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Symmetric Functions of Binary Products of Fibonacci and Orthogonal Polynomials

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Abstract. In this paper, we introduce a new operator in order to derive some new symmetric properties of Fibonacci numbers and Chebychev polynomials of first and second kind. By making use of the new operator defined in this paper, we give some new generating functions for Fibonacci numbers and Chebychev polynomials of first and second kinds.

1. Introduction and preliminaries

In mathematics, orthogonal polynomials consist of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product. The most widely used orthogonal polynomials are the classical orthogonal polynomials, consisting of the Hermite polynomials, the Laguerre polynomials, the Jacobi polynomials together with their special cases the Gegenbauer polynomials, the Chebyshev polynomials, and the Legendre polynomials, *cf*. [8], [9].

Recent works including the symmetric properties of some known special polynomials, e.g., Bernoulli polynomials, Euler polynomials, Genocchi polynomials and others, have been extensively investigated. For details, see, [1], [2], [11], [12], [13], [14], [15], [19], [20], [21].

In this contribution, we shall define a new useful operator denoted by $L_{e_1e_2}^{-k}$ for which we can formulate, extend and prove new results based on our previous ones, see [5], [6], [7]. In order to determine generating functions for Fibonacci numbers and Tchebychev polynomials of the first and second kind, we combine between our indicated past techniques and these presented polishing approaches.

We shall handle functions on different sets of indeterminates (called alphabets, though we shall mostly use commutative indeterminates for the moment). A symmetric function of an alphabet A is a function of the letters which is invariant under permutation of the letters of A. Taking an extra indeterminate z, one has two fundamental series

$$\lambda_z(A) = \Pi_{a\in A}(1+za),\ \sigma_z(A) = \frac{1}{\Pi_{a\in A}(1-za)}$$

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the expansion of which gives the elementary symmetric functions $\Lambda_n(A)$ and the complete functions $S_n(A)$:

$$\lambda_z(A) = \sum_{n=0}^{\infty} \Lambda_n(A)z^n, \ \sigma_z(A) = \sum_{n=0}^{\infty} S_n(A)z^n$$

Let us now start at the following definition.

Definition 1.1. *Let A and B be any two alphabets, then we give $S_n(A - B)$ by the following form:*

$$\frac{\Pi_{b\in B}(1 - zb)}{\Pi_{a\in A}(1 - za)} = \sum_{n=0}^{\infty} S_n(A - B)z^n = \sigma_z(A - B) \tag{1.1}$$

with the condition $S_n(A - B) = 0$ for $n < 0$ (see [3]).

Corollary 1.2. *Taking $A = 0$ in (1.1) gives*

$$\Pi_{b\in B}(1 - zb) = \sum_{n=0}^{\infty} S_n(-B)z^n = \lambda_z(-B). \tag{1.2}$$

Further, in the case $A = 0$ or $B = 0$, we have

$$\sum_{n=0}^{\infty} S_n(A - B)z^n = \sigma_z(A) \times \lambda_z(-B). \tag{1.3}$$

Thus,

$$S_n(A - B) = \sum_{k=0}^{n} S_{n-k}(A)S_k(-B) \qquad \text{(see [4])}. \tag{1.4}$$

Since the summation is, indeed, limited to a finite number of nonzero terms, we have

$$S_n(x - B) = x^n S_0(-B) + x^{n-1}S_1(-B) + x^{n-2}S_2(-B) + \cdots ,$$

where $S_k(-B)$ are the coefficients of polynomials $S_n(x - B)$ for $0 < k < n$.

Notice that $S_k(-B) = 0$ for $k > n$. Let $B = \{b, b, ..., b\}$ be an alphabet of cardinality n, we have

$$S_n(x - b - b - b - ... - b) = (x - b)^n. \text{ (see [8])}$$

Choosing $B = \{1, 1, 1, ..., 1\}$ yields to two binomial coefficients as

$$S_k(-n) = (-1)^k \binom{n}{k} \text{ and } S_k(n) = \binom{n + k - 1}{k}. \tag{1.5}$$

By combining (1.4) with (1.5), we obtain

$$S_k(A - kx) = S_k(A) - \binom{k}{1} x^1 S_{k-1}(A) + \cdots \pm \binom{k}{k} x^k.$$

Definition 1.3. *Let g be any function on $\mathbb{R}^n$, then we consider the divided difference operator as the following form*

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1, \cdots, x_i, x_{i+1}, \cdots x_n) - g^{\sigma}(x_1, \cdots, x_i, x_{i+1}, \cdots x_n)}{x_i - x_{i+1}},$$

where g^{σ} is given by

$$g^{\sigma}(x_1, \cdots, x_i, x_{i+1}, \cdots x_n) = g(x_1, \cdots x_{i-1}, x_{i+1}, x_i, x_{i+2} \cdots x_n)$$

(see [23]).

Definition 1.4. *The symmetrizing operator $L^{-k}_{e_1 e_2}$ is defined by [7]*

$$L^{-k}_{e_1 e_2} f = \frac{e_2^k f(e_1) - e_1^k f(e_2)}{(e_1 e_2)^k (e_1 - e_2)} \quad (k \in \mathbb{N}).$$

2. The Main Results

In this part, we are now in a position to provide Lemma 1 for Theorem 1. Also we derive the new generating functions of the products of some known polynomials.

Lemma 2.1. *Let $E = \{e_1, e_2\}$, we define the operator $L^{-k}_{e_1e_2}$ as follows:*

$$L^{-k}_{e_1e_2} f\,(e_1) = \frac{-S_{k-1}(E)}{e_1^k e_2^k} f(e_1) + \frac{e_1^k}{e_1^k e_2^k} \partial_{e_1e_2} f(e_1)$$

for all $k \in \mathbb{N}$ (see [7]).

Theorem 2.2. *Let E and A be two alphabets, respectively, $\{e_1, e_2\}$ and $\{a_1, a_2, ...\}$, then we have*

$$\frac{\sum\limits_{n=0}^{\infty} S_n\,(-A)\,\partial_{e_1e_2}(e_1^{k+n})z^n}{\lambda_{e_1z}(-A) \times \lambda_{e_2z}(-A)} = \sum_{n=0}^{k-1} S_n\,(A)\, e_1^n e_2^n \partial_{e_1e_2}(e_1^{k-n}) z^n - e_1^k e_2^k z^{k+1} \sum_{n=0}^{\infty} S_{n+k+1}\,(A)\,\partial_{e_1e_2}(e_1^{n+1}) z^n, \tag{2.1}$$

for all $k \in \mathbb{N}$.

Proof. By applying the operator $L^{-k}_{e_1e_2}$ to the series $f(e_1) = \left(\sum\limits_{n=0}^{\infty} S_n\,(-A)\, e_1^n z^n\right)^{-1} = (\lambda_{e_1z}(-A))^{-1}$, we have

$$\begin{aligned}
L^{-k}_{e_1e_2} f(e_1) &= L^{-k}_{e_1e_2}\left(\frac{1}{\sum\limits_{n=0}^{\infty} S_n\,(-A)\, e_1^n z^n}\right) \\
&= \frac{\frac{e_2^k}{\sum_{n=0}^{\infty} S_n(-A) e_1^n z^n} - \frac{e_1^k}{\sum_{n=0}^{\infty} S_n(-A) e_2^n z^n}}{e_1^k e_2^k\,(e_1 - e_2)} \\
&= \frac{\sum\limits_{n=0}^{\infty} S_n\,(-A)\, e_2^{n+k} z^n - \sum\limits_{n=0}^{\infty} S_n\,(-A)\, e_1^{n+k} z^n}{e_1^k e_2^k\,(e_1 - e_2)\,(\lambda_{e_1z}(-A) \times \lambda_{e_2z}(-A))} \\
&= \frac{-1}{e_1^k e_2^k}\left(\frac{\sum\limits_{n=0}^{\infty} S_n\,(-A)\,\partial_{e_1e_2}(e_1^{k+n}) z^n}{\lambda_{e_1z}(-A) \times \lambda_{e_2z}(-A)}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
L^{-k}_{e_1e_2} f(e_1) &= L^{-k}_{e_1e_2}\left(\sum_{n=0}^{\infty} S_n\,(A)\, e_1^n z^n\right) \\
&= \frac{e_2^k \sum\limits_{n=0}^{\infty} S_n\,(A)\, e_1^n z^n - e_1^k \sum\limits_{n=0}^{\infty} S_n\,(A)\, e_2^n z^n}{e_1^k e_2^k\,(e_1 - e_2)} \\
&= \frac{1}{e_1^k e_2^k}\left(\sum_{n=0}^{\infty} S_n\,(A)\, \frac{e_2^k e_1^n - e_1^k e_2^n}{e_1 - e_2} z^n\right)
\end{aligned}$$

$$= \frac{1}{e_1^k e_2^k}\left(\sum_{n=0}^{k-1} S_n(A)\frac{e_2^k e_1^n - e_1^k e_2^n}{e_1 - e_2} z^n + \sum_{n=k+1}^{\infty} S_n(A)\frac{e_2^k e_1^n - e_1^k e_2^n}{e_1 - e_2} z^n\right)$$

$$= \frac{-1}{e_1^k e_2^k}\left(\sum_{n=0}^{k-1} S_n(A) e_1^n e_2^n \partial_{e_1e_2}(e_1^{k-n}) z^n - e_1^k e_2^k z^{k+1}\sum_{n=0}^{\infty} S_{n+k+1}(A)\,\partial_{e_1e_2}(e_1^{n+1}) z^n\right).$$

Thus, this completes the proof. □

3. On The Generating Functions

In this part, we now derive the new generating functions of the products of some known polynomials. Indeed, we consider Theorem 1 in order to derive Fibonacci numbers and Tchebychev polynomials of second kind if $k = 1$.

Theorem 3.1. *Let E and A be two alphabets, respectively, $\{e_1, e_2\}$ and $\{a_1, a_2\}$, then we have*

$$\sum_{n=0}^{\infty} S_{n+2}(a_1 + a_2)\partial_{e_1e_2}(e_1^{n+1}) z^n = \frac{e_1e_2a_1^2a_2^2z^2 - a_1a_2(e_1 + e_2)(a_1 + a_2)z + (a_1 + a_2)^2 - a_1a_2}{\lambda_{e_1z}(-A) \times \lambda_{e_2z}(-A)}. \tag{3.1}$$

In the case $E = \{1, y\}$ and $A = \{1, x\}$, with substituting $e_1 = a_1 = 1$, $e_2 = x$ and $a_2 = y$ in (3.1), we have

$$\sum_{n=0}^{\infty} S_{n+2}(1 + x)S_n(1 + y)z^n = \frac{xy^2z^2 - x(1 + x)(1 + y)z + (1 + x)^2 - x}{(1 - z)(1 - xz)(1 - yz)(1 - xyz)} \tag{3.2}$$

representing a new generating function of the products $S_{n+2}(1 + x)S_n(1 + y)$.

Corollary 3.2. *For $n \in \mathbb{N}$, we have*

$$S_{n+2}(1 + x) = S_n(1 + x) + x(x + 1)S_n(x).$$

Proof. From (3.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} S_{n+2}(1 + x)S_n(1 + y)z^n &= \frac{xy^2z^2 - x(1 + x)(1 + y)z + (1 + x)^2 - x}{(1 - z)(1 - zx)(1 - zy)(1 - zxy)}\\ &= \frac{1 - xyz^2}{(1 - z)(1 - zx)(1 - zy)(1 - zxy)} + \frac{x(1 + x)}{(1 - zx)(1 - zxy)}.\end{aligned}$$

Since

$$\sum_{n=0}^{\infty} S_n(1 + x)S_n(1 + y)z^n = \frac{1 - xyz^2}{(1 - z)(1 - zx)(1 - zy)(1 - zxy)} \quad \text{(see [7])}$$

we have

$$\sum_{n=0}^{\infty} S_{n+2}(1 + x)S_n(1 + y)z^n = \sum_{n=0}^{\infty} S_n(1 + x)S_n(1 + y)z^n + \sum_{n=0}^{\infty} x(1 + x)S_n(x)S_n(1 + y)z^n.$$

By comparing the coefficients of z^n on the both sides of the above, we have

$$S_{n+2}(1 + x) = S_n(1 + x) + x(x + 1)S_n(x)$$

as desired. □

For the case $E = \{e_1, -e_2\}$ and $A = \{a_1, -a_2\}$ with replacing e_2 by $-e_2$, a_2 by $-a_2$ in (3.1), we have

$$\sum_{n=0}^{\infty} S_{n+2}(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n = \frac{-e_1e_2a_1^2a_2^2z^2 + a_1a_2(e_1 - e_2)(a_1 - a_2)z + (a_1 - a_2)^2 + a_1a_2}{(1 - a_1e_1z)(1 + a_2e_1z)(1 + a_1e_2z)(1 - a_2e_2z)}. \quad (3.3)$$

By the expression of (3.3), we have some corollaries as follows:
Assuming

$$\begin{cases} a_1 - a_2 = 1, \\ a_1a_2 = 1, \end{cases} \text{and} \begin{cases} e_1 - e_2 = 1, \\ e_1e_2 = 1, \end{cases}$$

in (3.3) gives

$$\begin{aligned} \sum_{n=0}^{\infty} S_{n+2}(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n &= \frac{2 + z - z^2}{1 - z - 4z^2 - z^3 + z^4} \\ &= \sum_{n=0}^{\infty} F_{n+2}F_nz^n \end{aligned}$$

representing a new generating function for the binary product of Fibonacci numbers F_n. From this application, we can state the following corollary.

Corollary 3.3. *The following identity holds true:*

$$F_{n+2}F_n = S_{n+2}(a_1 + [-a_2])S_n(e_1 + [-e_2]).$$

Also we give a new generating function for the binary product of Fibonacci numbers by the following theorem.

Theorem 3.4. *For $n \in \mathbb{N}$, the generating function of the binary product of Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} F_{n+1}F_nz^n = \frac{1}{1 - 2z - 2z^2 + z^3}.$$

Proof. We have

$$\begin{aligned} \sum_{n=0}^{\infty} F_{n+2}F_nz^n &= \sum_{n=0}^{\infty}(F_{n+1} + F_n)F_nz^n \\ &= \sum_{n=0}^{\infty} F_{n+1}F_nz^n + \sum_{n=0}^{\infty} F_n^2z^n. \end{aligned}$$

Since

$$\sum_{n=0}^{\infty} F_n^2z^n = \frac{1 - z^2}{1 - z - 4z^2 - z^3 + z^4} \quad \text{(see [8])}$$

we have

$$\sum_{n=0}^{\infty} F_{n+2}F_nz^n = \frac{1 + z}{1 - z - 4z^2 - z^3 + z^4} + \frac{1 - z^2}{1 - z - 4z^2 - z^3 + z^4},$$

therefore

$$\sum_{n=0}^{\infty} F_{n+1}F_nz^n = \frac{1}{1 - 2z - 2z^2 + z^3}.$$

□

Upon setting

$$\begin{cases} a_1 - a_2 = 2, \\ a_1 a_2 = 1, \end{cases} \text{ and } \begin{cases} e_1 - e_2 = 2, \\ e_1 e_2 = 1, \end{cases}$$

in (3.3) yields

$$\begin{aligned} \sum_{n=0}^{\infty} S_{n+1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n &= \frac{5z + 4z^2 - z^3}{1 - 4z - 10z^2 - 4z^3 + z^4} \\ &= \sum_{n=0}^{\infty} P_{n+2} P_n z^n \end{aligned}$$

that means a new generating function of the binary product of Pell numbers F_n. By comparing the coefficients z^n on both sides of (3.5), we have the following corollary.

Corollary 3.5. *The following identity holds true:*

$$P_{n+2} P_n = S_{n+1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2]).$$

The generating function of the binary product of Pell numbers is given by means of the following theorem.

Theorem 3.6. *For $n \in \mathbb{N}$, the generating function of the product of Pell numbers is given by*

$$\sum_{n=0}^{\infty} P_{n+1} P_n z^n = \frac{2z + 2z^2}{1 - 4z - 10z^2 - 4z^3 + z^4}.$$

Proof. We have

$$\begin{aligned} \sum_{n=0}^{\infty} P_{n+2} P_n z^n &= \sum_{n=0}^{\infty} (2P_{n+1} + P_n) P_n z^n \\ &= 2 \sum_{n=0}^{\infty} P_{n+1} P_n z^n + \sum_{n=0}^{\infty} P_n^2 z^n. \end{aligned}$$

Since

$$\sum_{n=0}^{\infty} P_n^2 z^n = \frac{z - z^3}{1 - 4z - 10z^2 - 4z^3 + z^4} \quad \text{(see [6])}$$

we have

$$\sum_{n=0}^{\infty} P_{n+1} P_n z^n = \frac{1}{2} \left(\sum_{n=0}^{\infty} P_{n+2} P_n z^n - \sum_{n=0}^{\infty} P_n^2 z^n \right),$$

therefore

$$\sum_{n=0}^{\infty} P_{n+1} P_n z^n = \frac{2z + 2z^2}{1 - 4z - 10z^2 - 4z^3 + z^4}.$$

□

By taking

$$e_1 - e_2 = 1,\ e_1e_2 = 1,\ 4a_1a_2 = -1,$$

and changing

$$(a_1 - a_2) \text{ to } 2(a_1 - a_2)$$

in (3.3), we derive a new generating function involving the product of Fibonacci numbers with Tchebychev polynomial of second kind

$$\begin{aligned}\sum_{n=0}^{\infty} S_{n+2}(2a_1 + [-2a_2])S_n(e_1 + [-e_2])z^n &= \frac{-1 + 4(a_1 - a_2)^2 - 2(a_1 - a_2)z - z^2}{P_{UF}} \\ &= \sum_{n=0}^{\infty} U_{n+2}(a_1 - a_2)F_n z^n\end{aligned}$$

with

$$P_{UF} = 1 - 2(a_1 - a_2)z + (3 - 4(a_1 - a_2)^2)z^2 + 2(a_1 - a_2)z^3 + z^4.$$

We deduce the following both corollary and theorem.

Corollary 3.7. *The following identity holds true:*

$$S_{n+2}(2a_1 + [-2a_2])S_n(e_1 + [-e_2]) = U_{n+2}(a_1 - a_2)F_n.$$

Theorem 3.8. *We have the following a new generating function of the product of Fibonacci numbers and Tchebychev polynomials of second kind as*

$$\sum_{n=0}^{\infty} U_{n+1}(a_1 - a_2)F_n z^n = \frac{2(a_1 - a_2) - z}{P_{UF}}.$$

Proof. For the proof of this theorem, the following generating function is first considered:

$$\begin{aligned}\sum_{n=0}^{\infty} U_{n+2}(a_1 - a_2)F_n z^n &= \frac{-1 + 4(a_1 - a_2)^2 - 2(a_1 - a_2)z - z^2}{P_{UF}} \\ &= -\frac{1 + z^2}{P_{UF}} - \frac{-4(a_1 - a_2)^2 + 2(a_1 - a_2)z}{P_{UF}}.\end{aligned}$$

Foata [14] derived the following generating function

$$\sum_{n=0}^{\infty} U_n(a_1 - a_2)F_n z^n = \frac{1 + z^2}{P_{UF}}.$$

By making use of the generating function given by Foata, we see the following series manipulation:

$$\begin{aligned}\sum_{n=0}^{\infty} U_{n+2}(a_1 - a_2)F_n z^n &= -\sum_{n=0}^{\infty} U_n(a_1 - a_2)F_n z^n - 2(a_1 - a_2)\frac{-2(a_1 - a_2) + z}{P_{UF}} \\ \sum_{n=0}^{\infty} U_{n+2}(a_1 - a_2)F_n z^n + \sum_{n=0}^{\infty} U_n(a_1 - a_2)F_n z^n &= -2(a_1 - a_2)\frac{-2(a_1 - a_2) + z}{P_{UF}} \\ \sum_{j=0}^{\infty}[U_{n+2}(a_1 - a_2) + U_n(a_1 - a_2)]F_n z^n &= -2(a_1 - a_2)\frac{-2(a_1 - a_2) + z}{P_{UF}} \\ \sum_{j=0}^{\infty} 2(a_1 - a_2)U_{n+1}(a_1 - a_2)F_n z^n &= -2(a_1 - a_2)\frac{-2(a_1 - a_2) + z}{P_{UF}}.\end{aligned}$$

Thus it is proved by the following generating function.

$$\sum_{n=0}^{\infty} U_{n+1}(a_1 - a_2)F_n z^n = \frac{2(a_1 - a_2) - z}{P_{UF}}.$$

□

Choosing a_i and e_i for $i = 1, 2$ such that

$$e_1 e_2 = a_1 a_2 = -1$$

and changing

$(a_1 - a_2)$ to $2(a_1 - a_2)$ and $(e_1 - e_2)$ to $2(e_1 - e_2)$

in (3.3), we get a new generating function involving the square of Tchebychev polynomials of second kind

$$\begin{aligned}\sum_{n=0}^{\infty} S_{n+2}(2a_1 + [-2a_2])S_n(2e_1 + [-2e_2])z^n &= \frac{z^2 - 1 + 4(a_1 - a_2)^2 - 4(a_1 - a_2)(e_1 - e_2)z}{P_{UU}} \\ &= \sum_{n=0}^{\infty} U_{n+2}(a_1 - a_2)U_n(e_1 - e_2)z^n\end{aligned}$$

with

$$P_{UU} = 1 - 4(e_1 - e_2)(a_1 - a_2)z + (4(a_1 - a_2)^2 + 4(e_1 - e_2)^2 - 2)z^2 - 4(e_1 - e_2)(a_1 - a_2)z^3 + z^4.$$

Thus we get the following both corollary and theorem.

Corollary 3.9. *The following identity holds true:*

$$S_{n+2}(2a_1 + [-2a_2])S_n(2e_1 + [-2e_2]) = U_{n+2}(a_1 - a_2)U_n(e_1 - e_2).$$

Theorem 3.10. *We have a new generating function of the product of Tchebychev polynomials of second kind as*

$$\sum_{n=0}^{\infty} U_{n+1}(a_1 - a_2)U_n(e_1 - e_2)z^n = \frac{2(a_1 - a_2) - 2(e_1 - e_2)z}{P_{UU}}.$$

Proof. We have,

$$\begin{aligned}\sum_{n=0}^{\infty} U_{n+2}(a_1 - a_2)U_n(e_1 - e_2)z^n &= \sum_{n=0}^{\infty}[2(a_1 - a_2)U_{n+1}(a_1 - a_2) - U_n(a_1 - a_2)]U_n(e_1 - e_2)z^n \\ &= 2(a_1 - a_2)\sum_{n=0}^{\infty} U_{n+1}(a_1 - a_2)U_n(e_1 - e_2)z^n - \sum_{n=0}^{\infty} U_n(a_1 - a_2)U_n(e_1 - e_2)z^n.\end{aligned}$$

Since

$$\sum_{n=0}^{\infty} U_n(a_1 - a_2)U_n(e_1 - e_2)z^n = \frac{1 - z^2}{P_{UU}} \quad \text{(see [8, 9])}$$

so

$$\sum_{n=0}^{\infty} U_{n+2}(a_1 - a_2)U_n(e_1 - e_2)z^n = 2(a_1 - a_2)\frac{2(a_1 - a_2) - 2(e_1 - e_2)z}{P_{UU}} - \frac{1 - z^2}{P_{UU}},$$

therefore

$$\sum_{n=0}^{\infty} U_{n+1}(a_1 - a_2)U_n(e_1 - e_2)z^n = \frac{2(a_1 - a_2) - 2(e_1 - e_2)z}{P_{UU}}.$$

□

Furthermore,

$$S_{n+1}(2a_1+[-2a_2])=\frac{(2a_1)^{n+2}-(-2a_2)^{n+2}}{2a_1+2a_2} \quad \text{(see [10])}$$

we have a mixed generating function including the product of Tchebychev polynomials of the second and first kinds

$$\sum_{n=0}^{\infty} T_{n+2}(a_1-a_2)U_n(e_1-e_2)z^n=\frac{-1+2(a_1-a_2)^2-2(a_1-a_2)(e_1-e_2)z+z^2}{P_{TU}},$$

with

$$P_{TU}=1-4(e_1-e_2)(a_1-a_2)z+(4(a_1-a_2)^2+4(e_1-e_2)^2-2)z^2-4(e_1-e_2)(a_1-a_2)z^3+z^4.$$

Thus we conclude with the following theorem.

Theorem 3.11. *We have a new generating function including Tchebychev polynomials of the second and first kinds*

$$\sum_{n=0}^{\infty} T_{n+1}(a_1-a_2)U_n(e_1-e_2)z^n=\frac{(a_1-a_2)-2(e_1-e_2)z+(a_1-a_2)z^2}{P_{TU}}.$$

Proof. we see that

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n+2}(a_1-a_2)U_n(e_1-e_2)z^n &= \sum_{n=0}^{\infty}\left[2(a_1-a_2)T_{n+1}(a_1-a_2)-T_n(a_1-a_2)\right]U_n(e_1-e_2)z^n\\ &= 2(a_1-a_2)\sum_{n=0}^{\infty} T_{n+1}(a_1-a_2)U_n(e_1-e_2)z^n-\sum_{n=0}^{\infty} T_n(a_1-a_2)U_n(e_1-e_2)z^n.\end{aligned}$$

On the other hand, we know that

$$\sum_{n=0}^{\infty} T_n(a_1-a_2)U_n(e_1-e_2)z^n=\frac{1-2(a_1-a_2)(e_1-e_2)z+(2(a_1-a_2)^2-1)z^2}{P_{TU}} \quad \text{(see [8, 9])}$$

from which it follows

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n+2}(a_1-a_2)U_n(e_1-e_2)z^n &= \frac{2(a_1-a_2)\left[(a_1-a_2)(1+z^2)-2(e_1-e_2)z\right]}{P_{TU}}-\\ &\quad \frac{1-2(a_1-a_2)(e_1-e_2)z+(2(a_1-a_2)^2-1)z^2}{P_{TU}},\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} T_{n+1}(a_1-a_2)U_n(e_1-e_2)z^n=\frac{(a_1-a_2)-2(e_1-e_2)z+(a_1-a_2)z^2}{P_{TU}}.$$

□

4. Conclusion

In this paper, we have derived Theorem 1 by making use of symmetrizing operator given by Definition 3. By making use of Theorem 1, we have obtained Theorem 2 which is led to a new generating function for a class of new family of complete functions. By the Theorem 2, we have written some new generating functions for the binary products of Fibonacci numbers, Pell numbers and Tchebychev polynomials of the first and second kinds.

In our forthcoming investigation, we plan to establish further results and properties associated with some generalized forms of the above-mentioned families of polynomials.

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