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Generalized Open Sets and Selection Properties

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Abstract. We define and study new weak versions of the classical Menger covering property. For this we use α-open and θ-open covers of a topological space. Relations of these properties with known weak versions of the Menger property are examined. In this way we complement the study of weak covering properties defined by selection principles.

1. Introduction

Throughout the paper, (X, τ), or shortly X, will denote a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of (X, τ), $\mathrm{Int}(A)$ and $\mathrm{Cl}(A)$ will denote the interior and the closure of A, respectively, while τ_A denotes the subspace topology on A inherited from (X, τ). If $f : X \to Y$ is a mapping between spaces X and Y, $\mathcal{F}$ a collection of subsets of X, and $\mathcal{G}$ a collection of subsets of Y, then $\bigcup \mathcal{F}$ and $\bigcap \mathcal{F}$ denote the union and intersection of all elements of $\mathcal{F}$, respectively, $f(\mathcal{F})$ denotes the set $\{f(F) : F \in \mathcal{F}\}$, and $f^{\leftarrow}(\mathcal{G})$ denotes the set $\{f^{\leftarrow}(G) : G \in \mathcal{G}\}$. The most of notation and terminology are as in the book [9].

Several weak variants of selection principles occur in the mathematical literature and have been extensively studied in the last few years by a number of authors. This investigation goes in two directions: 1) the closure operator is applied in the definition of a selection property [2–4, 6–8, 13–15, 30, 35], and 2) sequences of open covers are replaced by sequences of covers by some generalized open sets [18, 19, 33, 34]. In this paper we continue the study in the second direction by using (mainly) covers by α-open and θ-open sets. The properties we are going to define and investigate are related to the classical Menger covering property: a space X has the *Menger property* (or X is a *Menger space*) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \bigcup_{n \in \mathbb{N}} \bigcup \mathcal{V}_n$.

The following generalizations of open sets will be used for definitions of variations on the Menger property: semi-open, α-open, θ-open sets.

A subset A of a topological space X is said to be:

- *semi-open* if there is an open set $U \subset X$ such that $U \subset A \subset \mathrm{Cl}(U)$, or equivalently, if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ [21];
- *α-open* if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, or equivalently, if $A = U \setminus N$, where U is open and N is nowhere dense in X [27], or equivalently, if there exists an open set U such that $U \subset A \subset \mathrm{Int}(\mathrm{Cl}(U))$;
- *θ-open* if for each $x \in A$ there is an open set $U \subset X$ such that $x \in U \subset \mathrm{Cl}(U) \subset A$ [37].

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Note that we have

$$\text{clopen} \Rightarrow \theta\text{–open} \Rightarrow \text{open} \Rightarrow \alpha\text{–open} \Rightarrow \text{semi–open}.$$

Diagram 1

Definition 1.1. A space (X, τ) is said to be *semi-Menger* [33] (resp. *mildly Menger* [18], *θ-Menger*, *α-Menger*) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open (resp. clopen, θ-open, α-open) covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n = X$.

Evidently we have

$$\text{semi–Menger} \Rightarrow \alpha\text{–Menger} \Rightarrow \text{Menger} \Rightarrow \theta\text{–Menger} \Rightarrow \text{mildly Menger}.$$

Diagram 2

All these properties can be written in a form of the following selection property of the Menger type. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of some families of subsets of a space X. The space X satisfies the selection principle $\mathsf{S}_{\text{fin}}(\mathcal{A}, \mathcal{B})$ if for each sequence $(A_n : n \in \mathbb{N})$ in $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subset A_n$ and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$ (see [16, 17]).

Let $\mathcal{O}, \mathcal{CO}, s\text{-}\mathcal{O}, \alpha\text{-}\mathcal{O}, \theta\text{-}\mathcal{O}, z\text{-}\mathcal{O}$ denote the collection of all open, clopen, semi-open, α-open, θ-open covers, covers by cozero sets of a space X, respectively. Then the Menger, mildly Menger, semi-Menger, α-Menger, θ-Menger, z-Menger property of X is the property $\mathsf{S}_{\text{fin}}(\mathcal{O}, \mathcal{O})$, $\mathsf{S}_{\text{fin}}(\mathcal{CO}, \mathcal{CO})$, $\mathsf{S}_{\text{fin}}(s\text{–}\mathcal{O}, s\text{–}\mathcal{O})$, $\mathsf{S}_{\text{fin}}(\alpha\text{–}\mathcal{O}, \alpha\text{–}\mathcal{O})$, $\mathsf{S}_{\text{fin}}(\theta\text{–}\mathcal{O}, \theta\text{–}\mathcal{O})$, $\mathsf{S}_{\text{fin}}(z\text{–}\mathcal{O}, z\text{–}\mathcal{O})$, respectively.

Another classical selection principle (of Rothberger type) is the following. $\mathcal{A}$ and $\mathcal{B}$ are again collections of some families of subsets of a space X. The space X satisfies the selection principle $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ if for each sequence $(A_n : n \in \mathbb{N})$ in $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$ and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$ [16, 17]. Then the symbols $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$, $\mathsf{S}_1(\mathcal{CO}, \mathcal{CO})$, $\mathsf{S}_1(s\text{-}\mathcal{O}, s\text{-}\mathcal{O})$, $\mathsf{S}_1(\alpha\text{-}\mathcal{O}, \alpha\text{-}\mathcal{O})$, $\mathsf{S}_1(\theta\text{-}\mathcal{O}, \theta\text{-}\mathcal{O})$, $\mathsf{S}_1(z\text{-}\mathcal{O}, z\text{-}\mathcal{O})$, denote that X has the Rothberger, mildly Rothberger, semi-Rothberger, α-Rothberger, θ-Rothberger, z-Rothberger property, respectively.

In this article we discuss only Menger-type properties; Rothebger-type properties can be studied in a similar way with suitable modifications.

2. α-Menger Spaces

Recall first that the family of α-open sets of a space (X, τ) is a topology on X, denoted by τ_α [27], and that $\tau \subset \tau_\alpha$. For the real line $\tau \neq \tau_\alpha$. The role of α-open sets has been investigated in many papers (see, for instance, [1, 25, 28, 29, 32].

Evidently, a space (X, τ) is α-Menger if and only if (X, τ_α) is Menger, and, as we mentioned already, each α-Menger space is Menger.

Example 2.1. 1. There is a Menger space which is not α-Menger.

Let X be any uncountable set and F a fixed finite subset of X. Define topology τ on X by $\tau = \{X, \emptyset, F\}$. This space is compact, so Menger. All sets of the form $F \cup \{p\}$, $p \in X \setminus F$, are α-open. The α-cover $\mathcal{U} = \{F \cup \{p\} : p \in X \setminus F\}$ does not contain a countable subcover, so that (X, τ) cannot be α-Menger.

2. The real line (and any infinite set) with the cofinite topology is α-Menger.

3. Every semi-compact space (each semi-open cover has a finite subcover) is hereditarily α-Menger.

[It follows from the fact that every semi-compact space is hereditarily semi-compact [12], hence hereditarily semi-Menger. But every semi-Menger space is α-Menger.]

Example 2.2. α-Mengerness is not a hereditary property.

Let X be the space in [9, Example 1.1.8]. X is an infinite set and x_0 a point in X. Define on X the topology $\tau = \{U \subset X : x_0 \notin U\} \cup \{U \subset X : \text{ the set } X \setminus U \text{ is finite}\}$. All singletons $\{x\}$, $x \in X \setminus \{x_0\}$ are α-open. This space is α-Menger as it is easily checked. However, the subspace $Y = X \setminus \{x_0\}$ is not. Consider the α-open cover $\mathcal{U} = \{\{x\} : x \in Y\}$ of Y. This cover contains no a countable subcover, so that Y cannot be α-Menger.

Remark 2.3. Example 2.2 actually shows that an α-open subspace of an α-Menger space need not be α-Menger.

Recall that a subset A of a topological space X is *pre-open* [26] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, and that every α-open set if pre-open.

Proposition 2.4. *α-Mengerness is preserved by pre-open α-closed subspaces* (*hence by α-clopen subspaces*).

Proof. Let A be a pre-open α-closed subspace of an α-Menger space X. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of α-open covers of (A, τ_A). It is easy to see that every α-open subset of a pre-open subspace A of X is the intersection of an α-open subset of X with A. Therefore, for each $n \in \mathbb{N}$ and each $U \in \mathcal{U}_n$ there is H_U α-open in X such that $U = A \cap H_U$. Set $\mathcal{H}_n = \{H_U : U \in \mathcal{U}_n\} \cup \{X \setminus A\}$, $n \in \mathbb{N}$. Then $(\mathcal{H}_n : n \in \mathbb{N})$ is a sequence of α-open covers of X and thus there is a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that $\mathcal{W}_n$ is a finite subset of H_n for each n, and $X = \bigcup_{n \in \mathbb{N}} \bigcup W_n$. If we put for each n, $\mathcal{V}_n = \{U : H_U \in \mathcal{W}_n\}$ we obtain the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ witnessing for $(\mathcal{U}_n : n \in \mathbb{N})$ that (A, τ_A) is α-Menger. $\square$

We give now definitions of generalized continuous (open) mappings related to α-open sets.

A mapping $f : X \to Y$ is called:

1. *α-continuous* [26] (*α-irresolute* [24]) if the preimage of any open (α-open) subset of Y is α-open in X;
2. *α-open* (*strongly α-open*) if the image of any α-open subset of X is α-open (open) in Y.

Theorem 2.5. *An α-continuous image of an α-Menger space is a Menger space.*

Proof. Let $f : X \to Y$ be an α-continuous mapping from an α-Menger space X onto a topological space Y. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. Since f is α-continuous, setting $\mathcal{U}_n = f^{\leftarrow}(\mathcal{V}_n)$, $n \in \mathbb{N}$, we get the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of α-open covers of X. For each n, there is a finite subset $\mathcal{H}_n$ of $\mathcal{U}_n$ such that $X = \bigcup_{n \in \mathbb{N}} \bigcup \mathcal{H}_n$. Let $\mathcal{W}_n = f(\mathcal{H}_n)$, $n \in \mathbb{N}$. Then the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ testifies for $(\mathcal{V}_n : n \in \mathbb{N})$ that Y is a Menger space. $\square$

Similarly, we have the following fact.

Theorem 2.6. *An α-irresolute image of an α-Menger space is also α-Menger.*

Theorem 2.7. *Let (X, τ) be an α-Menger space. Then:*

(1) (X, τ) *is an α-continuous image of a Menger space;*

(2) (X, τ) *is an α-open preimage of a Menger space.*

Proof. Since (X, τ) is α-Menger, the space (X, τ_α) is Menger. The identity mapping $1_X : (X, \tau_\alpha) \to (X, \tau)$ is α-continuous. On the other hand, $1_X : (X, \tau) \to (X, \tau_\alpha)$ is α-open (because τ and τ_α have same α-open sets). $\square$

Theorem 2.8. *For a space (X, τ) the following are equivalent:*

(1) *X is α-Menger;*

(2) *X admits a strongly α-open bijection onto a Menger space Y.*

Proof. (1) ⇒ (2) Since (X, τ) is α-Menger the space (X, τ_α) is Menger. The identity mapping $1_X : (X, \tau) \to (X, \tau_\alpha)$ is clearly strongly α-open and bijective.

(2) ⇒ (1) Let $f : X \to Y$ be a strongly α-open bijection from a space X onto a Menger space Y. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of α-open covers of (X, τ). Then $(f(\mathcal{U}_n) : n \in \mathbb{N})$ is a sequence of open covers of Y. Choose for each n a finite subset $\mathcal{V}_n$ such that $Y = \bigcup_{n\in\mathbb{N}} \bigcup f(\mathcal{V}_n)$. Evidently, then $X = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n$, i.e. X is α-Menger. □

The following two theorems give interesting properties of α-Mengerness and its relations with semi-Mengerness.

Theorem 2.9. *α-Mengerness is a semi-topological property.*

Proof. Let $f : X \to Y$ be a semi-homeomorphism from an α-Menger space (X, τ) onto a space (Y, σ) (this means that f is a bijection and images and preimages of semi-open sets are semi-open). Then f is a homeomorphism from (X, τ_α) onto (Y, σ_α) [5]. As (X, τ_α) is Menger, and Mengerness is a topological property, one concludes that (Y, σ_α) is Menger. Therefore, (Y, σ) is α-Menger. □

A space (X, τ) is *semi-Lindelöf* if any semi-open cover of X has a countable subcover.

Theorem 2.10. *Let (X, τ) be a semi-Lindelöf space. If the space (X, τ_α) is hereditarily Menger, then (X, τ) is semi-Menger.*

Proof. Let $(\mathcal{S}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Since X is semi-Lindelöf, one can assume that all these covers are countable. For each n and each $S \in \mathcal{S}_n$ there is an open set $U_S \subset X$ such that $U_S \subset S \subset \mathrm{Cl}(U_S)$. Let $\mathcal{U}_n = \{U_S : S \in \mathcal{S}_n\}$. The set $O_n = \cup\mathcal{U}_n$ is open and dense in X, for each $n \in \mathbb{N}$, hence $X \setminus O_n$ is closed and nowhere dense, so discrete. Since (X, τ) is Lindelöff, being semi-Lindelöf, it follows that the sets $X \setminus O_n$ are all countable. (It follows from the fact that the extent $e(X)$ of a Lindelöf space is countable [9].) By assumption, the subspace $Y = X \setminus \bigcup_{n\in\mathbb{N}} O_n$ is Menger so that there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each n, and $Y \subset \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n$. It remains to cover the countable set $\bigcup_{n\in\mathbb{N}} O_n$ by finite sets $\mathcal{W}_n \subset \mathcal{U}_n$ (which is possible). Set $\mathcal{H}_n = \mathcal{V}_n \cup \mathcal{W}_n$, $n \in \mathcal{N}$. For each $H \in \mathcal{H}_n$ take a set $G_H \in \mathcal{S}_n$ with $H \subset G_H$. Let $\mathcal{G}_n = \{G_H : H \in \mathcal{H}_n\}$. The sequence $(\mathcal{G}_n : n \in \mathbb{N})$ witnesses for $(\mathcal{S}_n : n \in \mathbb{N})$ that X is semi-Menger. □

We end this section with an observation.

Remark 2.11. It is known that there is a game associated to the selection principle $\mathsf{S}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$. It is a game, denoted by $\mathsf{G}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$, for two players ONE and TWO who play a round for each natural number. In the n-th round ONE chooses an element $A_n \in \mathcal{A}$ and TWO responds by choosing a finite $B_n \subset A_n$. Two wins if and only if $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$. As we mentioned, (X, τ_α) is a topological space for a space (X, τ). Since these two spaces have the same α-open sets, by the classical result of Hurewicz (see [16]) we have the following:

A space (X, τ) is α-Menger, i.e. satisfies $\mathsf{S}_{\mathrm{fin}}(\alpha\text{–}\mathcal{O}, \alpha\text{–}\mathcal{O})$, if and only if the player ONE has no winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\alpha\text{–}\mathcal{O}, \alpha\text{–}\mathcal{O})$.

3. θ-Menger Spaces

In this section we examine some properties of θ-Menger spaces and their relationships with some known weaker forms of the Menger property. θ-open and θ-closed sets and the θ-closure operator play an important role in several branches of topology, in particular in cardinal invariants theory and the theory of absolutes (see [10, 11, 20, 22, 23, 31]).

Recall that a space (X, τ) is said to be *weakly Menger* [6] (resp. *almost Menger* [14]) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subset \mathcal{U}_n$ and $X = \mathrm{Cl}\left(\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n\right)$ (resp. $X = \bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{Cl}(V) : V \in \mathcal{V}_n\}$). For a study of these classes of spaces see [2–4, 7, 13, 15, 30, 35]. A space X is *quasi-Menger* [8] (qM) if for each closed set $F \subset X$ and each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of F by sets open in X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each n, $\mathcal{V}_n \subset \mathcal{U}_n$ and $\mathrm{Cl}\left(\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n\right) \supset F$.

We define now another Menger-type covering property.

Definition 3.1. A space (X, τ) is *nearly Menger* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{Int}(\mathrm{Cl}(V)) : V \in \mathcal{V}_n\}$.

Recall that a subset A of a space (X, τ) is *regular open* [9] if it is the interior of a closed subset of X. Denote by $\mathcal{RO}$ the set of all regular open subsets of a space X.

Theorem 3.2. *For a topological space X the following are equivalent:*

(1) *X is nearly Menger;*

(2) *X satisfies $\mathsf{S}_{\mathrm{fin}}(\mathcal{RO}, \mathcal{RO})$.*

Proof. (1) $\Rightarrow$ (2) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of regular open covers of X. So, we have actually a sequence of open covers of X. Since X is nearly Menger there is a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{Int}(\mathrm{Cl}(V) : V \in \mathcal{U}_n\}$. Since $\mathrm{Int}(\mathrm{Cl}(V)) = V$ for each n and each $V \in \mathcal{V}_n$ we conclude that (2) is satisfied.

(2) $\Rightarrow$ (1) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Define for each $n \in \mathbb{N}$, $\mathcal{V}_n = \{\mathrm{Int}(\mathrm{Cl}(U)) : U \in \mathcal{U}_n\}$. Then $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of regular open covers of X. By (2) there is a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ for each $n \in \mathbb{N}$, and $X = \bigcup_{n\in\mathbb{N}} \bigcup\{W : W \in \mathcal{W}_n\}$. Pick for each n and each $W \in \mathcal{W}_n$ a set $U_W \in \mathcal{U}_n$ with $W = \mathrm{Int}(\mathrm{Cl}(U_W))$ and set $\mathcal{H}_n = \{U_W : W \in W_n\}$, a finite subset of $\mathcal{U}_n$. Since $X = \bigcup_{n\in\mathbb{N}} \bigcup\{W : W \in \mathcal{W}_n\} = \bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{Int}(\mathrm{Cl}(U_W)) : W \in \mathcal{W}_n\}$ we conclude that X is nearly Menger. $\square$

We are going back to θ-Menger spaces.

The collection of θ-open subsets of a space (X, τ) form a new topology τ_θ on X weaker than the original topology τ. (In regular spaces $\tau_\theta = \tau$.)

Because $\tau_\theta \subset \tau$, any Menger space is θ-Menger. In the class of regular spaces these two classes coincide.

Example 3.3. There is a Urysohn θ-Menger space which is not Menger.

Let X be the Euclidean plane with the deleted radius topology [36]: a subbasis for a topology τ on X is the set of all open discs minus horizontal diameters other than center. The closure of any open set $U \subset (X, \tau)$ is the usual Euclidean closure of U, hence θ-open sets in (X, τ) are open sets in X with the usual Euclidean topology. It follows that (X, τ) is a θ-Menger space. On the other hand, (X, τ) is not a Menger space because it is not Lindelöf (see [36]).

Remark 3.4. The (counter)example 78 (Half-Disc Topology) in [36] is another example of a Uryshon θ-Menger space that is not Menger.

The following theorem describes a relation between almost Menger and θ-Menger spaces.

Theorem 3.5. *Every almost Menger space X is θ-Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of θ-open covers of X. Let $x \in X$. Then for each $n \in \mathbb{N}$ there is $U_{x,n} \in \mathcal{U}_n$ with $x \in U_{x,n}$, and since $U_{x,n}$ is θ-open there is an open set $W_{x,n}$ such that $x \in W_{x,n} \subset \mathrm{Cl}(W_{x,n}) \subset U_{x,n}$. Therefore, for each $n \in \mathbb{N}$, the set $\mathcal{W}_n = \{W_{x,n} : x \in X\}$ is an open cover of X. Since X is almost Menger there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each n, $\mathcal{V}_n \subset \mathcal{W}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{Cl}(V) : V \in \mathcal{V}_n\}$. For each $n \in \mathbb{N}$ and each $V \in \mathcal{V}_n$ take the element $U_V \in \mathcal{U}_n$ containing $\mathrm{Cl}(V)$ and let $\mathcal{H}_n = \{U_V : V \in \mathcal{V}_n\}$. Then the sequence $(\mathcal{H}_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathcal{B}_n)$ that X is θ-Menger. $\square$

The following diagram shows relationships among known and new classes of weak versions of the Menger property.

$$\begin{tikzcd}[arrows=Rightarrow]
s\text{–M} \arrow[r] & \alpha\text{–M} \arrow[r] & \text{M} \arrow[r] \arrow[d] & \text{nearly M} \arrow[r] & \text{almost M} \arrow[r] \arrow[d] & \theta\text{–M} \arrow[r] & z\text{–M} \\
 & & \text{q–M} \arrow[rr] & & \text{weakly M} & &
\end{tikzcd}$$

Diagram 3

Remark 3.6. In [37], the *θ-closure operator* was introduced: if A is a subset of a space X, then $\mathrm{Cl}_\theta(A) = \{x \in X : \text{ for each neighbourhood } U \text{ of } x, \mathrm{Cl}(U) \cap A \neq \emptyset\}$. It would be interesting to use this operator to define and investigate the following two classes of spaces. A space X is said to be *θ-almost Menger* (respectively, *θ-weakly Menger*) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{Cl}_\theta(\mathrm{Cl}(V)) : V \in \mathcal{V}_n\}$ (respectively, $X = \mathrm{Cl}_\theta\left(\mathrm{Cl}\left(\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n\right)\right)$). Observe that every almost Menger space is θ-almost Menger, and every weakly Menger space is θ-weakly Menger.

In some special classes of spaces some properties in Diagram 3 coincide. Recall that a space (X, τ) is said to be *almost regular* if for each regular closed subset A of X and each $x \in X \setminus A$ there are disjoint open sets U and V such that $A \subset U$ and $x \in V$.

Theorem 3.7. *An almost regular θ-Menger space is nearly Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Because X is almost regular, by [20, Theorem 2.5], putting $\mathcal{H}_n = \{\mathrm{Int}(\mathrm{Cl}(U) : U \in \mathcal{U}_n\}$, we get the sequence $(\mathcal{H}_n : n \in \mathbb{N})$ of θ-open covers of X. Since X is θ-Menger there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that $\mathcal{V}_n \subset \mathcal{H}_n$, $n \in \mathbb{N}$, and $X = \bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{Int}(\mathrm{Cl}(V)) : V \in \mathcal{V}_n\}$. This means X is nearly Menger. $\square$

Corollary 3.8. *In the class of almost regular spaces almost Mengerness, near Mengerness and θ-Mengerness coincide.*

We do not know an almost Menger space which is not nearly Menger.

Example 3.9. θ-Mengerness is not a hereditary property.

Let (X, τ) be Example 78 in [36] (see Remark 3.4). This space is θ-Menger and has a discrete uncountable subspace L which cannot be θ-Menger.

However we have the following result.

Theorem 3.10. *A clopen subspace A of a θ-Menger space X is θ-Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of θ-open covers of (A, τ_A). Since A is clopen in X, for each $n \in \mathbb{N}$ and each $U \in \mathcal{U}_n$ there exists a θ-open set $V_U \subset X$ such that $U = A \cap V_U$. Let $\mathcal{V}_n = \{V_U : U \in \mathcal{U}_n\} \cup \{X \setminus A\}$. Then $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of θ-open covers of X. θ-Mengerness of X implies the existence of a sequence $(\mathcal{F}_n : n \in \mathbb{N})$ with $\mathcal{F}_n$ is a finite subset of $\mathcal{V}_n$ for each $n \in \mathbb{N}$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{F}_n$. Set $\mathcal{H}_n = \mathcal{F}_n \setminus \{X \setminus A\}$, $n \in \mathbb{N}$. Clearly, the sequence $(\mathcal{H}_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that (A, τ_A) is θ-Menger. $\square$

Clearly, a space (X, τ) is θ-Menger if and only if (X, τ_θ) is Menger. It is known that the Menger property is not finitely productive. One can easily conclude that the product of two θ-Menger spaces need not be θ-Menger, even the square of a θ-Menger space need not be θ-Menger. However we have the following fact which is given without proof.

Theorem 3.11. *The product of a θ-Menger space and a compact space is θ-Menger.*

Let $\mathcal{U}$ be a θ-open cover of a space X. Call $\mathcal{U}$ a *θ-ω-cover* of X if each finite subset of X is contained in an element U in $\mathcal{U}$ and $X \notin \mathcal{U}$. The family of θ-ω-covers of a space X denote by θ-Ω.

In the following theorem we use the fact that the product of θ-open sets $U \subset X$ and $V \subset Y$ is θ-open in $X \times Y$ [23].

Theorem 3.12. *If each finite power of a space X is θ-Menger, then X satisfies $\mathsf{S}_{\mathsf{fin}}(\theta\text{–}\Omega, \theta\text{–}\Omega)$.*

Proof. Suppose that X^m is θ-Menger for each $m \in \mathbb{N}$. Fix $k \in \mathbb{N}$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of θ-ω-covers of X. Set $\mathcal{V}_n = \{U^k : U \in \mathcal{U}_n\}$, $n \in \mathbb{N}$. Then we have the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of θ-open covers of X^k. Since X^k is θ-Menger, there is a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ of finite sets such that for each n, $\mathcal{W}_n \subset \mathcal{V}_n$ and $\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{W}_n$ covers X^k. For each n and each $W \in \mathcal{W}_n$ take $U_W \in \mathcal{U}_n$ so that $U_W^k = W$ and put $\mathcal{H}_n = \{U_W : W \in \mathcal{W}_n\}$. We claim that $\{U_W : W \in \mathcal{H}_n, n \in \mathbb{N}\}$ is a θ-ω-cover of X.

Take a finite set $K = \{x_1, \ldots, x_p\} \subset X$. Then $x = (x_1, \ldots, x_p) \in X^p$ so that there is some $n \in \mathbb{N}$ and $U \in \mathcal{H}_n$ such that $x \in U^p$. Then $F \subset U$, which means that X satisfies $\mathsf{S}_{\mathsf{fin}}(\theta\text{–}\Omega, \theta\text{–}\Omega)$. □

We will discuss now preservation of θ-Mengerness under several kinds of mappings.

Let $f : X \to Y$ be a mapping between spaces X and Y. Then:

1. f is *θ-continuous* [10, 11] (see also [31]) (*strongly θ-continuous* [22]) if for each $x \in X$ and each open set V containing $f(x)$ there is an open set U containing x such that $f(\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$ ($f(\mathrm{Cl}(U)) \subset V$).
2. f is *faintly continuous* if for each $x \in X$ and each θ-open set V containing $f(x)$ there is an open set U containing x such that $f(U) \subset V$, or, equivalently, if the preimage $f^{\leftarrow}(V)$ is open in X for each V θ-open in Y [23].
3. f is *almost continuous* if the preimage of any regular open set $V \subset Y$ is open in X.
4. f is *weakly continuous* if for each $x \in X$ and each neighbourhood V of $f(x)$ there is a neighbourhood U of x such that $f(U) \subset \mathrm{Cl}(V)$.

Relations among these mappings are given in the following diagram:

$$\begin{array}{ccccccccccc} \text{strong } \theta\text{-cont.} & \Rightarrow & \text{cont.} & \Rightarrow & \text{almost cont.} & \Rightarrow & \theta\text{-cont.} & \Rightarrow & \text{weak cont.} & \Rightarrow & \text{faint cont.} \\ & & & & & & \Uparrow & & & & \\ & & & & & & \alpha\text{-cont.} & & & & \end{array}$$

Diagram 4

In almost regular spaces the last four classes of mappings in the first line of the above diagram coincide, and in regular spaces all the six classes coincide.

Theorem 3.13. *A θ-continuous image $Y = f(X)$ of a θ-Menger space X is also θ-Menger.*

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of θ-open covers of Y. Since f is θ-continuous, $\mathcal{U}_n := \{f^{\leftarrow}(V) : V \in \mathcal{V}_n\}$ is a θ-open cover of X for each $n \in \mathbb{N}$. Use the fact X is θ-Menger and find a sequence $(\mathcal{H}_n : n \in \mathbb{N})$ of finite sets such that for each n, $\mathcal{H}_n \subset \mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup\{H : H \in \mathcal{H}_n\}$. For each $H \in \mathcal{H}_n$ there is a set $V_H \in \mathcal{V}_n$ such that $H = f^{\leftarrow}(V_H)$. Set $\mathcal{W}_n = \{V_H : H \in \mathcal{H}_n\}$. We get the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ of finite sets with $\mathcal{W}_n \subset \mathcal{V}_n$, $n \in \mathbb{N}$, such that

$$Y = f(X) = f\left(\bigcup_{n\in\mathbb{N}} \bigcup \{H : H \in \mathcal{H}_n\}\right) = \bigcup_{n\in\mathbb{N}} \bigcup \{V_H : V_H \in \mathcal{W}_n\}.$$

Therefore, Y is θ-Menger. □

Since any continuous mapping is θ-continuous we get

Corollary 3.14. *A continuous image of a θ-Menger space is also θ-Menger.*

The next theorem gives an interesting characterization of the θ-Menger property and its connection with the Menger property.

Theorem 3.15. *For a space (X, τ) the following are equivalent:*

(1) *(X, τ) is θ-Menger;*

(2) *(X, τ_θ) is Menger;*

(3) *(X, τ) is a faintly continuous open image of a Menger space Y.*

Proof. (1) $\Leftrightarrow$ (2) follows from the definitions of Menger and θ-Menger spaces.

(1) $\Rightarrow$ (3) Since $\tau_\theta \subset \tau$, the mapping $1_X : (X, \tau_\theta) \to (X, \tau)$ is open and faintly continuous, and (X, τ_θ) is a Menger space.

(3) $\Rightarrow$ (1) Let $f : (Y, \sigma) \to (X, \tau))$ be a faintly continuous mapping from a Menger space Y onto a space X. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of θ-open covers of X. For each n, set $\mathcal{W}_n = f^{\leftarrow}(\mathcal{U}_n)$. Then $(\mathcal{W}_n : n \in \mathbb{N})$ is a sequence of open covers of Y. Since Y is Menger, there is a sequence $(\mathcal{H}_n : n \in \mathbb{N})$ of finite sets such that for each n, $\mathcal{H}_n \subset \mathcal{W}_n$ and $Y = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{H}_n$. For each n end each $H \in \mathcal{H}_n$ pick $U_H \in \mathcal{U}_n$ with $f^{\leftarrow}(U_H) = H$ and put $\mathcal{V}_n = \{U_H : H \in \mathcal{H}_n\}$, a finite subset of $\mathcal{U}_n$. Clearly, then $X = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n$ which shows that X is θ-Menger. $\square$

Theorem 3.16. *The strongly θ-continuous image of a θ-Menger space is a Menger space.*

Proof. Let $f : X \to Y$ be a strongly θ-continuous mapping from a θ-Menger space X onto a space Y. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. Take some $x \in X$. For each $n \in \mathbb{N}$, $f(x) \in V$ for some $V \in \mathcal{V}_n$. Since f is strongly θ-continuous, there is an open set $U_{x,n} \subset X$ such that $f(\mathrm{Cl}(U_{x,n})) \subset V$. Then $x \in \mathrm{Cl}(U_{x,n}) \subset f^{\leftarrow}(V)$, i.e. $f^{\leftarrow}(V)$ is θ-open. Therefore, $\mathcal{U}_n := \{f^{\leftarrow}(V) : V \in \mathcal{V}_n\}$ is a θ-open cover of X for each $n \in \mathbb{N}$. Use the fact X is θ-Menger and find a sequence $(\mathcal{H}_n : n \in \mathbb{N})$ of finite sets such that for each n, $\mathcal{H}_n \subset \mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup \{H : H \in \mathcal{H}_n\}$. For each $H \in \mathcal{H}_n$ there is a set $V_H \in \mathcal{V}_n$ such that $H = f^{\leftarrow}(V_H)$. If we set $\mathcal{W}_n = \{V_H : H \in \mathcal{H}_n\}$, $n \in \mathbb{N}$, we get the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ of finite sets with $\mathcal{W}_n \subset \mathcal{V}_n$, $n \in \mathbb{N}$, such that

$$Y = f(X) = f\left(\bigcup_{n\in\mathbb{N}} \bigcup \{H : H \in \mathcal{H}_n\}\right) = \bigcup_{n\in\mathbb{N}} \bigcup \{V_H : V_H \in \mathcal{W}_n\}.$$

Therefore, Y is Menger. $\square$

Remark 3.17. In the proof of (3) $\Rightarrow$ (1) in Theorem 3.15 we actually proved: If $f : X \to Y$ is a faintly continuous surjection and X is a Menger space, then Y is θ-Menger.

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