Filomat 33:4 (2019), 1259–1266
https://doi.org/10.2298/FIL1904259R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Total Normalcy of Knots

Svetozar R. Rančić[a], Marija S. Najdanović[b], Ljubica S. Velimirović[a]

[a]*Faculty of Science and Mathematics, University of Niš, Niš, Serbia*
[b]*University of Priština - Kosovska Mitrovica, Faculty of Sciences, Department of Mathematics, 38220 Kosovska Mitrovica, Serbia*

Abstract. In this paper we consider first order infinitesimal bending of curves and knots. The total normalcy of the knot during the first order infinitesimal bending is discussed and expressions for the first variation of the total normalcy are given. Some examples aimed to illustrate infinitesimal bending of knots are shown using figures. Colors are used to illustrate normalcy values at different points of bent knots and the total normalcy is numerically calculated.

1. Introduction and preliminaries

The study of knots and their properties is known as knot theory. There are different aspects of the studying of knots. Apart from the mathematical point of view, this topic is of interest in the context of different areas in sciences like biology, chemistry, physics, computer graphics (see [1–3, 5–10, 15, 17]). Under the term knot we mean a closed, self-avoiding curve in a 3-dimensional space.

Since in this paper we are interested in a study of infinitesimal bending of knots, we will give a brief overview of the results on the infinitesimal bending of curves. More information about infinitesimal bending of curves and surfaces one can get from [4, 11–14, 16, 18–21].

Definition 1.1. *Let us consider continuous regular curve*

$$C : \mathbf{r} = \mathbf{r}(u), \quad u \in J \subseteq \mathcal{R} \tag{1}$$

included in a family of the curves

$$C_\epsilon : \tilde{\mathbf{r}}(u,\epsilon) = \mathbf{r}_\epsilon(u) = \mathbf{r}(u) + \epsilon\mathbf{z}(u), \quad u \in J, \quad (\epsilon \geq 0, \quad \epsilon \to 0), \tag{2}$$

where u is a real parameter and we get C for $\epsilon = 0$ ($C = C_0$). A family of curves C_ϵ is **an infinitesimal bending of a curve** *C if*

$$ds_\epsilon^2 - ds^2 = o(\epsilon), \tag{3}$$

where $\mathbf{z} = \mathbf{z}(u)$, $\mathbf{z} \in C^1$, *is* **the infinitesimal bending field** *of the curve C.*

2010 *Mathematics Subject Classification*. Primary 53A04, 53C45, 57M25; Secondary 57M27, 78A25
Keywords. (first order infinitesimal bending, first variation, total normalcy, curve, knot.)
Received: 12 December 2018; Accepted: 23 January 2019
Communicated by Mića S. Stanković
Research supporteed by Serbian Ministry of Science under project 174012)
Email addresses: rancicsv@yahoo.com (Svetozar R. Rančić), marija.najdanovic@pr.ac.rs (Marija S. Najdanović), vljubica@pmf.ni.ac.rs (Ljubica S. Velimirović)

Theorem 1.2. [4] *A necessary and sufficient condition for* $\mathbf{z}(u)$ *to be an infinitesimal bending field of a curve C is to be*

$$d\mathbf{r}\cdot d\mathbf{z}=0, \tag{4}$$

where $\cdot$ *stands for the scalar product in* $\mathcal{R}^3$. □

Theorem 1.3. [19] *The infinitesimal bending field for the curve C is*

$$\mathbf{z}(u)=\int[p(u)\mathbf{n}_1(u)+q(u)\mathbf{n}_2(u)]\,du, \tag{5}$$

where $p(u)$ *and* $q(u)$ *are arbitrary integrable functions and vectors* $\mathbf{n}_1(u)$ *and* $\mathbf{n}_2(u)$ *are respectively unit principal normal and binormal vector fields of the curve C.* □

Geometric magnitudes are changing under infinitesimal bending and that changing is described by the variation. We define the variation of geometric magnitudes according to [18].

Definition 1.4. *Let* $\mathcal{A}=\mathcal{A}(u)$ *be a magnitude that characterizes a geometric property on the curve C and* $\mathcal{A}_\epsilon=\mathcal{A}_\epsilon(u)$ *the corresponding magnitude on the curve* C_ϵ *being an infinitesimal bending of the curve C, and set*

$$\Delta\mathcal{A}=\mathcal{A}_\epsilon-\mathcal{A}=\epsilon\,\delta\mathcal{A}+\epsilon^2\,\delta^2\mathcal{A}+\dots\epsilon^n\,\delta^n\mathcal{A}+\dots \tag{6}$$

The coefficients $\delta\mathcal{A},\delta^2\mathcal{A},\dots,\delta^n\mathcal{A},\dots$ *are* **the first, the second, ..., the** n**th variation** *of the geometric magnitude* $\mathcal{A}$*, respectively under the infinitesimal bending* C_ϵ *of the curve C.*

In this paper we will only consider the first variation. The first variation obviously satisfies

$$\delta\mathcal{A}=\frac{d}{d\epsilon}\mathcal{A}_\epsilon(u)\Big|_{\epsilon=0}. \tag{7}$$

In addition

$$\textbf{a)}\quad \delta(\mathcal{A}\mathcal{B})=\mathcal{A}\delta\mathcal{B}+\mathcal{B}\delta\mathcal{A},\quad \textbf{b)}\quad \delta\Big(\frac{\partial\mathcal{A}}{\partial u}\Big)=\frac{\partial(\delta\mathcal{A})}{\partial u},\quad \textbf{c)}\quad \delta(d\mathcal{A})=d(\delta\mathcal{A}). \tag{8}$$

Below we will consider a regular curve

$$C:\mathbf{r}=\mathbf{r}(s)=r[u(s)],\quad \mathbf{r}:[0,L]\to\mathcal{R}^3, \tag{9}$$

of the class C^α, $\alpha\geq 3$, parameterized by the arc length s. The unit tangent to the curve is given by $\mathbf{t}=\mathbf{r}'$, where prime denotes a derivative with respect to arc length s. Clearly, $\mathbf{t}'$ is orthogonal to $\mathbf{t}$, but $\mathbf{t}''$ is not. The classical Frenet equations

$$\mathbf{t}'=k\mathbf{n}_1,\quad \mathbf{n}_1'=-k\mathbf{t}+\tau\mathbf{n}_2,\quad \mathbf{n}_2'=-\tau\mathbf{n}_1, \tag{10}$$

describe the construction of an orthonormal basis $\{\mathbf{t},\mathbf{n_1},\mathbf{n_2}\}$ along a curve, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are respectively unit principal normal and binormal vector fields of the curve. We choose an orientation with $\mathbf{n}_2=\mathbf{t}\times\mathbf{n}_1$. k and τ are respectively the curvature and the torsion.

Consider an infinitesimal bending of the curve (9),

$$C_\epsilon:\tilde{\mathbf{r}}(s,\epsilon)=\mathbf{r}_\epsilon(s)=\mathbf{r}(s)+\epsilon\mathbf{z}(s). \tag{11}$$

As the vector field $\mathbf{z}$ is defined in the points of the curve (9), it can be presented in the form

$$\mathbf{z}=z\mathbf{t}+z_1\mathbf{n}_1+z_2\mathbf{n}_2, \tag{12}$$

where $z\mathbf{t}$ is a tangential and $z_1\mathbf{n}_1+z_2\mathbf{n}_2$ is a normal component, z,z_1,z_2 are the functions of s.

Theorem 1.5. [11] *A necessary and sufficient condition for the field* $\mathbf{z}$*, (12), to be an infinitesimal bending field of the curve C, (9), is*

$$z'-kz_1=0, \tag{13}$$

where k is the curvature of C.

It is easy to see that under infinitesimal bending of the curve C, a unit vector of the orthonormal basis and its variation are orthogonal. Also, the variation of the line element ds is equal to zero, i. e. $\delta(ds)=0$.

2. Total normalcy of knots under infinitesimal bending

The total normalcy is a geometric quantity which measures the binormal indicatrix of a knot. It is defined by the equation

$$\omega = \int_C \sqrt{k^2(s) + \tau^2(s)}\, ds, \tag{14}$$

where k and τ are the curvature and the torsion of the knot C, respectively.

Let us describe the behavior of the total normalcy under infinitesimal bending, i. e. let us find its variation.

According to (7), the variation of the total normalcy satisfies the following equation

$$\delta\omega = \frac{d}{d\epsilon} w_\epsilon \Big|_{\epsilon=0} = \frac{d}{d\epsilon} \int_0^L \sqrt{(k+\epsilon\delta k)^2 + (\tau + \epsilon\delta\tau)^2}\,(ds + \epsilon\delta(ds))\Big|_{\epsilon=0},$$

i. e.

$$\delta\omega = \int_0^L \frac{k\delta k + \tau\delta\tau}{\sqrt{k^2+\tau^2}} ds. \tag{15}$$

It is necessary to find the variations of the curvature and the torsion under infinitesimal bending. Below we will do that according to [11].

Since $\delta\mathbf{t} = \delta\mathbf{r}' = (\delta\mathbf{r})' = \mathbf{z}'$ and $\delta\mathbf{t}' = (\delta\mathbf{t})'$, using (12), (13) and Frenet equations we obtain

$$\delta\mathbf{t} = (z_1' - \tau z_2 + kz)\,\mathbf{n}_1 + (z_2' + \tau z_1)\,\mathbf{n}_2 \tag{16}$$

and

$$\begin{aligned}\delta\mathbf{t}' = &-k(kz + z_1' - \tau z_2)\,\mathbf{t} + (k'z + z_1'' + (k^2-\tau^2)z_1 - 2\tau z_2' - \tau' z_2)\,\mathbf{n}_1 \\ &+ (k\tau z + 2\tau z_1' + \tau' z_1 + z_2'' - \tau^2 z_2)\,\mathbf{n}_2.\end{aligned} \tag{17}$$

Taking a variation of the first equation in (10) we obtain $\delta\mathbf{t}' = \delta k\,\mathbf{n}_1 + k\,\delta\mathbf{n}_1$ wherefrom we have $\delta k = \mathbf{n}_1 \cdot \delta\mathbf{t}'$. This leads to

$$\delta k = k'z + z_1'' + (k^2 - \tau^2)z_1 - 2\tau z_2' - \tau' z_2. \tag{18}$$

after using (17).

In order to determine the variation of the torsion, let us take a variation of the Frenet equation for $\mathbf{n}_1'$ and dot with $\mathbf{n}_2$. We have

$$\delta\tau = k\mathbf{n}_2 \cdot \delta\mathbf{t} + \mathbf{n}_2 \cdot \delta\mathbf{n}_1'. \tag{19}$$

Further, after a little transformation of the previous equation using the Frenet equations we obtain

$$\delta\tau = k\mathbf{n}_2 \cdot \delta\mathbf{t} + \Big(\frac{1}{k}\mathbf{n}_2 \cdot \delta\mathbf{t}'\Big)' \tag{20}$$

Substituting (16) and (17) into (20) and using (13) we obtain

$$\delta\tau = z\tau' + k(z_2' + 2\tau z_1) + \Big\{\frac{1}{k}\Big[2\tau z_1' + \tau' z_1 + z_2'' - \tau^2 z_2\Big]\Big\}'. \tag{21}$$

Let's go back to the variation of the total normalcy (15). We will consider separately the following integrals:

$$I_1 = \int_0^L \frac{k\delta k}{\sqrt{k^2+\tau^2}} ds, \quad I_2 \int_0^L \frac{\tau\delta\tau}{\sqrt{k^2+\tau^2}} ds.$$

Using (18) one obtains

$$I_1 = \int_0^L \frac{k}{\sqrt{k^2+\tau^2}}(k'z + z_1'' + (k^2-\tau^2)z_1 - 2\tau z_2' - \tau' z_2)ds$$
$$= \int_0^L \Big(\frac{kk'}{\sqrt{k^2+\tau^2}}z + \frac{k}{\sqrt{k^2+\tau^2}}z_1'' + \frac{k(k^2-\tau^2)}{\sqrt{k^2+\tau^2}}z_1 - \frac{2k\tau}{\sqrt{k^2+\tau^2}}z_2' - \frac{k\tau'}{\sqrt{k^2+\tau^2}}z_2\Big)ds.$$

After a few transformations of the second and the fourth term in the previous integral we obtain

$$\begin{aligned} I_1 = \int_0^L \Big\{ \frac{kk'}{\sqrt{k^2+\tau^2}}z + \Big[\Big(\frac{k}{\sqrt{k^2+\tau^2}}\Big)'' + \frac{k(k^2-\tau^2)}{\sqrt{k^2+\tau^2}}\Big]z_1 + \Big[\Big(\frac{2k\tau}{\sqrt{k^2+\tau^2}}\Big)' \\ - \frac{k\tau'}{\sqrt{k^2+\tau^2}}\Big]z_2\Big\}ds + \int_0^L \Big[\frac{k}{\sqrt{k^2+\tau^2}}z_1' - \Big(\frac{k}{\sqrt{k^2+\tau^2}}\Big)' z_1 - \frac{2k\tau}{\sqrt{k^2+\tau^2}}z_2\Big]' ds. \end{aligned} \tag{22}$$

The second integral is according to (21)

$$I_2 = \int_0^L \frac{\tau}{k^2+\tau^2}\Big\{z\tau' + k(z_2' + 2\tau z_1) + \Big[\frac{1}{k}\Big(2\tau z_1' + \tau' z_1 + z_2'' - \tau^2 z_2\Big)\Big]'\Big\},\ ds$$

wherefrom, after a few transformations, we obtain

$$\begin{aligned} I_2 = \int_0^L \Big\{ \frac{\tau\tau'}{\sqrt{k^2+\tau^2}}z + \Big[\frac{2k\tau^2}{\sqrt{k^2+\tau^2}} + \Big(\Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)'\frac{2\tau}{k}\Big)' - \Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)'\frac{\tau'}{k}\Big]z_1 \\ + \Big[-\Big(\frac{k\tau}{\sqrt{k^2+\tau^2}}\Big)' - \Big(\Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)'\frac{1}{k}\Big)'' + \Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)'\frac{\tau^2}{k}\Big]z_2\Big\}ds \\ + \int_0^L \Big\{\frac{\tau}{k\sqrt{k^2+\tau^2}}(2\tau z_1' + \tau' z_1 + z_2'' - \tau^2 z_2) - \Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)'\frac{2\tau}{k}z_1 \\ - \Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)'\frac{1}{k}z_2' + \frac{k\tau}{\sqrt{k^2+\tau^2}}z_2 + \Big(\Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)'\frac{1}{k}\Big)'z_2\Big\}'\ ds \end{aligned} \tag{23}$$

Since

$$\frac{kk' + \tau\tau'}{\sqrt{k^2+\tau^2}}z = (\sqrt{k^2+\tau^2})'z = (\sqrt{k^2+\tau^2}z)' - \frac{k(k^2+\tau^2)}{\sqrt{k^2+\tau^2}}z_1$$

after using the necessary and sufficient condition for the infinitesimal bending, $z' = kz_1$, we obtain

$$\begin{aligned} \delta\omega = \int_0^L \Big\{\Big[\Big(\frac{k}{\sqrt{k^2+\tau^2}}\Big)'' + \Big(\Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)'\frac{2\tau}{k}\Big)' - \Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)'\frac{\tau'}{k}\Big]z_1 \\ + \Big[\Big(\frac{k}{\sqrt{k^2+\tau^2}}\Big)'\tau - \Big(\Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)'\frac{1}{k}\Big)'' + \Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)'\frac{\tau^2}{k}\Big]z_2\Big\}ds \\ + \int_0^L \Big\{\sqrt{k^2+\tau^2}z + \frac{k}{\sqrt{k^2+\tau^2}}z_1' - \Big(\frac{k}{\sqrt{k^2+\tau^2}}\Big)'z_1 - \frac{k\tau}{\sqrt{k^2+\tau^2}}z_2 \\ + \frac{\tau}{k\sqrt{k^2+\tau^2}}(2\tau z_1' + \tau' z_1 + z_2'' - \tau^2 z_2) - \Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)'\frac{2\tau}{k}z_1 \\ - \Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)'\frac{1}{k}z_2' + \Big(\Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)'\frac{1}{k}\Big)'z_2\Big\}'\ ds. \end{aligned} \tag{24}$$

In the case of the infinitesimal bending of knots we specify the condition $\mathbf{z}(0) = \mathbf{z}(L)$ for the infinitesimal bending field in order to get a family of closed curves. Also, we suppose that the knot, as well as the infinitesimal bending field are sufficiently smooth. Keeping this in mind we have the following theorem.

Theorem 2.1. *Under infinitesimal bending of a knot C, the variation of its total normalcy is*

$$\begin{aligned}\delta\omega = \int_0^L \Big\{ \Big[\Big(\frac{k}{\sqrt{k^2+\tau^2}}\Big)'' + \Big(\Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)' \frac{2\tau}{k}\Big)' - \Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)' \frac{\tau'}{k} \Big] z_1 \\ + \Big[\Big(\frac{k}{\sqrt{k^2+\tau^2}}\Big)' \tau - \Big(\Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)' \frac{1}{k}\Big)'' + \Big(\frac{\tau}{\sqrt{k^2+\tau^2}}\Big)' \frac{\tau^2}{k} \Big] z_2 \Big\} ds\end{aligned} \tag{25}$$

where k and τ are the curvature and the torsion of C, respectively.

3. Normalcy of infinitesimally bent knots

Here we will give some examples aimed to illustrate infinitesimal bending on knots defined by a simple parametric representation. Infinitesimal bending has an influence on the knotted curve, on its shape and geometrical magnitudes. Our aim is to visualise changes in shape and normalcy. In following figures colors are used to indicate the values of normalcy at different points of the knots, together with color-values scale. In addition the total normalcy is also calculated.

We start from knot representations as a curve in $\mathcal{R}^3$. Then, according to (2) we apply the bending of the first order given by (5). Bending field is defined by integral whose sub integral function includes arbitrary functions: p and q. The curve is visualized as polygonal line which connect points on curve. At every such point, as well as, every subdivision point for the purpose of numerically integral calculation we should calculate functions: the curve, first, second and third derivative, both normals of the curve, curvature and torsion. p and q, also local values of normalcy. This is necessary to obtain transformed shape of curve and for aimed curve coloring. Instead of using existing software packages capable of symbolic and numeric calculations also with some visualization features, we decided to develop our own software tool using*Microsoft* Visual C++. We are dealing, according to (2) and (5), with arbitrary mathematical functions, so tool is developed for manipulating explicitly defined functions. It starts from usual symbolic definitions as a string, then parsing it to obtain an internal, tree like, form. For the purpose of efficiency the function is parsed once, then calculated many times. We also have some additional important benefits of the tree like form: make derivatives, combine more function to obtain a compound function like sub integral function for infinitesimal bending field. Our tool has not possibility to calculate integral symbolically, instead we are using ability for fast calculation of sub integral function. Those values are used to calculate needed integral numerically, according to $F(x) = \int_0^x f(x)dx$.

Knot visualization and obtaining 3D model is done by using *OpenGL*. In the following examples the knot is represented as a tube around a curve. It looks like a rope, but without examination any physical characteristics of the rope.

3.1. *Trefoil knot*

A trefoil knot is given by the parametric equations: $x = sin(u)+2sin(2u)$, $y = cos(u)-2cos(2u)$, $z = -sin(3u)$. The basic and infinitesimally bent trefoil knots are given in Figs. 1 and 2.

The bending fields are defined by: $p(u) = cos(3u)$ and $q(u) = sin(3u)$.

The numerically calculated total normalcy of the trefoil knot curve is 15.626719, 15.875856 and 16.6498 for $\epsilon = 0$, $\epsilon = 0.5$ and $\epsilon = 1$ respectively.

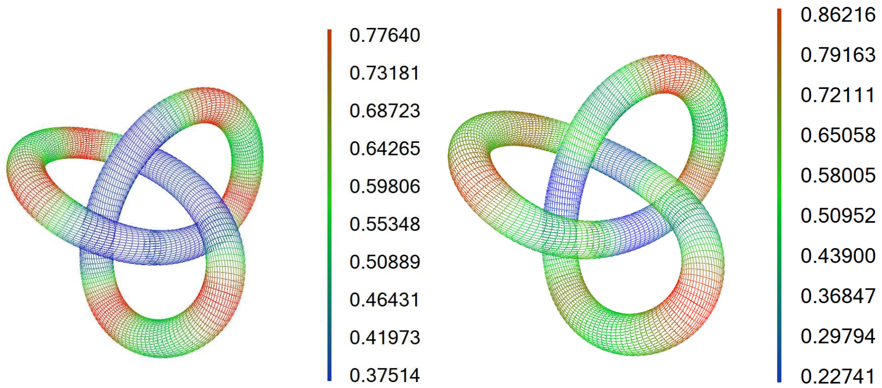

Figure 1: Trefoil knot: basic and infinitesimally bent with $\epsilon = 0.5$.

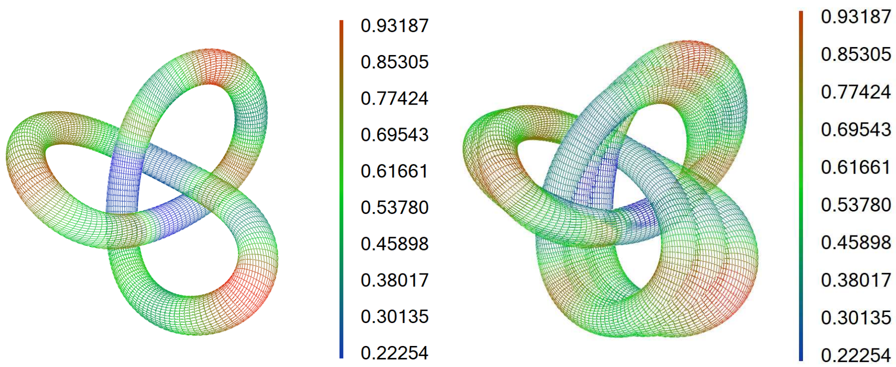

Figure 2: Trefoil knot: infinitesimally bent with $\epsilon = 1$ and all knots together.

A second example of the trefoil knot is knot given by the parametric equations: $x = (cos(2u) + 2) * cos(3u)$, $y = (cos(2u) + 2) * sin(3u)$, $z = -sin(2u)$, which is a kind of torus knot, obtained for $p = 3$, $q = 2$, topologically equivalent to the trefoil knot. The basic and infinitesimally bent knots are given in Fig. 3.

The bending fields are defined by: $p(u) = cos(6u)$ and $q(u) = sin(6u)$.

The numerically calculated total normalcy of the $p = 3$, $q = 2$ torus knot is 19.558 and 32.0013 for $\epsilon = 0$ and $\epsilon = 1.8$ respectively.

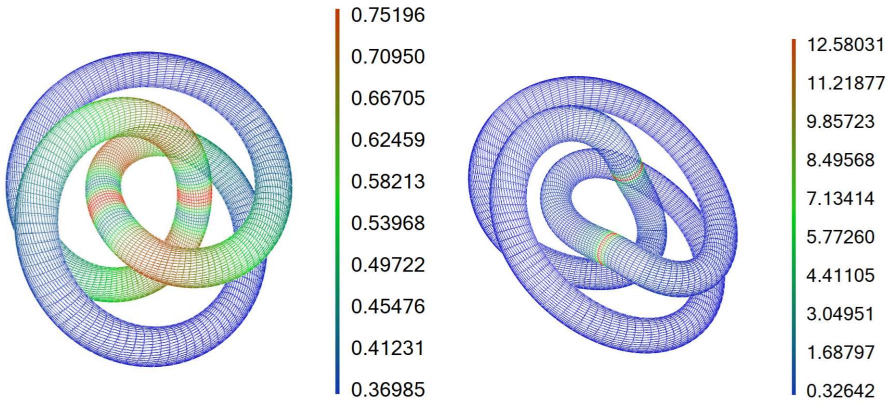

Figure 3: $p = 3, q = 2$ torus knot: basic and infinitesimally bent with $\epsilon = 1.8$.

3.2. *Figure eight knot*

A figure eight knot is given by the parametric equations: $x = (2+cos(2u))*cos(3u)$, $y = (2+cos(2u))*sin(3u)$, $z = sin(4u)$. The basic and infinitesimally bent figure eight knot are given in Figs. 4 and 5.

The bending fields are defined by: $p(u) = cos(6u)$ and $q(u) = sin(6u)$.

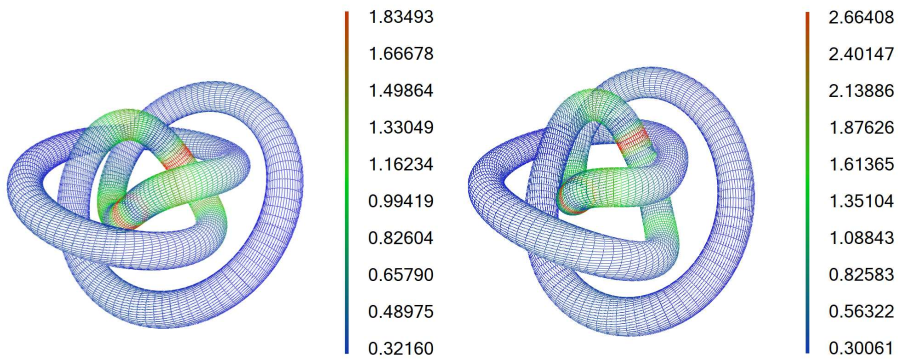

Figure 4: Figure eight knot: basic and infinitesimally bent with $\epsilon = 1.3$.

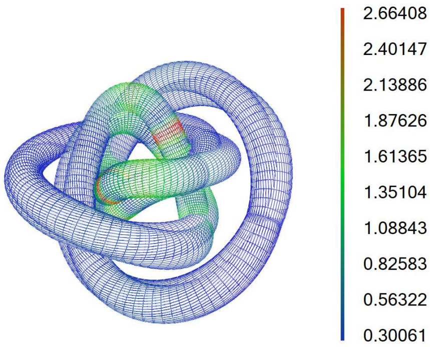

Figure 5: Figure eight knot: basic and infinitesimally bent with $\epsilon = 1.3$ together.

The numerically calculated total normalcy of the figure eight knot curve is 23.724 and 28.583 for $\epsilon = 0$ and $\epsilon = 1.3$ respectively.

Acknowledgement

The authors were supported from the research project 174012 of the Serbian Ministry of Science.

References

[1] J. Baranska, S. Przybyl, P. Pieranski, *Curvature and torsion of the tight closed trefoil knot*,The european Physical Journal B 66 (2008), 547–556, DOI:10.1140/epjb/e2008-00443-y
[2] K. Borsuk, *Sur la courbure totale des courbes fermées*, Annales de la Soc. Polonaise 20 (1947), 251-265.
[3] G. R. Buck, J. K. Simon, *Total curvature and packing of knots*, Topology and its Applications 154 (2007) 192–204.
[4] N. V. Efimov, *Kachestvennye voprosy teorii deformacii poverhnostei*, UMN 3.2 (1948) 47–158.
[5] M. I. Fáry, *Sur la Courbure Totale d'une Courbe Gauche Faisant un Nœud*, Bulletin de la Soc. Math. de France LXXVII (1949) 128–138.
[6] W. Fenchel, *Uber Krummung und Windung geschlossener Raumkurven*, Math. Ann. 101 (1929) 238–252.
[7] L.H. Kauffman, *Knots and Physics*, Series on Knots and Everything 1, World Scientific, 1991.
[8] L.H. Kauffman, *Knots and Applications*, Series on Knots and Everything 6, World Scientific, 1995.
[9] N. C. Lim, S. E. Jackon, *Molecular knots in biology and chemistry*, J. Phys.: Condens. Matter 27 (2015) 354101 (35pp) doi:10.1088/0953-8984/27/35/354101
[10] J. W. Milnor, *On the Total Curvature of Knots*, The Annals of Mathematics, 2nd Ser., Vol 52. (1950) 248–257.
[11] M. S. Najdanović, *Infinitesimal bending influence on the Willmore energy of curves*, Filomat 29:10 (2015) 2411 – 2419.
[12] M. S. Najdanović, Lj. S. Velimirović, *On the Willmore energy of curves under second order infinitesimal bending*, Miskolc Mathematical Notes 17 (2) (2017) 979–987.
[13] M. S. Najdanović, Lj. S. Velimirović, *Second order infinitesimal bending of curves*, Filomat 31:13 (2017), 4127–4137.
[14] M. Najdanović, Lj. Velimirović, *Infinitesimal bending of curves on the ruled surfaces*, The University Thought-Publication in Natural Sciences 8 (1), (2018), 46–51.
[15] C. Oberti, R. L. Ricca, *On torus knots and unknots*, Journal of Knot Theory and Its Ramifications, Vol. 25, No. 6 (2016) 1650036 (18 pages) DOI: 10.1142/S021821651650036X
[16] S. Rančić, Lj. Velimirović, M. Zlatanović, *Curvebend graphical tool for presentation of infinitesimal bending of curves*, Filomat **23:2** (2009), 108–116.
[17] R.L. Ricca, *Applications of knot theory in fluid mechanics*, In Knot Theory (ed. V.F.R. Jones et al.), pp. 321-346, Banach Center Publs. 42, Polish Academy of Sciences, Warsaw, 1998.
[18] I. Vekua, *Obobschennye analiticheskie funkcii*, Moskva, 1959.
[19] Lj. Velimirović, *Change of geometric magnitudes under infinitesimal bending*, Facta Universitates, Vol.3, N^o11 (2001)135–148.
[20] Lj. Velimirović, *Infinitesimal bending of curves*, Matematicki bilten Skopje, Makedonija 25(LI), (2001) 25–36.
[21] Lj. Velimirović, *Infinitesimal bending*, University of Niš, Faculty of Science and Mathematics, 2009.