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New Extragradient Methods with Non-Convex Combination for Pseudomonotone Equilibrium Problems with Applications in Hilbert Spaces

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Abstract. In the literature, the most authors modify the viscosity methods or hybrid projection methods to construct the strong convergence algorithms for solving the pseudomonotone equilibrium problems. In this paper, we introduce some new extragradient methods with non-convex combination to solve the pseudomonotone equilibrium problems in Hilbert space and prove the strong convergence for the constructed algorithms. Our algorithms are very different with the existing ones in the literatures. As the application, the fixed point theorems for strict pseudo-contraction are considered. Finally, some numerical examples are given to show the effectiveness of the algorithms.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ be a bifunction with $f(x, x) = 0$ for all $x \in C$. The equilibrium problem [1–3] for f is to find $z \in C$ such that

$$f(z, y) \geq 0, \quad \forall y \in C.$$

Many problems in physics, optimization, and economics can be reduced to find the solutions of equilibrium problems. The set of all solutions of the equilibrium problem is denoted by $EP(f, C)$, i.e.,

$$EP(f, C) = \{z \in C : f(z, y) \geq 0, \forall y \in C\}.$$

The bifunction f is said to be pseudomonotone if

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0, \quad \forall x, y \in C.$$

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In 2008, Tran et al. [4] introduced an extragradient method to solve a pseudomonotone equilibrium problem in $\mathbb{R}^n$. The extragradient method is: given $x_0 \in C$, find successively y_n and x_{n+1} by

$$\begin{cases} y_n = \operatorname{argmin}_{y\in C}\{\lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ x_{n+1} = \operatorname{argmin}_{y\in C}\{\lambda_n f(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \end{cases}$$

where $\{\lambda_n\} \subset (0, 1]$ and f satisfies a Lipschitz-type condition. The authors proved that the iterative scheme $\{x_n\}$ converges to some $x^* \in EP(f, C)$.

In [5], Anh introduced the following algorithm to find a solution of equilibrium problem for pseudomonotone bifunction f which is also the fixed point of a nonexpansive mapping T in Hilbert space:

$$\begin{cases} y_n = \operatorname{argmin}_{y\in C}\{\lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ t_n = \operatorname{argmin}_{y\in C}\{\lambda_n f(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tt_n, \end{cases}$$

where $x_0 \in C$ is a fixed point, $\{\alpha_n\}, \{\lambda_n\} \subset (0, 1)$, and f satisfies a Lipschitz-type condition. The author proved the strong convergence of $\{x_n\}$ provided $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and some assumptions on $\{\alpha_n\}$ and $\{\lambda_n\}$.

In 2012, Vuong et al. [6] constructed a hybrid projection algorithm for finding the common element of fixed point set of a pseudo-contraction S and solution set of an equilibrium problem on pseudomonotone bifunction f by the following manner: $x_0 \in C$ and

$$\begin{cases} y_n = \operatorname{argmin}_{y\in C}\{\lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ z_n = \operatorname{argmin}_{y\in C}\{\lambda_n f(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ t_n = \alpha_n x_n + (1 - \alpha_n)[\beta_n z_n + (1 - \beta_n)Sz_n], \\ C_n = \{z \in C : \|t_n - z\| \le \|x_n - z\|\}, \\ D_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap D_n} x_0, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\} \subset (0, 1)$, and f satisfies a Lipschitz-type condition. The strong convergence of $\{x_n\}$ was proved.

In [7], Anh and Thi constructed an Armijo-type method for pseudomotone equilibrium problems in Hilbert spaces. Note that the bifunction f is not required to satisfy the Lipschitz-type condition. Very recently, Dinh and Kim [8] introduced the strong and weak convergence algorithms to solve the equilibrium problem. It is especially worthy to mention that in the results of Dinh and Kim, any restriction of monotonicity on the bifunction is not required. On the recent results for pseudomotone equilibrium problems, the interested readers also may refer to [9–12].

In the literatures, most of authors modify the hybrid projection methods or viscosity methods to obtain the strong convergence of the iterative algorithms for pseudomonotone equilibrium problems; see [6, 9–18]. In [19], a non-convex combination iterative algorithm for pseudomonotone equilibrium problem and fixed point problem was introduced and the strong convergence was proved. In this paper, we construct two new extragradient methods with non-convex combination to solve the pseudomonotone equilibrium problems in Hilbert space and prove the strong convergence for the constructed algorithms. The algorithms designed in this paper are very different with the existing ones in the literatures. As the application, the fixed point theorems for strict pseudo-contraction are considered. Finally, some numerical examples are given to illustrate the constructed algorithms.

2. Preliminaries

Let H be a Hilbert space and C be a nonempty closed subset of H. Let $f : C \times C \to \mathbb{R}$ be a bifunction. f is said to be

(a1) γ-strong monotone ($\gamma > 0$) on C if for each $x, y \in C$, one has

$$f(x,y) + f(y,x) \leq -\gamma\|x-y\|^2;$$

(a2) monotone on C if for each $x, y \in C$, one has

$$f(x,y) + f(y,x) \leq 0;$$

(a3) pseudomonotone on C if for each $x, y \in C$, one has

$$f(x,y) \geq 0 \quad \text{implies} \quad f(y,x) \leq 0;$$

Obviously, we have $(a1) \Rightarrow (a2) \Rightarrow (a3)$.

The bifunction f is said to satisfy a Lipschitz-type condition [20] on C if there exist the constants $c_1 > 0$ and $c_2 > 0$ such that

$$f(x,z) \leq f(x,y) + f(y,z) + c_1\|x-y\|^2 + c_2\|y-z\|^2, \ \ \forall x, y, z \in C.$$

Let $F : C \to H$ is an L-Lipschitz continuous mapping. Then $f(x,y) = \langle F(x), y - x\rangle$ satisfies the Lipschitz-type condition with the constants $c_1 = c_2 = \frac{L}{2}$; see [4, 5, 20]. Another example is the following Cournot-Nash oligopolistic market equilibrium model:

$$f(x,y) = \langle Px + Qy + q, y - x\rangle, \ \ \forall x, y \in \mathbb{R}^n,$$

where $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ is symmetric positive semidefinite and $Q - P$ is negative semidefinite. Then f satisfies the Lipschitz-type condition with the constants $c_1 = c_2 = \frac{1}{2}\|P - Q\|$; see [4].

In this paper, assume the bifunction f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$ and f is pseudomonotone on C;

(A2) f satisfies a Lipschitz-type condition on C;

(A3) for each $x \in C$, $y \to f(x,y)$ is convex and subdifferentiable;

(A4) $f(x,y)$ is jointly weakly continuous on $C \times C$.

It is easy to prove that $EP(f,C)$ is weakly closed and convex under the conditions (A1), (A3) and (A4).

Example 2.1 Let $H = \mathbb{R}^n$ ($n \geq 2$) and $C = \{(x_1, \cdots, x_n) : x_1, \cdots, x_n \geq 1\}$. Define $f : C \times C \to \mathbb{R}$ by $f(x,y) = \sum_{i=2}^{n}(y_i - x_i)\|x\|$ for each $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in C$. Obviously, f is pseudomonotone but not monotone. We show that f satisfies a Lipschitz-type condition. In fact, for all $x, y, z \in C(z = (z_1, \cdots, z_n))$, we have

$$\begin{aligned}
f(x,z) &= \sum_{i=2}^{n}(z_i - x_i)\|x\| = \sum_{i=2}^{n}(z_i - y_i)\|x\| + \sum_{i=2}^{n}(y_i - x_i)\|x\| \\
&= \sum_{i=2}^{n}(z_i - y_i)(\|x\| - \|y\|) + \sum_{i=2}^{n}(y_i - x_i)\|x\| + \sum_{i=2}^{n}(z_i - y_i)\|y\| \\
&\le \sum_{i=2}^{n}|z_i - y_i|\|x - y\| + \sum_{i=2}^{n}(y_i - x_i)\|x\| + \sum_{i=2}^{n}(z_i - y_i)\|y\| \\
&\le (n-1)\|z - y\|\|x - y\| + \sum_{i=2}^{n}(y_i - x_i)\|x\| + \sum_{i=2}^{n}(z_i - y_i)\|y\| \\
&\le \frac{n-1}{2}(\|z - y\|^2 + \|x - y\|^2) + \sum_{i=2}^{n}(y_i - x_i)\|x\| + \sum_{i=2}^{n}(z_i - y_i)\|y\| \\
&= \frac{n-1}{2}(\|z - y\|^2 + \|x - y\|^2) + f(x,y) + f(y,z).
\end{aligned}$$

It follows that f satisfies the Lipschitz-type condition with the constants $c_1 = c_2 = \frac{n-1}{2}$ and satisfies the above conditions (A1)-(A4).

Lemma 2.1 ([4, 5, 19, 21]) *Assume that $EP(f,C) \neq \emptyset$. Let $x \in C$. Assume that $y,t \in C$ are the solutions of the following strongly convex problems:*

$$y = \operatorname{argmin}\{\frac{1}{2}\|z - x\|^2 + \lambda f(x,z) : z \in C\},$$
$$t = \operatorname{argmin}\{\frac{1}{2}\|a - x\|^2 + \lambda f(y,a) : a \in C\},$$

where $\lambda > 0$. Then, the following hold:

$$\lambda[f(x,z) - f(x,y)] \ge \langle y - x, y - z\rangle, \quad \forall z \in C,$$

and

$$\|t - w\|^2 \le \|x - w\|^2 - (1 - 2\lambda c_1)\|x - y\|^2 - (1 - 2\lambda c_2)\|t - y\|^2, \quad \forall w \in EP(f,C).$$

Lemma 2.2 *For each point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H onto C. For $x \in H$ and $z \in C$, $z = P_C x$ if and only if $\langle x - z, z - y\rangle \ge 0$, for all $y \in C$.*

The more information of EP(f,C) on the metric projection can be found in [22, Section 3].

Lemma 2.3 *Let H be a real Hilbert space. For all $x,y \in H$, the following hold:*

(b1) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$;

(b2) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$, *for all* $t \in [0,1]$.

Lemma 2.4 ([23]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad \forall n \in \mathbb{N},$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfy the conditions:

$$\lim_{n\to\infty}\gamma_n = 0, \quad \sum_{n=1}^{\infty}\gamma_n = \infty, \text{ and } \limsup_{n\to\infty}\delta_n \le 0.$$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 ([24]) *Let* $\{a_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $a_{n_i} < a_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Then there exists a nondecreasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \to \infty$ *as* $k \to \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$*:*

$$a_{m_k} \le a_{m_k+1} \text{ and } a_k \le a_{m_k+1}.$$

In fact, $m_k = \max\{j \le k : a_j < a_{j+1}\}$.

3. Main results

Let H be a real Hilbert space and $C \subset H$ be a nonempty closed convex subset. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (i)-(iv). Assume that $EP(f,C) \neq \emptyset$. Put $x^* = P_{EP(f,C)}\theta$, where θ denotes the zero element in H.

We first give the following algorithm that strongly converges to the element x^*.

Algorithm 3.1 Initialization Choose $\{\lambda_n\} \subset [\delta_1, \delta_2]$ with $0 < \delta_1 < \delta_2 < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$, $\{\gamma_n\} \subset [\gamma, 1)$ with $\gamma > 0$, $\{\alpha_n\} \subset (0,1)$. Take $x_1 \in C$. Set $n = 1$.

Step 1 Solve the strongly convex problems:

$$\begin{cases} y_n = \operatorname{argmin}\{\frac{1}{2}\|y - x_n\|^2 + \lambda_n f(x_n, y) : y \in C\}, \\ t_n = \operatorname{argmin}\{\frac{1}{2}\|t - x_n\|^2 + \lambda_n f(y_n, t) : t \in C\}. \end{cases}$$

Step 2 If $y_n = x_n$, then $x_n \in EP(f,C)$, stop; otherwise, go to Step 3.

Step 3 Generate $x_{n+1} = P_C[\alpha_n(1-\gamma_n)x_n + (1-\alpha_n)t_n]$. Set $n = n+1$ and go to Step 1.

Remark 3.1 Obviously, if $y_n = x_n$ for some $n \in \mathbb{N}$, from Lemma 2.1 it follows that $f(x_n, z) \ge 0$ for all $z \in C$ and hence $x_n \in EP(f,C)$.

In the next process of showing the convergence of Algorithm 3.1, assume that the stop criterion at Step 2 can not be satisfied for all $n \in \mathbb{N}$.

Lemma 3.1 *Assume that* $\{x_n\}$ *is bounded. If* $\|x_n - y_n\| \to 0$ *as* $n \to \infty$*, then*

$$\limsup_{n\to\infty}\langle -x^*, x_n - x^*\rangle \le 0.$$

Proof. Since $\{x_n\}$ is bounded, then we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ weakly converges to some $x \in C$ and

$$\limsup_{n\to\infty}\langle -x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle -x^*, x_{n_k} - x^*\rangle = \langle -x^*, x - x^*\rangle. \tag{3.1}$$

Since $\|x_n - y_n\| \to 0$, we have

$$y_{n_k} \rightharpoonup x, \text{ as } k \to \infty,$$

where $\rightharpoonup$ denotes the weak convergence. By Lemma 2.1 with $x = x_{n_k}$ and $y = y_{n_k}$, we have

$$f(x_{n_k}, z) - f(x_{n_k}, y_{n_k}) \ge \frac{1}{\lambda_{n_k}}\Big\langle y_{n_k} - x_{n_k}, y_{n_k} - z\Big\rangle, \ \ \forall z \in C.$$

Letting $k \to \infty$, by the hypothesis on $\{\lambda_n\}$, (i) and (iv) we get

$$f(x,z) \ge 0, \ \ \forall z \in C.$$

It follows that $x \in EP(f, C)$. Finally, by (3.1) and Lemma 2.2 we get

$$\begin{aligned}\limsup_{n\to\infty}\langle -x^*, x_n - x^*\rangle &= \lim_{k\to\infty}\langle -x^*, x_{n_k} - x^*\rangle = \langle -x^*, x - x^*\rangle \\ &= \langle \theta - P_{EP(f,C)}\theta, x - P_{EP(f,C)}\theta\rangle \\ &\le 0.\end{aligned}$$

This completes the proof. □

Theorem 3.1 *If the sequence $\{\alpha_n\}$ satisfies the following conditions:*

$$\lim_{n\to\infty}\alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty}\alpha_n = \infty,$$

then $\{x_n\}$ converges strongly to the element x^*.

Proof. We first show that $\{x_n\}$ is bounded. By Lemma 2.1 and Lemma 2.3 (b2) we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|P_C[\alpha_n(1-\gamma_n)x_n + (1-\alpha_n)t_n] - P_C x^*\|^2 \\ &\le \|\alpha_n(1-\gamma_n)x_n + (1-\alpha_n)t_n - x^*\|^2 \\ &\le \alpha_n\|(1-\gamma_n)x_n - x^*\|^2 + (1-\alpha_n)\|t_n - x^*\|^2 \\ &\le \alpha_n\Big[(1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\|x^*\|^2\Big] + (1-\alpha_n)\Big[\|x_n - x^*\|^2 \\ &\qquad - (1-2\lambda_n c_1)\|x_n - y_n\|^2 - (1-2\lambda_n c_2)\|y_n - t_n\|^2\Big] \\ &= (1-\alpha_n\gamma_n)\|x_n - x^*\|^2 + \alpha_n\gamma_n\|x^*\|^2 - (1-\alpha_n)\Big((1-2\lambda_n c_1)\|x_n - y_n\|^2 \\ &\qquad + (1-2\lambda_n c_2)\|y_n - t_n\|^2\Big) \\ &\le \max\{\|x_n - x^*\|^2, \|x^*\|^2\} \le \cdots \le \max\{\|x_1 - x^*\|^2, \|x^*\|^2\}, \quad \forall n \in \mathbb{N}.\end{aligned} \tag{3.2}$$

It follows that $\{x_n\}$ is bounded. So are $\{y_n\}$ and $\{t_n\}$.

Put

$$M_0 = \sup\Big\{\gamma_n\|x^*\|^2 + (1-2\lambda_n c_1)\|x_n - y_n\|^2 + (1-2\lambda_n c_2)\|y_n - t_n\|^2 : n \in \mathbb{N}\Big\}. \tag{3.3}$$

By (3.2) and (3.3) we get

$$\begin{aligned}&(1-2\lambda_n c_1)\|x_n - y_n\|^2 + (1-2\lambda_n c_2)\|y_n - t_n\|^2 \\ &\qquad \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_0, \quad \forall n \in \mathbb{N}.\end{aligned} \tag{3.4}$$

Set $h_n = \alpha_n x_n + (1-\alpha_n)t_n$ for each $n \in \mathbb{N}$. By Lemma 2.1 and Lemma 2.3 (b2) we have

$$\|h_n - x^*\|^2 \le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|t_n - x^*\|^2 \le \|x_n - x^*\|^2, \quad \forall n \in \mathbb{N}. \tag{3.5}$$

Note that

$$\begin{aligned}x_{n+1} &= P_C[h_n - \alpha_n\gamma_n x_n] \\ &= P_C[(1-\alpha_n\gamma_n)h_n + \alpha_n\gamma_n(1-\alpha_n)(t_n - x_n)], \quad \forall n \in \mathbb{N}.\end{aligned}$$

From Lemma 2.3 (b1) and (3.5) it follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|P_C[(1-\alpha_n\gamma_n)h_n+\alpha_n\gamma_n(1-\alpha_n)(t_n-x_n)]-P_Cx^*\|^2\\ &\le \left\|(1-\alpha_n\gamma_n)(h_n-x^*)+\alpha_n\gamma_n[(1-\alpha_n)(t_n-x_n)-x^*]\right\|^2\\ &\le (1-\alpha_n\gamma_n)\|h_n-x^*\|^2+2\alpha_n\gamma_n\langle(1-\alpha_n)(t_n-x_n)-x^*,(1-\alpha_n\gamma_n)h_n\\ &\quad +\alpha_n\gamma_n(1-\alpha_n)(t_n-x_n)-x^*\rangle\\ &= (1-\alpha_n\gamma_n)\|h_n-x^*\|^2+2\alpha_n\gamma_n(1-\alpha_n)\langle t_n-x_n,(1-\alpha_n\gamma_n)h_n\\ &\quad +\alpha_n\gamma_n(1-\alpha_n)(t_n-x_n)-x^*\rangle+2\alpha_n\gamma_n\langle -x^*,(1-\alpha_n)(t_n-x_n)\\ &\quad +(1-\alpha_n\gamma_n)x_n-x^*\rangle\\ &= (1-\alpha_n\gamma_n)\|h_n-x^*\|^2+2\alpha_n\gamma_n(1-\alpha_n)\langle t_n-x_n,(1-\alpha_n\gamma_n)h_n\\ &\quad +\alpha_n\gamma_n(1-\alpha_n)(t_n-x_n)-x^*\rangle+2\alpha_n\gamma_n(1-\alpha_n)\langle -x^*,t_n-x_n\rangle\\ &\quad +2\alpha_n\gamma_n\langle -x^*,x_n-x^*\rangle+2(\alpha_n\gamma_n)^2\langle x^*,x_n\rangle\\ &\le (1-\alpha_n\gamma_n)\|x_n-x^*\|^2+2\alpha_n\gamma_n(1-\alpha_n)\langle t_n-x_n,(1-\alpha_n\gamma_n)h_n\\ &\quad +\alpha_n\gamma_n(1-\alpha_n)(t_n-x_n)-x^*\rangle+2\alpha_n\gamma_n(1-\alpha_n)\langle -x^*,t_n-x_n\rangle\\ &\quad +2\alpha_n\gamma_n\langle -x^*,x_n-x^*\rangle+2(\alpha_n\gamma_n)^2\langle x^*,x_n\rangle,\quad \forall n\in\mathbb{N}.\end{aligned}\tag{3.6}$$

The rest of the proof will be divided into the following parts:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\|x_n - x^*\|\}$ is convergent. This together with the hypothesis on $\{\lambda_n\}$, $\{\alpha_n\}$ and (3.4) gives

$$\lim_{n\to\infty}\|x_n-y_n\| = \lim_{n\to\infty}\|y_n-t_n\| = 0$$

and hence

$$\|x_n-t_n\| \le \|x_n-y_n\|+\|y_n-t_n\| \to 0, \text{ as } n\to\infty. \tag{3.7}$$

By $\|x_n - y_n\| \to 0$ and Lemma 3.1 we have

$$\limsup_{n\to\infty}\langle -x^*, x_n-x^*\rangle \le 0. \tag{3.8}$$

The conclusion follows from the hypothesis on $\{\alpha_n\}$, $\{\gamma_n\}$, (3.6), (3.7), (3.8) and Lemma 2.4.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i}-x^*\| < \|x_{n_i+1}-x^*\|$$

for all $i \in \mathbb{N}$.

Then, by Lemma 2.5 there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\|x_{m_k}-x^*\| \le \|x_{m_k+1}-x^*\| \quad \text{and} \quad \|x_k-x^*\| \le \|x_{m_k+1}-x^*\|$$

for all $k \in \mathbb{N}$. This with (3.4) gives

$$\begin{aligned}&(1-2\lambda_{m_k}c_1)\|x_{m_k}-y_{m_k}\|^2+(1-2\lambda_{m_k}c_2)\|y_{m_k}-t_{m_k}\|^2\\ &\quad \le \|x_{m_k}-x^*\|^2-\|x_{m_k+1}-x^*\|^2+\alpha_{m_k}M_0,\ \ \forall k\in\mathbb{N}.\end{aligned}$$

By the hypothesis on $\{\lambda_n\}$ and $\{\alpha_n\}$, we have

$$\lim_{k\to\infty}\|x_{m_k}-y_{m_k}\| = \lim_{k\to\infty}\|y_{m_k}-t_{m_k}\| = 0. \tag{3.9}$$

Hence

$$\|x_{m_k}-t_{m_k}\| \le \|x_{m_k}-y_{m_k}\|+\|y_{m_k}-t_{m_k}\| \to 0, \text{ as } k\to\infty. \tag{3.10}$$

By (3.9) and Lemma 3.1 we get

$$\limsup_{k\to\infty}\langle -x^*, x_{m_k} - x^*\rangle \le 0. \tag{3.11}$$

Now from (3.6) we have

$$\begin{aligned}\|x_{m_k+1} - x^*\|^2 \le (1-\alpha_{m_k}\gamma_{m_k})\|x_{m_k} - x^*\|^2 &+ 2\alpha_{m_k}\gamma_{m_k}(1-\alpha_{m_k})\langle t_{m_k} - x_{m_k}, (1-\alpha_{m_k}\gamma_{m_k})h_{m_k} \\ &+ \alpha_{m_k}\gamma_{m_k}(1-\alpha_{m_k})(t_{m_k} - x_{m_k}) - x^*\rangle + 2\alpha_{m_k}\gamma_{m_k}(1-\alpha_{m_k})\langle -x^*, t_{m_k} - x_{m_k}\rangle \\ &+ 2\alpha_{m_k}\gamma_{m_k}\langle -x^*, x_{m_k} - x^*\rangle + 2(\alpha_{m_k}\gamma_{m_k})^2\langle x^*, x_{m_k}\rangle, \quad \forall k \in \mathbb{N}.\end{aligned} \tag{3.12}$$

Since $\|x_{m_k} - x^*\| \le \|x_{m_k+1} - x^*\|$, we have

$$\begin{aligned}&\alpha_{m_k}\gamma_{m_k}\|x_{m_k} - x^*\|^2 \\ \le\ &\|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + 2\alpha_{m_k}\gamma_{m_k}(1-\alpha_{m_k})\langle t_{m_k} - x_{m_k}, (1-\alpha_{m_k}\gamma_{m_k})h_{m_k} \\ &+ \alpha_{m_k}\gamma_{m_k}(1-\alpha_{m_k})(t_{m_k} - x_{m_k}) - x^*\rangle + 2\alpha_{m_k}\gamma_{m_k}(1-\alpha_{m_k})\langle -x^*, t_{m_k} - x_{m_k}\rangle \\ &+ 2\alpha_{m_k}\gamma_{m_k}\langle -x^*, x_{m_k} - x^*\rangle + 2(\alpha_{m_k}\gamma_{m_k})^2\langle x^*, x_{m_k}\rangle \\ \le\ &2\alpha_{m_k}\gamma_{m_k}(1-\alpha_{m_k})\langle t_{m_k} - x_{m_k}, (1-\alpha_{m_k}\gamma_{m_k})h_{m_k} + \alpha_{m_k}\gamma_{m_k}(1-\alpha_{m_k})(t_{m_k} - x_{m_k}) - x^*\rangle \\ &+ 2\alpha_{m_k}\gamma_{m_k}(1-\alpha_{m_k})\langle -x^*, t_{m_k} - x_{m_k}\rangle + 2\alpha_{m_k}\gamma_{m_k}\langle -x^*, x_{m_k} - x^*\rangle \\ &+ 2(\alpha_{m_k}\gamma_{m_k})^2\langle x^*, x_{m_k}\rangle, \quad \forall k \in \mathbb{N}.\end{aligned}$$

Since $\alpha_{m_k}\gamma_{m_k} > 0$, we get

$$\begin{aligned}\|x_{m_k} - x^*\|^2 \le\ &2(1-\alpha_{m_k})\langle t_{m_k} - x_{m_k}, (1-\alpha_{m_k}\gamma_{m_k})h_{m_k} + \alpha_{m_k}\gamma_{m_k}(1-\alpha_{m_k})(t_{m_k} - x_{m_k}) - x^*\rangle \\ &+ 2(1-\alpha_{m_k})\langle -x^*, t_{m_k} - x_{m_k}\rangle + 2\langle -x^*, x_{m_k} - x^*\rangle + 2\alpha_{m_k}\gamma_{m_k}\langle x^*, x_{m_k}\rangle, \quad \forall k \in \mathbb{N}.\end{aligned}$$

Since $\alpha_{m_k} \to 0$, it follows from (3.10), (3.11) that $\|x_{m_k} - x^*\| \to 0$ as $k \to \infty$. This with (3.10)-(3.12) implies that

$$\lim_{k\to\infty}\|x_{m_k+1} - x^*\| = 0.$$

But $\|x_k - x^*\| \le \|x_{m_k+1} - x^*\|$ for all $k \in \mathbb{N}$. So, we conclude that $x_k \to x^*$ as $k \to \infty$. The proof is complete. □

Remark 3.2 In [25], the authors introduced an iterative algorithm for solving a pseudomonotone equilibrium problem and fixed point problem and proved a strong convergence theorem for the proposed algorithm. If both the demicontractive mapping S and Lipschitz continuous and strongly monotone mapping F are the identity mapping I, i.e., $F = S = I$, then $x_{n+1} = \beta_n(1-\alpha_n)t_n + (1-\beta_n)t_n (:= (1-\alpha_n\beta_n)t_n)$ in Algorithm 1 of [25]. The manner of computing x_{n+1} at each step is similar with the one of Algorithm 3.1 that they are computed by a non-convex combination.

By a light modification on Algorithm 3.1, we give the following algorithm:

Algorithm 3.2 Initialization Choose $\{\lambda_n\} \subset [\delta_1, \delta_2]$ with $0 < \delta_1 < \delta_2 < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$, $\{\gamma_n\} \subset [\gamma, 1)$ with $\gamma > 0$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. Take $x_1 \in H$. Set $n = 1$.

Step 1 Solve the strongly convex problems:

$$\begin{cases} y_n = \mathrm{argmin}\{\frac{1}{2}\|y - P_C x_n\|^2 + \lambda_n f(P_C x_n, y) : y \in C\}, \\ t_n = \mathrm{argmin}\{\frac{1}{2}\|t - P_C x_n\|^2 + \lambda_n f(y_n, t) : t \in C\}. \end{cases}$$

Step 2 If $y_n = x_n$, then $x_n \in EP(f, C)$, stop; otherwise, go to Step 3.

Step 3 Generate $x_{n+1} = \alpha_n(1-\gamma_n)x_n + (1-\alpha_n)(\beta_n t_n + (1-\beta_n)x_n)$. Set $n = n+1$ and go to Step 1.

Remark 3.3 Obviously, if $y_n = x_n$ for some $n \in \mathbb{N}$, then $x_n \in C$ and hence $y_n = P_C x_n$. By Lemma 2.1 it follows that

$$f(x_n, z) = f(P_C x_n, z) \geq 0, \quad \forall z \in C.$$

Hence $x_n \in EP(f, C)$.

In the next process of showing the convergence of Algorithm 3.2, assume that the stop criterion at Step 2 can not be satisfied for all $n \in \mathbb{N}$.

Lemma 3.2 *Assume that $\{x_n\}$ is bounded. If $\|P_C x_n - y_n\| \to 0$ and $\|x_n - y_n\| \to 0$ as $n \to \infty$, then*

$$\limsup_{n\to\infty} \langle -x^*, x_n - x^* \rangle \leq 0.$$

Proof. Since $\{x_n\}$ is bounded, then we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ weakly converges to some $x \in H$ and

$$\limsup_{n\to\infty} \langle -x^*, x_n - x^* \rangle = \lim_{k\to\infty} \langle -x^*, x_{n_k} - x^* \rangle = \langle -x^*, x - x^* \rangle. \tag{3.13}$$

Since $\|x_n - y_n\| \to 0$ as $n \to \infty$ and $\{y_n\} \subset C$, it follows that $y_{n_k} \rightharpoonup x \in C$ as $k \to \infty$. Further by the hypothesis that $\|P_C x_n - y_n\| \to 0$ we have $P_C x_{n_k} \rightharpoonup x$ as $k \to \infty$. From Lemma 2.1 it follows that

$$f(P_C x_{n_k}, z) - f(P_C x_{n_k}, y_{n_k}) \geq \frac{1}{\lambda_{n_k}} \Big\langle y_{n_k} - P_C x_{n_k}, y_{n_k} - z \Big\rangle, \;\; \forall z \in C.$$

Letting $k \to \infty$, by the hypothesis on $\{\lambda_n\}$, (i) and (iv) we get

$$f(x, z) \geq 0, \;\; \forall z \in C.$$

It follows that $x \in EP(f, C)$. Finally, by (3.13) and Lemma 2.2 the desired result is obtained. This completes the proof. □

Theorem 3.2 *If the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:*

$$\lim_{n\to\infty} \alpha_n = 0, \; \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0,$$

then $\{x_n\}$ converges strongly to the element x^*.

Proof. We first show that $\{x_n\}$ is bounded. By Lemma 2.1 and Lemma 2.3 (b2) we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n((1-\gamma_n)x_n - x^*) + (1-\alpha_n)(\beta_n(t_n - x^*) + (1-\beta_n)(x_n - x^*))\|^2 \\
&\leq \alpha_n\|(1-\gamma_n)x_n - x^*\|^2 + (1-\alpha_n)\Big[\beta_n\|t_n - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 \\
&\quad - \beta_n(1-\beta_n)\|t_n - x_n\|^2\Big] \\
&\leq \alpha_n\Big[(1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\|x^*\|^2\Big] + (1-\alpha_n)\Big[\|x_n - x^*\|^2 \\
&\qquad - (1-2\lambda_n c_1)\beta_n\|P_C x_n - y_n\|^2 - (1-2\lambda_n c_2)\beta_n\|y_n - t_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\|t_n - x_n\|^2\Big] \\
&= (1-\alpha_n\gamma_n)\|x_n - x^*\|^2 + \alpha_n\gamma_n\|x^*\|^2 - (1-\alpha_n)\Big[(1-2\lambda_n c_1)\beta_n\|P_C x_n - y_n\|^2 \\
&\qquad + (1-2\lambda_n c_2)\beta_n\|y_n - t_n\|^2 + \beta_n(1-\beta_n)\|t_n - x_n\|^2\Big] \\
&\leq \max\{\|x_n - x^*\|^2, \|x^*\|^2\} \leq \cdots \leq \max\{\|x_1 - x^*\|^2, \|x^*\|^2\}
\end{aligned} \tag{3.14}$$

for all $n \in \mathbb{N}$. It follows that $\{x_n\}$ is bounded. So are $\{y_n\}$ and $\{t_n\}$.

By (3.14) we get

$$\begin{aligned}&(1-2\lambda_n c_1)\beta_n\|P_C x_n - y_n\|^2 + (1-2\lambda_n c_2)\beta_n\|y_n - t_n\|^2 + \beta_n(1-\beta_n)\|t_n - x_n\|^2\\&\quad\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_0,\ \forall n\in\mathbb{N},\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}M_0 = \sup\Big\{&\gamma_n\|x^*\|^2 + (1-2\lambda_n c_1)\beta_n\|P_C x_n - y_n\|^2 + (1-2\lambda_n c_2)\beta_n\|y_n - t_n\|^2\\&+\beta_n(1-\beta_n)\|t_n - x_n\|^2 : n\in\mathbb{N}\Big\}.\end{aligned}$$

Set $h_n = \alpha_n x_n + (1-\alpha_n)v_n$, where $v_n = \beta_n t_n + (1-\beta_n)x_n$, for each $n\in\mathbb{N}$. By Lemma 2.1 and Lemma 2.3 (b2) we have

$$\begin{aligned}\|h_n - x^*\|^2 &\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2\\&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\Big(\beta_n\|t_n - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2\Big)\\&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\Big(\beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2\Big)\\&= \|x_n - x^*\|^2,\quad \forall n\in\mathbb{N}.\end{aligned} \tag{3.16}$$

Note that

$$\begin{aligned}x_{n+1} &= h_n - \alpha_n\gamma_n x_n\\&= (1-\alpha_n\gamma_n)h_n + \alpha_n\gamma_n(1-\alpha_n)(v_n - x_n),\quad \forall n\in\mathbb{N}.\end{aligned}$$

From Lemma 2.3 (b1) and (3.16) it follows that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n\gamma_n)h_n + \alpha_n\gamma_n(1-\alpha_n)(v_n - x_n) - x^*\|^2\\&= \Big\|(1-\alpha_n\gamma_n)(h_n - x^*) + \alpha_n\gamma_n[(1-\alpha_n)(v_n - x_n) - x^*]\Big\|^2\\&\le (1-\alpha_n\gamma_n)\|h_n - x^*\|^2 + 2\alpha_n\gamma_n(1-\alpha_n)\langle v_n - x_n, x_{n+1} - x^*\rangle\\&\quad + 2\alpha_n\gamma_n\langle -x^*, x_{n+1} - x^*\rangle\\&\le (1-\alpha_n\gamma_n)\|x_n - x^*\|^2 + 2\alpha_n\gamma_n(1-\alpha_n)\langle v_n - x_n, x_{n+1} - x^*\rangle\\&\quad + 2\alpha_n\gamma_n\langle -x^*, x_{n+1} - x^*\rangle\\&= (1-\alpha_n\gamma_n)\|x_n - x^*\|^2 + 2\alpha_n\gamma_n(1-\alpha_n)\beta_n\langle t_n - x_n, x_{n+1} - x^*\rangle\\&\quad + 2\alpha_n\gamma_n\langle -x^*, x_{n+1} - x^*\rangle.\end{aligned} \tag{3.17}$$

The rest of the proof will be divided into the following parts:

Case 1. Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\|x_n - x^*\|\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\|x_n - x^*\|\}$ is convergent. This together with the hypothesis on $\{\lambda_n\}, \{\alpha_n\}, \{\beta_n\}$ and (3.15) gives

$$\lim_{n\to\infty}\|P_C x_n - y_n\| = \lim_{n\to\infty}\|y_n - t_n\| = \lim_{n\to\infty}\|x_n - t_n\| = 0. \tag{3.18}$$

By (3.18) that shows $\|x_n - y_n\|\to 0$ and Lemma 3.2 we have

$$\limsup_{n\to\infty}\langle -x^*, x_n - x^*\rangle \le 0. \tag{3.19}$$

Now, by (3.17)-(3.19) and Lemma 2.4 we can conclude that $\|x_n - x^*\|\to 0$ as $n\to\infty$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - x^*\| < \|x_{n_i+1} - x^*\|$$

for all $i\in\mathbb{N}$. Then, by Lemma 2.5 there exists a nondecreasing sequence $\{m_k\}\subset\mathbb{N}$ such that $m_k\to\infty$,

$$\|x_{m_k} - x^*\| \le \|x_{m_k+1} - x^*\| \quad\text{and}\quad \|x_k - x^*\| \le \|x_{m_k+1} - x^*\|$$

for all $k \in \mathbb{N}$.

Under this case the proof process is similar with Case 1 and Case 2 in the proof lines of Theorem 3.1. Hence we omit the rest proof lines. This completes the proof. □

Remark 3.4 In Algorithm 3.2, at each step x_{n+1} is not necessary in the subset C since the non-convex combination of x_n with t_n is involved. However, the sequence $\{x_n\}$ is proved to asymptotically fall into the subset C and converge to the solution of the pseudomonotone equilibrium problem. The manners of constructing the algorithms in this paper are new and different with the existing ones in the literature.

4. Applications

Let H be a Hilbert space and C be a nonempty closed and convex subset of H. Let $T : C \to C$ be a mapping. T is said to be a κ-strict pseudo-contraction [26] if there exists a constant $\kappa \in (0, 1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + \kappa||(x - Tx) - (y - Ty)||^2, \ \ \forall x, y \in C, \tag{4.1}$$

which is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq ||x - y||^2 - \frac{1-\kappa}{2}||(x - Tx) - (y - Ty)||^2, \ \ \forall x, y \in C. \tag{4.2}$$

Example 4.1 Let $H = l^2$ and $C = \{x = (x_1, x_2, \cdots, x_n, \cdots) \in l^2 : x_1 \geq 0, x_i \in \mathbb{R}, \ \forall\ i = 2, 3, \cdots\}$. Define the mapping $T : C \to C$ by $Tx = (\frac{x_1}{2}, -3x_2, -3x_3, \cdots)$ for all $x = (x_1, x_2, x_3, \cdots, x_n, \cdots) \in C$. Then T is a $\frac{1}{2}$-strict pseudo-contraction. In fact, for each $x = (x_1, x_2, \cdots, x_n, \cdots), y = (y_1, y_2 \cdots, y_n, \cdots) \in C$, we have

$$\begin{aligned}
||Tx - Ty||^2 &= ||(\frac{x_1 - y_1}{2}, 3(x_2 - y_2), \cdots, 3(x_n - y_n), \cdots)||^2 \\
&= \frac{(x_1 - y_1)^2}{4} + 9\sum_{i=2}^{\infty}(x_i - y_i)^2 \\
&\leq ||x - y||^2 + \frac{1}{2}||(I - T)x - (I - T)y||^2.
\end{aligned}$$

Hence T is a $\frac{1}{2}$-strict pseudo-contraction.

Denote the set of fixed points of T by $Fix(T)$. In [27, 28], the authors proved that $Fix(T)$ is closed and convex. Many fixed point theorems for strict pseudo-contractions have been introduced in the literature; see [29–32].

Let $f(x, y) = \langle x - Tx, y - x\rangle$ for all $x, y \in C$. It is easy to see that $EP(f, C) = Fix(T)$. If T is weakly continuous, then f satisfies the conditions (A4). Obviously, f satisfies the condition (A3). We prove that f is pseudomonotone on C. Assume that $x, y \in C$ are such that $f(x, y) = \langle x - Tx, y - x\rangle \geq 0$. By (4.2) we have

$$\begin{aligned}
f(y, x) = \langle y - Ty, x - y\rangle &= \langle y - x + x - Tx + Tx - Ty, x - y\rangle \\
&= -||x - y||^2 + \langle x - Tx, x - y\rangle + \langle Tx - Ty, x - y\rangle \\
&\leq -||x - y||^2 + \langle Tx - Ty, x - y\rangle \quad (\text{since } \langle x - Tx, y - x\rangle \geq 0) \\
&\leq -||x - y||^2 + ||x - y||^2 - \frac{1-\kappa}{2}||(x - Tx) - (y - Ty)||^2 \\
&= -\frac{1-\kappa}{2}||(x - Tx) - (y - Ty)||^2 \\
&\leq 0.
\end{aligned}$$

Hence f is pseudomonotone. It is clear that $f(x, x) = 0$ for all $x \in C$. It follows that f satisfies the condition (A1).

Now we show that f satisfies the condition (A2). Since T is $\frac{2-\kappa}{1-\kappa}$-Lipschitzian continuous [27], we have

$$\begin{aligned}
f(x,z) &= \langle x - Tx, z - x\rangle = \langle x - Tx, y - x\rangle + \langle x - Tx, z - y\rangle \\
&= f(x,y) + f(y,z) + \langle x - Tx + Ty - y, z - y\rangle \\
&\le f(x,y) + f(y,z) + (\|x-y\| + \|Ty - Tx\|)\|z-y\| \\
&\le f(x,y) + f(y,z) + \|x-y\|\|z-y\| + \frac{2-\kappa}{1-\kappa}\|x-y\|\|z-y\| \\
&\le f(x,y) + f(y,z) + \frac{1}{2}(\|x-y\|^2 + \|z-y\|^2) + \frac{2-\kappa}{2(1-\kappa)}(\|x-y\|^2 + \|z-y\|^2) \\
&= f(x,y) + f(y,z) + \frac{3-2\kappa}{2(1-\kappa)}(\|x-y\|^2 + \|z-y\|^2),\ \forall x,y,z \in C.
\end{aligned}$$

Hence f is Lipschitz-type continuous on C with the constants $c_1 = c_2 = \frac{3-2\kappa}{2(1-\kappa)}$. Therefore, f satisfies the condition (A2).

In Algorithm 3.1, at Step 1 the strongly convex programs needed to solve are of the forms

$$\begin{aligned}
y_n &= \arg\min\{\frac{1}{2}\|y - x_n\|^2 + \lambda_n f(x_n, y) : y \in C\} \\
&= \arg\min\{\frac{1}{2}\|y - x_n\|^2 + \lambda_n\langle x_n - Tx_n, y - x_n\rangle : y \in C\} \\
&= \arg\min\{\frac{1}{2}\|y - (x_n - \lambda_n(x_n - Tx_n))\|^2 : y \in C\} \\
&= P_C((1-\lambda_n)x_n + \lambda_n Tx_n) = (1-\lambda_n)x_n + \lambda_n Tx_n
\end{aligned}$$

and

$$\begin{aligned}
t_n &= \arg\min\{\frac{1}{2}\|t - x_n\|^2 + \lambda_n f(y_n, t) : t \in C\} \\
&= \arg\min\{\frac{1}{2}\|t - x_n\|^2 + \lambda_n\langle y_n - Ty_n, t - x_n\rangle + \lambda_n\langle y_n - Ty_n, x_n - y_n\rangle : t \in C\} \\
&= \arg\min\{\frac{1}{2}\|t - x_n\|^2 + \lambda_n\langle y_n - Ty_n, t - x_n\rangle : t \in C\} \\
&= \arg\min\{\frac{1}{2}\|t - (x_n - \lambda_n(y_n - Ty_n))\|^2 : t \in C\} \\
&= P_C(x_n - \lambda_n(y_n - Ty_n)).
\end{aligned}$$

By the results in Section 3, we get the following fixed point theorems:

Theorem 4.1 *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a weakly continuous κ-strict pseudo-contraction with $Fix(T) \neq \emptyset$. Generate the sequence $\{x_n\}$ by the following manner: $x_1 \in C$ and*

$$\begin{cases} y_n = (1-\lambda_n)x_n + \lambda_n Tx_n, \\ t_n = P_C(x_n - \lambda_n(y_n - Ty_n)), \\ x_{n+1} = P_C[\alpha_n(1-\gamma_n)x_n + (1-\alpha_n)t_n], \end{cases} \tag{4.3}$$

where $\{\lambda_n\} \subset [\delta_1, \delta_2]$ with $0 < \delta_1 < \delta_2 < \frac{3-2\kappa}{4(1-\kappa)}$, $\{\gamma_n\} \subset [\gamma, 1)$ with $\gamma > 0$, $\{\alpha_n\} \subset (0,1)$. If $\{\alpha_n\}$ satisfies the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty,$$

then the sequence $\{x_n\}$ generated by (4.3) strongly converges to the element $x^* = P_{Fix(T)}\theta$, where θ denotes the zero element in H.

Theorem 4.2 *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a weakly continuous κ-strict pseudo-contraction with $Fix(T) \neq \emptyset$. Generate the sequence $\{x_n\}$ by the following manner: $x_1 \in H$ and*

$$\begin{cases} y_n = (1-\lambda_n)P_C x_n + \lambda_n T P_C x_n, \\ t_n = P_C(x_n - \lambda_n(y_n - Ty_n)), \\ x_{n+1} = \alpha_n(1-\gamma_n)x_n + (1-\alpha_n)(\beta_n t_n + (1-\beta_n)x_n), \end{cases} \tag{4.4}$$

where $\{\lambda_n\} \subset [\delta_1, \delta_2]$ with $0 < \delta_1 < \delta_2 < \frac{3-2\kappa}{4(1-\kappa)}$, $\{\gamma_n\} \subset [\gamma, 1)$ with $\gamma > 0$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ and \ \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0,$$

then the sequence $\{x_n\}$ generated by (4.4) strongly converges to the element $x^* = P_{Fix(T)}\theta$, where θ denotes the zero element in H.

Remark 4.1 In [33], the authors introduced an iterative algorithm in which a non-convex combination is involved to approximate the fixed point of a strict pseudo-contraction. However, the subset C is required to be a cone in [33] . In Theorems 4.1 and 4.2, the subset C is not restricted to be a cone. From the viewpoint, Theorems 4.1 and 4.2 are the improvement of the result of [33].

5. Numerical examples

In this section, we give some examples to illustrate the algorithms in this paper. We perform the algorithms by Matlab R2008a running on a PC Desktop with Core(TM) i3CPU M550 3.20GHz with 4GB Ram.

First we give the following example to illustrate the effectiveness of Algorithm 3.1 and Algorithm 3.2.

Example 5.1 Let $H = \mathbb{R}^5$ and $C = \{(x_1, \cdots, x_5) : x_1 \geq -1, x_i \geq 1, i = 2, \cdots, 5\}$. Let $f(x,y) = \sum_{i=2}^{5}(y_i - x_i)\|x\|$ for all $x = (x_1, x_2, \cdots, x_5), y = (y_1, y_2, \cdots, y_5) \in C$. Then f is Lipschitz-type continuous with the constants $c_1 = c_2 = 2$ and satisfies the conditions (i)-(iv). Obviously, $EP(f, C) = \{(x_1, 1, \cdots, 1) : x_1 \geq -1\}$.

For given $\epsilon > 0$, if $r_n = \|x_n - y_n\| \leq \epsilon$ we call x_n an ϵ-solution of the pseudomonotone equilibrium problem. The program will stop when $r_n \leq \epsilon$. In this example, we take $\epsilon = 10^{-4}$.

Put the sequences $\alpha_n = \frac{1}{40n}, \beta_n = \frac{1}{4} + \frac{1}{4n}, \gamma_n = \frac{1}{5} + \frac{1}{5n}$ and $\lambda_n = \frac{1}{30} + \frac{1}{30n}$ for all $n \geq 1$. The Table 1 and Table 2 show that the programs stop after 28 steps and 104 steps for Algorithm 3.1 and Algorithm 3.2, respectively.

Table 1 Algorithm 3.1 with $\epsilon = 10^{-4}$ and $x_1 = (2, 3, 2, 5, 2)$

Iter (n)	x_n^1	x_n^2	x_n^3	x_n^4	x_n^5
1	2.00000	3.00000	2.00000	5.00000	2.0000
2	1.96500	2.55768	1.57518	4.52268	1.57518
3	1.95395	2.27261	1.29564	4.22656	1.29564
4	1.94797	2.04093	1.06693	3.98889	1.06694
5	1.94402	1.83651	1.00000	3.78053	1.00000
6	1.94110	1.64989	1.00000	3.59099	1.00000
7	1.93881	1.47664	1.00000	3.41546	1.00000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
26	1.92286	1.00000	1.00000	1.09982	1.00000
27	1.92246	1.00000	1.00000	1.00361	1.00000
28	1.92207	1.00000	1.00000	1.00000	1.00000
Cpu times	3.065262 s				

Table 2 Algorithm 3.2 with $\epsilon = 10^{-4}$ and $x_1 = (2,3,2,5,2)$

Iter (n)	x_n^1	x_n^2	x_n^3	x_n^4	x_n^5
1	2.00000	3.00000	2.00000	5.00000	2.0000
2	1.96500	2.55768	1.57518	4.52268	1.57518
3	1.95395	2.27261	1.29564	4.22656	1.29564
4	1.94797	2.04093	1.06693	3.98889	1.06694
5	1.94402	1.83651	0.99825	3.78053	0.99825
6	1.94111	1.64989	0.99849	3.59099	0.99849
7	1.93883	1.47663	0.99881	3.41546	0.99881
8	1.93695	1.31412	0.99903	3.25105	0.99903
9	1.93537	1.16052	0.999918	3.09589	0.99918
10	1.93397	1.01460	1.00000	2.94859	1.00000
100	1.90916	0.99995	0.99995	0.99995	0.99995
101	1.90906	0.99995	0.99995	0.99995	0.99995
102	1.90887	0.99995	0.99995	0.99995	0.99995
103	1.90887	0.99995	0.99995	0.99995	0.99995
104	1.90878	0.99995	0.99995	0.99995	0.99995
Cpu times	12.350364 s				

Example 5.2 We consider the classical Cournot-Nash model (see [34]) and use the same H, C, f, A, χ, μ, α with [7, Example 1] for the comparison. That is, let $H = \mathbb{R}^5$,

$$C = \begin{cases} x \in \mathbb{R}^5_+, \\ x_1 + x_2 + x_3 + 2x_4 + x_5 \leq 10, \\ 2x_1 + x_2 - x_3 + x_4 + 3x_5 \leq 15, \\ x_1 + x_2 + x_3 + x_4 + 0.5x_5 \geq 4, \end{cases}$$

and $f(x,y) = \langle Bx + \chi^5(y+x) + \mu - \alpha, y - x\rangle$, where

$$B = \begin{pmatrix} 0 & \chi & \chi & \chi & \chi \\ \chi & 0 & \chi & \chi & \chi \\ \chi & \chi & 0 & \chi & \chi \\ \chi & \chi & \chi & 0 & \chi \\ \chi & \chi & \chi & \chi & 0 \end{pmatrix},$$

$\chi = 3$, $\alpha = (2,2,2,2,2)^T$ and $\mu = (3,4,5,7,6)^T$. It is known that f is Lipschitz-type continuous with the constants $c_1 = c_2 = 6$ and f satisfies the conditions (A1)-(A4).

In [7, Example 1], the initial point chosen is $x_1 = (1,2,1,1,1)^T \in C$ and $\epsilon = 10^{-4}$. For the comparison, we also use the same x_1 and ϵ. We perform Algorithm 3.1 with the following cases of the sequences $\{\alpha_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$:

Case 1: $\alpha_n = \frac{1}{40\sqrt{n}}$, $\gamma_n = \frac{1}{5} + \frac{1}{2n}$ and $\lambda_n = \frac{1}{30} + \frac{1}{30n}$;

Case 2: $\alpha_n = \frac{1}{n}$, $\gamma_n = \frac{e^n}{1+e^n}$, $\lambda_n = \frac{\ln(2+\frac{1}{n})}{8}$;

Case 3: $\alpha_n = \frac{1}{40n}$, $\gamma_n = \frac{n}{1+n}$, $\lambda_n = \frac{n}{25+5n}$.

The following tables show the iterative results and cpu times.

Table 3 Computed results and cpu times for Algorithm 3.1 with Case 1

Iteration (n)	x_n^1	x_n^2	x_n^3	x_n^4	x_n^5
1	1.00000	2.00000	1.00000	1.00000	1.00000
2	0.93582	0.97076	0.93190	0.92799	0.46705
3	0.94634	0.94613	0.94229	0.93825	0.45394
4	0.94692	0.94495	0.94281	0.93869	0.45326
5	0.94695	0.94490	0.94283	0.93872	0.45320
Cpu times	0.282469 s				

Table 4 Computed results and cpu times for Algorithm 3.1 with Case 2

Iteration (n)	x_n^1	x_n^2	x_n^3	x_n^4	x_n^5
1	1.00000	2.00000	1.00000	1.00000	1.00000
2	0.86208	1.13102	0.86208	0.86208	0.56551
3	0.93861	0.95602	0.93658	0.93454	0.46851
4	0.94489	0.94404	0.94213	0.93938	0.45909
5	0.94547	0.94404	0.94242	0.93931	0.45751
Cpu times	1.355334 s				

Table 5 Computed results and cpu times for Algorithm 3.1 with Case 3

Iteration (n)	x_n^1	x_n^2	x_n^3	x_n^4	x_n^5
1	1.00000	2.00000	1.00000	1.00000	1.00000
2	0.92605	0.99371	0.92226	0.91845	0.47905
3	0.94616	0.94682	0.94204	0.93786	0.45424
4	0.94692	0.94489	0.94285	0.93870	0.45326
5	0.94689	0.94490	0.94282	0.93876	0.45326
6	0.94696	0.94497	0.94286	0.93863	0.45314
Cpu times	1.568336 s				

The Table 1 from [7, Example 1] shows that after 9 iterations the approximate solution is

$$x_9 = (0.9467, 0.9447, 0.9426, 0.9405, 0.4510).$$

From the Table 3-Table 5 we see that the result x_n is very near with the one in [7, Example 1] when the program stops.

On the other hand, for each $n \in \mathbb{N}$, let $s_n = \inf_{y \in C} f(x_n, y)$, where each x_n is generated by Algorithm 3.1. It is easy to see that for $x_n \in EP(f, C)$ if and only if $s_n = 0$ and $x_n \notin EP(f, C)$ if and only if $s_n < 0$. Comparing with $\{r_n\}$, $\{s_n\}$ is also an important data by which we can obtain direct the sense of x_n approximating the solution of the pseudomonotone equilibrium problem. For given $\delta < 0$, we will stop program if $s_n > \delta$ and call x_n a δ-approximate solution. In this example, put $\delta = -10^{-8}$. The following table gives the cpu times and total iteration steps for $\{s_n\}$ with the initial point $x_1 = (1, 2, 1, 1, 1)^T$ and different $\{\alpha_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$.

Table 6 Cpu times and total iteration steps for $\{s_n\}$ with $\delta = -10^{-8}$ and different $\{\alpha_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$

	Cpu times (s)	Total iteration steps
$\alpha_n = \frac{1}{5n}, \gamma_n = \frac{1}{5}, \lambda_n = \frac{1}{30}$	4.849255	35
$\alpha_n = \frac{1}{10n}, \gamma_n = \frac{1}{5}, \lambda_n = \frac{1}{30}$	1.979621	22
$\alpha_n = \frac{1}{15n}, \gamma_n = \frac{1}{5}, \lambda_n = \frac{1}{30}$	1.553208	16
$\alpha_n = \frac{1}{20n}, \gamma_n = \frac{1}{5}, \lambda_n = \frac{1}{30}$	0.875901	9
$\alpha_n = \frac{1}{5n}, \gamma_n = \frac{1}{5} + \frac{1}{5n}, \lambda_n = \frac{1}{30} + \frac{1}{30n}$	3.261556	37
$\alpha_n = \frac{1}{5n}, \gamma_n = \frac{1}{5} + \frac{1}{10n}, \lambda_n = \frac{1}{30} + \frac{1}{40n}$	3.056605	34
$\alpha_n = \frac{1}{5n}, \gamma_n = \frac{1}{5} + \frac{1}{15n}, \lambda_n = \frac{1}{30} + \frac{1}{45n}$	2.649645	29
$\alpha_n = \frac{1}{5n}, \gamma_n = \frac{1}{5} + \frac{1}{20n}, \lambda_n = \frac{1}{30} + \frac{1}{50n}$	2.865101	32
$\alpha_n = \frac{1}{5n}, \gamma_n = \frac{1}{5} + \frac{1}{5n}, \lambda_n = \frac{1}{30} + \frac{1}{30n}$	3.042921	32
$\alpha_n = \frac{1}{10n}, \gamma_n = \frac{1}{5} + \frac{1}{10n}, \lambda_n = \frac{1}{30} + \frac{1}{40n}$	2.365104	24
$\alpha_n = \frac{1}{15n}, \gamma_n = \frac{1}{5} + \frac{1}{15n}, \lambda_n = \frac{1}{30} + \frac{1}{45n}$	1.548552	16
$\alpha_n = \frac{1}{20n}, \gamma_n = \frac{1}{5} + \frac{1}{20n}, \lambda_n = \frac{1}{30} + \frac{1}{50n}$	1.275078	14

Since the rate of $\{s_n\}$ converging to 0 describes the convergence rate of Algorithm 3.1, from the Table 6 we can roughly see that the convergence rate of Algorithm 3.1 has the more closed relation with $\{\alpha_n\}$ rather than $\{\gamma_n\}$ and $\{\lambda_n\}$. That is, the faster $\{\alpha_n\}$ converges to 0, the better the convergence rate of Algorithm 3.1 is.

6. Conclusion

In this paper, we have proposed two new extragradient methods with non-convex combination to solve the pseudomonotone equilibrium problems in Hilbert space. Under some simple conditions on the control sequences, the strong convergence for the constructed algorithms is obtained. The algorithms introduced in this paper are very different with the present ones in the literatures. As the application, we proved some fixed point theorems for strict pseudo-contractions. The efficiency of the proposed algorithms has been illustrated by some numerical experiments.

References

[1] K. Fan, A minimax inequality and applications, Math. Stud. 63 (1972) 127–149.
[2] E. Blume, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994) 123–145.
[3] L.D. Muu, W. Oettli, Convergence of an adaptive penalty scheme for finding constrained equilibria, Nonlinear Anal. 18 (1992), 1159–1166.
[4] D.Q. Tran, M.L. Dung, V.H. Nguyen, Extragradient algorithms extended to equilibrium problem, Optimization. 57 (2008) 749–776.
[5] P.N. Anh, A hybrid extragradient method extended to fixed point problems and equilibrium problems, Optimization. 63 (2013) 271–283.
[6] P.T. Vuong, J.J. Strodiot, V.H. Nguyen, Extragradient methods and linear algorithms for solving Ky Fan inequalities and fixed point problems, J. Optim. Theory Appl. 155 (2012), 605–627.
[7] P.N. Anh, H.A. Le Thi, An Armijo-type method for pseudomonotone equilibrium problems and its applications, J. Global. Optim. 57 (2013), 803–820.
[8] B. V. Dinh, D.S. Kim, Projection algorithms for solving nonmonotone equilibrium problems in Hilbert space, J. Comput. Appl. Math. 302 (2016), 106–117
[9] B.V. Dinh, L. D. Muu, A projection algorithm for solving pseudomonotone equilibrium problems and its application to a class of bilievel equilibria, Optimization. 64 (2015), 559–575.
[10] D.V. Hieu, L. D. Muu, P.K. Anh, Parallel hybrid extragradient methods for pseudomotone equilibrium problems and nonexpansive mappings, Numer Algor. 73 (2016), 197–217.
[11] L.Q. Thuy, P.K. Anh, L.D. Muu, T.N. Hai, Novel hybrid methods for pseudomonotone equilibrium problems and common fixed point problems, Numer. Funct. Anal. Optim. 38 (2017), 443–465.

[12] S.H. Wang, M.L. Zhao, P. Kumam, Y.J. Cho, A viscosity extragradient method for an equilibrium problem and fixed point problem in Hilbert space, J. Fixed Point Theory Appl. (2018) 20: 19.
[13] P.N. Anh, Strong convergence theorems for nonexpansive mappings and Ky Fan inequalities, J. Optim. Theory Appl. 154 (2012), 303–320.
[14] P.N. Anh, A hybrid extragradient method for pseudomonotone equilibrium problems and fixed point problems, Bull. Malays. Math. Sci. 36 (2013), 107–116.
[15] D.V. Hieu, An Inertial-Like Proximal Algorithm for Equilibrium Problems, Math. Meth. Oper Res. 88 (2018), 399–415.
[16] D.V. Hieu, Convergence analysis of a new algorithm for strongly pseudomonotone equilibrium problems, Numer Algorithms. 77 (2018), 983–1001.
[17] D.V. Hieu, New extragradient method for a class of equilibrium problems in Hilbert spaces, Appl. Anal. 97 (2018), 811–824.
[18] D. V. Hieu, Y.J. Cho, Y.B. Xiao, Modified extragradient algorithms for solving equilibrium problems, Optimization, 67 (2018), 2003–2029.
[19] D.V. Hieu, Strong convergence of a new hybrid algorithm for fixed point problems and equilibrium problems, Math Model Anal. 24 (2019), 1–19.
[20] G. Mastroeni, On auxiliary principle for equilibrium problems, Publicatione del Dipartimento di Mathematica dell, Universita di Pisa, 3: 12441258, 2000.
[21] D.V. Hieu, J.J. Strodiot, Strong convergence theorems for equilibrium problems and fixed point problems in Banach spaces, J. Fixed Point Theory Appl. (2018) 20:131.
[22] K. Goebel, S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York, 1984.
[23] H.K. Xu, Another control condition in an iterative method for nonexpansive mappings, Bull. Austral. Math. Soc. 65 (2002), 109–113.
[24] P.E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal. 16 (2008), 899–912.
[25] P.T. Vuong, J.J. Strodiot, V.H. Nguyen, On extragradient-viscosity methods for solving equilibrium and fixed point problems in a Hilbert space, Optimization, 64 (2015), 429–451.
[26] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20 (1967), 197–228.
[27] H.Y. Zhou, Convergence theorems of fixed points for λ-strict pseudo-contractions in Hilbert spaces, Nonlinear Anal. 69 (2008), 456–462.
[28] G. Marino, H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007), 33–346.
[29] G.L. Acedo, H.K. Xu, Iterative methods for strict pseudo-contractions in Hilbert spaces, Nonlinear Anal. 67 (2007), 2258–2271.
[30] C.E. Chidume, N. Shahzad, Weak convergence theorems for a finite family of strict pseudocontractions, Nonlinear Anal. 72 (2010), 1257–1265.
[31] M.O. Osilike, Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps, J. Math. Anal. Appl. 294 (2004), 73–81.
[32] C.E. Chidume, M. Abbas, B. Ali, Convergence of the Mann iteration algorithm for a class of pseudocontractive mappings, Appl. Math. Comput. 194 (2007), 1–6.
[33] G. Marino, B. Scardamaglia, E. Karapinar, Strong convergence theorem for strict pseudo-contractions in Hilbert spaces, J. Inequal. Appl. (2016) 2016:134
[34] P.N. Anh, L.D. Muu, V.H. Nguyen, J.J. Strodiot, Using the Banach contraction principle to implement the proximal point method for multivalued monotone variational inequalities. J. Optim Theory Appl. 124 (2005), 285–306.