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Application of Shape Operator Under Infinitesimal Bending of Surface

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Abstract. In case of bendable surfaces it is useful to discuss the variation of magnitudes such as the shape operator. The shape operator is a good way to measure how a regular surface S bends in $\mathcal{R}^3$ by valuation how the surface normal ν changes from point to point. We considered the variation of shape operator under infinitesimal bending of surface given in an explicit form and its application in considering what happened with the elliptic, hyperbolic, parabolic kind of points under the infinitesimal bending of surface.

1. Introduction

The importance of shape of surfaces analysis is confirmed with wide applications. Many procedures in science, engineering and medicine produce data in the form of geometric shapes. Shape analysis of surfaces is important in medicine (neurologist), biochemistry (biopolymer),biophysics (shape of cell membrane), history (shape of stones and rocks), construction, contemporary architecture, computer graphics, 3D television, modern art, etc. There has been a significant amount of research and activity in the general area of shape analysis. By shape analysis we mean a set of tools for comparing, matching, deforming, and modeling shapes.

There are two rudimentary ways to characterize the shape of a surface S: to consider how the unit normal ν behaves as we move around (*Shape Operator*) and to compare S to a sphere (*Willmore energy*). Shape operator is a linear operator that calculates the bending of surface S.

Since the shape is an important feature of objects and can be immensely useful in characterizing objects, we point out the shape analysis considering the variation of quantities that characterize the shape itself. Fundamental functionals that measure the bending of a surface, are the shape operator, as a vector function, the normal curvature, as a real-valued function, with the principal curvatures as its extreme values.

It is known that the magnitudes depending on the first fundamental form are stationary under infinitesimal bending. Infinitesimal bending field of a Gaudi surface was determined in [3]. Variation of some geometric magnitudes under infinitesimal bending was considered in [2] and [1]. Variation of the Willmore energy under infinitesimal bending of a surface is studied in [5]. Variation of the shape operator was considered in [4].

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2. Preliminary of Curvature Based Functions

Shape Operator is a linear operator that calculates the bending of surface S.

Definition 2.1. *Let $S \subset \mathcal{R}^3$ be a regular surface, and let U be a surface normal to S defined in a neighborhood of a point $p \in S$. For a tangent vector v_p to S at p we put*

$$\underline{S}(v_p) = -D_v U. \tag{1}$$

*Then $\underline{S}$ is called the **shape operator**.*

The shape operator can be expressed in matrix form:

$$\underline{S} = \frac{1}{EG - F^2} \begin{pmatrix} LG - MF & ME - LF \\ MG - NF & NE - MF \end{pmatrix}. \tag{2}$$

While the shape operator is vector function that measures the bending of a surface, normal curvature is a real-valued function that does the same thing.

Definition 2.2. *Let v_p is an arbitrary non-zero tangent vector of surface $S \subset \mathcal{R}^3$ in $p \in S$. Then the **normal curvature** of S in direction v_p will be equal:*

$$k_n(v_p) = \frac{\underline{S}(v_p) \cdot v_p}{\|v_p\|^2} \ . \tag{3}$$

Next Lemma was proved in [6]:

Lemma 2.3. *The normal curvature of S at a given point $p \in S$, in direction of tangent vector $(u(t))_s$, can be expressed in form:*

$$k_n(t) = L\cos^2 t + 2M \sin t \cos t + N \sin^2 t, \tag{4}$$

where $(u(t))_s$ represents all directions:

$$(u(t))_s = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_s, \tag{5}$$

$t \in [0, 2\pi)$ and index $\mathbf{s}$ presents a vector in standard base $\{x_u, x_v\}$:

$$(x_u)_s = (1, 0), \qquad (x_v)_s = (0, 1).$$

The principal curvatures measure the maximum and minimum bending of a regular surface S at each point $p \in S$.

Definition 2.4. *Let $S \subset \mathcal{R}^3$ is regular surface and $p \in S$. The maximal and the minimal value of normal curvature k_n of S at point p we call **the principal curvatures** of S at point p and denote k_1 and k_2.*

2.1. Connections among the curvatures

Among the curvatures and shape operator there are some connections.

Definition 2.5. *Gaussian and mean curvatures are the functions $K, H : S \to R$ defined as:*

$$K(p) = det(\underline{S}(p)) \ , \qquad H(p) = \frac{1}{2} tr(\underline{S}(p)) \ . \tag{6}$$

Theorem 2.6. *Let k_1 and k_2 are the principal curvatures of regular surface $S \subset \mathcal{R}^3$. Gaussian curvature of surface S is in form:*

$$K = k_1 k_2 \ . \tag{7}$$

Mean curvature of surface S is in form:

$$H = \frac{1}{2}(k_1 + k_2) \ . \tag{8}$$

Depending on the sign of curvatures, there are four types of surface points.

Definition 2.7. *Let p is a point of regular surface $S \subset \mathcal{R}^3$. The point p is:*

- *p is* elliptic *if* $K(p) > 0 \qquad (k_1 k_2 > 0)$;
- *p is* hyperbolic *if* $K(p) < 0 \qquad (k_1 k_2 < 0)$;
- *p is* parabolic *if* $K(p) = 0$ *and* $\underline{S}(p) \neq 0 \qquad (k_1 = 0 \veebar k_2 = 0)$;
- *p is* planar *if* $K(p) = 0$ *and* $\underline{S}(p) = 0 \qquad (k_1 = k_2 = 0)$.

3. Variation of Curvatures Under Infinitesimal Bending of Surface

Geometric quantities change by infinitesimal bending and those changes can be measured with variation of geometric magnitudes. It is known that variations of some geometric magnitudes that depend on coefficients of the first fundamental form of the surface are zero under infinitesimal bending of the surface in $\mathcal{R}^3$. For instance, Christoffel's symbols, the first fundamental form, the determinant of the first and the second fundamental form, the area of a region on the surface, the Gaussian and the geodesic curvature are stationary under infinitesimal bending of a surface.

Let the surface S be a regular surface, parameterized by:

$$\mathbf{r}(u, v) = (u, v, f(u, v)) \ , \tag{9}$$

and infinitesimal bending field by:

$$\mathbf{z}(u, v) = (\xi(u, v), \eta(u, v), \zeta(u, v)). \tag{10}$$

Then, the infinitesimal bending of a surface S is given with:

$$\begin{aligned} S_\epsilon : \tilde{\mathbf{r}}(u, v, \epsilon) &= \mathbf{r}(u, v) + \epsilon \mathbf{z}(u, v) = \\ &= (u + \epsilon\xi(u, v), v + \epsilon\eta(u, v), f(u, v) + \epsilon\zeta(u, v)) \ . \end{aligned} \tag{11}$$

According to [7], the coefficients of the first and the second fundamental form of surface S_ϵ can be expressed with equations (12) and (13):

$$\begin{aligned} \tilde{E} &= \tilde{\mathbf{r}}_u \cdot \tilde{\mathbf{r}}_u = 1 + f_u^2 + \epsilon^2(\xi_u^2 + \eta_u^2 + \zeta_u^2) \ , \\ \tilde{F} &= \tilde{\mathbf{r}}_u \cdot \tilde{\mathbf{r}}_v = f_u f_v + \epsilon^2(\xi_u \xi_v + \eta_u \eta_v + \zeta_u \zeta_v) \ , \\ \tilde{G} &= \tilde{\mathbf{r}}_v \cdot \tilde{\mathbf{r}}_v = 1 + f_v^2 + \epsilon^2(\xi_v^2 + \eta_v^2 + \zeta_v^2) \ , \end{aligned} \tag{12}$$

$$\begin{aligned} \tilde{L} &= \frac{1}{\sqrt{\tilde{g}}}[\tilde{\mathbf{r}}_{uu}, \tilde{\mathbf{r}}_u, \tilde{\mathbf{r}}_v] = \frac{1}{\sqrt{\tilde{g}}}[f_{uu} + \epsilon\zeta_{uu}(1 + f_u^2 + f_v^2) + \epsilon^2 A_1 + \epsilon^3 A_2] \ , \\ \tilde{M} &= \frac{1}{\sqrt{\tilde{g}}}[\tilde{\mathbf{r}}_{uv}, \tilde{\mathbf{r}}_u, \tilde{\mathbf{r}}_v] = \frac{1}{\sqrt{\tilde{g}}}[f_{uv} + \epsilon\zeta_{uv}(1 + f_u^2 + f_v^2) + \epsilon^2 B_1 + \epsilon^3 B_2] \ , \\ \tilde{N} &= \frac{1}{\sqrt{\tilde{g}}}[\tilde{\mathbf{r}}_{vv}, \tilde{\mathbf{r}}_u, \tilde{\mathbf{r}}_v] = \frac{1}{\sqrt{\tilde{g}}}[f_{vv} + \epsilon\zeta_{vv}(1 + f_u^2 + f_v^2) + \epsilon^2 C_1 + \epsilon^3 C_2] \ . \end{aligned} \tag{13}$$

Functions: A_i, B_i, C_i, $i = 1, 2$, are obtained in development of corresponding determinants and

$$\tilde{g} = \tilde{E}\tilde{G} - \tilde{F}^2 = 1 + f_u^2 + f_v^2 + \epsilon^2 \ldots + \epsilon^4 \ldots . \tag{14}$$

Using the coefficients of the first and the second fundamental form of surface S_ϵ, we can consider the variation of curvatures under infinitesimal bending of surface.

In work [4] it was calculated the variation of the shape operator and found when it will be vanished.

Theorem 3.1. *Variation of shape operator of surface* (9) *under infinitesimal bending* (11) *is given with equations:*

$$\begin{aligned}
\delta\underline{S}(\mathbf{r}_u) =& \frac{1}{\sqrt{g}}\Big((\zeta_{uu}(1 + f_v^2) - \zeta_{uv} f_u f_v)\mathbf{r}_u + (\zeta_{uv}(1 + f_u^2) - \zeta_{uu} f_u f_v)\mathbf{r}_v\Big) + \\
&+ \frac{1}{g^{\frac{3}{2}}}\Big((f_{uu}(1 + f_v^2) - f_{uv} f_u f_v)\mathbf{z}_u + (f_{uv}(1 + f_u^2) - f_{uu} f_u f_v)\mathbf{z}_v\Big), \\
\delta\underline{S}(\mathbf{r}_v) =& \frac{1}{\sqrt{g}}\Big((\zeta_{uv}(1 + f_v^2) - \zeta_{vv} f_u f_v)\mathbf{r}_u + (\zeta_{vv}(1 + f_u^2) - \zeta_{uv} f_u f_v)\mathbf{r}_v\Big) + \\
&+ \frac{1}{g^{\frac{3}{2}}}\Big((f_{uv}(1 + f_v^2) - f_{vv} f_u f_v)\mathbf{z}_u + (f_{vv}(1 + f_u^2) - f_{uv} f_u f_v)\mathbf{z}_v\Big),
\end{aligned} \tag{15}$$

where $g = 1 + f_u^2 + f_v^2$.

Corollary 3.2. *Variation of the shape operator of surface* (9) *under infinitesimal bending* (11) *will be equal zero if second order partial derivatives of functions* $\boldsymbol{f}$ *and its bending field* $\mathbf{z}$ *are zero, i.e. these functions are linear.*

Variation of the normal and the principal curvatures was considered in [6].

Theorem 3.3. *Variation of the normal curvature of a surface* (9) *under infinitesimal bending* (11) *is given with equation:*

$$\delta k_n(t) = \sqrt{g}(\zeta_{uu}\cos^2 t + 2\zeta_{uv}\sin t\cos t + \zeta_{vv}\sin^2 t), \tag{16}$$

where $g = 1 + f_u^2 + f_v^2$.

Corollary 3.4. *Variation of the normal curvature of a surface* (9) *under infinitesimal bending* (11) *will be equal zero if the third coordinate of bending field* $\mathbf{z}$ *is linear function.*

Lemma 3.5. *Variation of the principal curvatures of a surface under infinitesimal bending of a surface, expressed by Gaussian and mean curvature:*

$$\delta k_{1,2} = \delta H \frac{\sqrt{H^2 - K} \pm H}{\sqrt{H^2 - K}} . \tag{17}$$

Theorem 3.6. *Variation of the principal curvatures of a surface* (9) *under infinitesimal bending of a surface* (11) *is given with equation:*

$$\delta k_{1,2} = \delta H \frac{\sqrt{\mathcal{A}} \pm \mathcal{B}}{\sqrt{\mathcal{A}}} , \tag{18}$$

where

$$\begin{aligned}
\mathcal{A} =& \Big(f_{uu}(1 + f_v^2) - f_{vv}(1 + f_u^2)\Big)^2 + \\
&+ 4\Big(f_{uv}(1 + f_v^2) - f_{vv} f_u f_v\Big)\Big(f_{uv}(1 + f_u^2) - f_{uu} f_u f_v\Big) , \\
\mathcal{B} =& f_{uu}(1 + f_v^2) - 2 f_{uv} f_u f_v + f_{vv}(1 + f_u^2) ,
\end{aligned} \tag{19}$$

and

$$\delta H = \frac{(1+f_u^2)\zeta_{vv} - 2f_u f_v \zeta_{uv} + (1+f_v^2)\zeta_{uu}}{2\sqrt{1+f_u^2+f_v^2}} \tag{20}$$

is from [7].

Corollary 3.7. *Variation of the principal curvatures of a surface* (9) *under infinitesimal bending of a surface* (11),*will be equal zero if the variation of the mean curvature vanishes.*

4. Kinds of points under the infinitesimal bending

Since the Gaussian curvature is stationary under infinitesimal bending of a surface, it is obviously that elliptic points stay elliptic and hyperbolic points stay hyperbolic. The thing is what happened with parabolic and planar points under the infinitesimal bending of a surface.

If we suppose that p is parabolic $K(p) = 0$ and $k_1 = 0 \veebar k_2 = 0$, for example, $k_1 = 0$ and $k_2 \neq 0$. Then, $\delta K = 0 \Leftrightarrow \delta(k_1 k_2) = k_1\delta k_2 + k_2 \delta k_1 = 0$, and must be $\delta k_1 = 0$, which means that $\tilde{k}_1 = 0$.

The conditions when δk_2 will be vanished are given in next Lemma, that can be proved using the variation of the principal curvatures:

Lemma 4.1. *If the variation of the mean curvature vanishes, parabolic point remains parabolic and planar point remains planar under the infinitesimal bending of the surface.*

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