mathbb{R}), $\{(22)\}$ (over \mathbb{R}) and $\{(zz)(\bar{z}\bar{z})\}$. For Lorentz signature, the permitted Segre types are $\{1, 1(11)\}$, $\{(1, 1)11\}$, $\{(1, 11)1\}$, $\{(1, 1)(11)\}$, $\{1, (111)\}$, $\{(1, 111)\}$ (where the comma separates off the eigenvalue corresponding to the timelike eigenvector from those associated with spacelike ones), $\{(11)z\bar{z}\}$, $\{(21)1\}$, $\{2(11)\}$, $\{(211)\}$ and $\{(31)\}$ (see [8], page 302). By the aid of the subalgebras of $o(4)$ in Table 1, we can now complete this investigation for positive definite signature. To get the associated Segre types for each subalgebra, it is first useful to note that if (5) holds for a simple F at p , then the blade of F is an eigenspace of $Ricc$ at p . For instance, if $F = x \wedge y$ (simple), then $x \wedge y$ is an eigenspace of $Ricc(p)$ and by using (5), one gets the possible Segre types as $\{11(11)\}$, $\{1(111)\}$, $\{(11)(11)\}$, $\{(1111)\}$. On the other hand, if $F = \alpha(x \wedge y) + \beta(z \wedge w)$ and $0 \neq \alpha \neq \pm\beta \neq 0$, then $x \wedge y$ and $z \wedge w$ are eigenspaces for $Ricc$ whilst if $\alpha = \beta \neq 0$ (and similarly for $\alpha = -\beta \neq 0$), then there exist invariant 2–spaces for $Ricc$ (see, e. g., [11]). By considering these results and the classification of second order symmetric tensors mentioned above, possible Segre types for $Ricc$ can be read from the fourth column of Table 1 for each subalgebra of $o(4)$. More clearly, the Segre types occurring in Table 1 are $\{11(11)\}$, $\{1(111)\}$, $\{(11)(11)\}$ and $\{(1111)\}$.

Now, if we restrict this examination to the special subalgebras of $\mathfrak{o}(4)$, the following theorem is obtained from Table 1:

Theorem 4.2. *Let (M, g) be a structure with M being a smooth, connected, 4–dimensional manifold and g being a positive definite metric on M . Suppose that \mathcal{O} is a proper and a dimensionally stable orbit of $\mathcal{K}(M)$. Then \mathcal{I}_p is special and the relationships between \mathcal{I}_p (where $p \in \mathcal{O}$) and the algebraic types of the Weyl and Ricci tensors are given as follows:*

(i) **Weyl tensor;**

◆ S_1 : $(\mathbf{D}, \mathbf{D}), (\mathbf{D}, \mathbf{O})^*, (\mathbf{O}, \mathbf{O})$.

◆ S_3 : (\mathbf{O}, \mathbf{O}) .

(ii) **Ricci tensor;**

◆ S_1 : $\{11(11)\}, \{1(111)\}, \{(11)(11)\}, \{(1111)\}$.

◆ S_3 : $\{1(111)\}, \{(1111)\}$.

We can also associate this examination with the dimension of \mathcal{I}_p . With the help of the above information, for all signatures together with their special subalgebras, we can state the following theorem:

Theorem 4.3. *Let (M, g) be a structure with M being a smooth, connected, 4–dimensional manifold admitting a smooth metric g . For any $p \in M$, the following conditions hold:*

(i) *If g has positive definite signature or neutral signature and if $\dim \mathcal{I}_p > 4$, then $C(p) \equiv 0$.*

(ii) *If g has Lorentz signature and if $\dim \mathcal{I}_p > 2$, then $C(p) \equiv 0$.*

(iii) *For all signatures, if $\dim \mathcal{I}_p \geq 4$, then Ricc is proportional to g at p .*

(iv) *If p lies in a dimensionally stable orbit and if $\dim \mathcal{I}_p \geq 3$, then $C(p) \equiv 0$.*

(v) *If p lies in a dimensionally stable orbit and if $\dim \mathcal{I}_p \geq 3$, then for Lorentz and neutral signatures, Ricc has Segre type either $\{(211)\}$ or $\{1(111)\}$ or $\{(1111)\}$ at p and for positive definite signature, its Segre type is either $\{1(111)\}$ or $\{(1111)\}$.*

5. Examples

In this section, some examples will be given about the concepts described in Sections 3 and 4.

Example 5.1. *Consider the following positive definite metric on $M = \mathbb{R}^4$ with the global coordinate system x, y, z, w*

$$e^{(x^2+y^2)}(dx^2 + dy^2) + e^{(z^2+w^2)}(dz^2 + dw^2).$$

In this case, $\mathcal{K}(M)$ is spanned by the Killing vector fields $X \equiv (y, -x, 0, 0)$ and $Y \equiv (0, 0, w, -z)$ and so, $\dim \mathcal{K}(M) = 2$. It is clear that origin is a zero for all members of $\mathcal{K}(M)$. At this point, \mathcal{I}_p is spanned by $x \wedge y$ and $z \wedge w$ so that it is isomorphic to S_2 which is not a special subalgebra. It can be shown that (M, g) is an Einstein space and at the origin, Weyl tensor is of type (\mathbf{D}, \mathbf{D}) . The proper orbits are either 1– or 2–dimensional. The 2–dimensional orbits are stable whilst 1–dimensional orbits are not dimensionally stable.

Example 5.2. *Consider the following positive definite metric on $M = \mathbb{R}^4$ with the global coordinate system x, y, z, w*

$$e^{(x^2+y^2+z^2)}(dx^2 + dy^2 + dz^2) + dw^2.$$

For this metric, $\mathcal{K}(M)$ is 4–dimensional spanned by the Killing vector fields $(0, 0, 0, 1)$, $(y, -x, 0, 0)$, $(z, 0, -x, 0)$ and $(0, z, -y, 0)$. Here, the orbits are either 1– or 3–dimensional. The 1–dimensional orbit \mathcal{O} is the submanifold $x = y = z = 0$ of M . Along this orbit, it can be checked that C vanishes and Ricc has Segre type $\{1(111)\}$. The Killing vector fields $(y, -x, 0, 0)$, $(z, 0, -x, 0)$ and $(0, z, -y, 0)$ vanish on \mathcal{O} and their Killing bivectors, respectively, give a contribution $x \wedge y$, $x \wedge z$ and $y \wedge z$ to \mathcal{I}_p . Therefore, \mathcal{I}_p is 3–dimensional and it is isomorphic to the Lie algebra S_3 which is special. The other orbits are 3–dimensional, dimensionally stable and given by $x^2 + y^2 + z^2 = c$ where $c > 0$ is a constant and at each p on any of these orbits \mathcal{I}_p is special (and 1–dimensional) and isomorphic to S_1 in Table 1.

Example 5.3. Let us now consider the following neutral metric on $M = \mathbb{R}^4$ with the global coordinate system x, y, s, t

$$e^{(x^2+y^2)}(dx^2 + dy^2) - e^{(s^2+t^2)}(ds^2 + dt^2).$$

This metric is the neutral signature equivalent of the metric given in Example 5.1. It admits a 2-dimensional Killing algebra spanned by the Killing vector fields $Y \equiv (0, 0, -t, s)$ and $Z \equiv (y, -x, 0, 0)$ which are orthogonal and their Lie bracket is zero, that is, $[Y, Z] = 0$. The origin is a zero for each member of $\mathcal{K}(M)$ and at this point \mathcal{I}_p is spanned by $x \wedge y$ and $s \wedge t$ which is isomorphic to the type $2(e)$ and which is not special (for the subalgebras of $o(2, 2)$, see, e. g., [16]). It can be shown that at the origin C vanishes and the Segre type of $Ricc$ is $\{(11)(11)\}$. The proper orbits are either 1- or 2-dimensional. The 2-dimensional orbits are dimensionally stable and may be spacelike, timelike, null or totally null. Moreover, the 1-dimensional orbits are not dimensionally stable and can be spacelike, timelike or null and they occur in the submanifolds $x = constant$, $y = constant$ and in the submanifolds $s = constant$, $t = constant$. At any point of these latter orbits, \mathcal{I}_p is 1-dimensional and it is of the special type $1(b)$. (For more examples in neutral signature, we refer to [16].)

Example 5.4. Consider the following Lorentz metric on $M = \mathbb{R}^4$ with the global coordinate system x, y, z, t ($t > 0$)

$$t^2(dx^2 + dy^2 - dt^2) + dz^2.$$

For this metric, $\dim \mathcal{K}(M) = 4$ and $\mathcal{K}(M)$ is spanned by the Killing vector fields $X = (1, 0, 0, 0)$, $Y = (0, 1, 0, 0)$, $Z = (0, 0, 1, 0)$, $W = (y, -x, 0, 0)$. The Killing vector fields X, Y and Z are non-zero and span a 3-dimensional subspace of the tangent space everywhere and W lies in this subspace. Thus, the orbits are everywhere 3-dimensional. The Ricci tensor satisfies $Ricc = \frac{2}{t} dt dt$. In addition to these, \mathcal{I}_p is 1-dimensional (since $\dim \mathcal{I}_p = \dim \mathcal{K}(M) - \dim \mathcal{O}_p = 4 - 3 = 1$), special and isomorphic to the type R_4 (for the subalgebras of $o(1, 3)$, see, e. g., [8, 22] and for more examples in space-times, we refer to [8, 9].)

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