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The Exact Spectral Asymptotic of the Logarithmic Potential on Harmonic Function Space

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Abstract. In this paper we consider the product of the harmonic Bergman projection $P_h : L^2(D) \to L^2_h(D)$ and the operator of logarithmic potential type defined by $Lf(z) = -\frac{1}{2\pi}\int_D \ln|z-\xi| f(\xi)dA(\xi)$, where D is the unit disc in $\mathbb{C}$. We describe the asymptotic behaviour of the eigenvalues of the operator $(P_hL)^*(P_hL)$. More precisely, we prove that

$$\lim_{n\to+\infty} n^2 s_n(P_hL) = \sqrt{\frac{4\pi^2}{3} - 1}.$$

1. Introduction and Notation

In connection with the Green function for the unit ball the logarithmic potential type operator appears as an important singular integral operator. Its properties, such as a norm boundedness, singular numbers and many others were investigated in numerous of papers in various Lebesgue space settings. Let $D = \{z||z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$. By $L^2(D)$ we denote the space of all complex-valued functions f defined on D such that the norm

$$||f|| = \left(\int_D |f(z)|^2 dA(z)\right)^{\frac{1}{2}}$$

is finite. Here dA denotes Lebesgue measure on D. The logarithmic potential type operator is then defined as

$$Lf(z) = -\frac{1}{2\pi}\int_D \ln|z-\xi| f(\xi)dA(\xi). \tag{1.1}$$

It is known that the operator $L : L^2(D) \to L^2(D)$ is bounded. Moreover, it is not hard to show that the operator $L : L^2(\Omega) \to L^2(\Omega)$ is a compact operator in the case where Ω is a bounded domain in complex plane $\mathbb{C}$ (see [2]). Moreover, L is a self-adjoint compact operator.

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Let us mention the relation between the operator L and the operator induced by the Green function. First of all, recall that Green's function of D is defined as

$$g(z,\xi) = \frac{1}{2\pi}\ln\left|\frac{1-z\bar{\xi}}{z-\xi}\right|, \ z,\xi \in D$$

and related integral operator induced by g is

$$(Gf)(z) = \int_D g(z,\xi)f(\xi)dA(\xi).$$

The relation between the logarithmic potential L and the operator G is then given by

$$(Lf)(z) = (Gf)(z) + \frac{1}{2\pi}\int_D \ln\frac{1}{|1-z\bar{\xi}|}f(\xi)dA(\xi).$$

Harmonic Bergman projection. Throughout the paper by $L^2_h(D)(L^2_a(D))$ we denote the harmonic (analytic) subspace of $L^2(D)$. Let $P_h(P_a)$ be the orthogonal projection of $L^2(D)$ onto $L^2_h(D)(L^2_a(D))$. The orthogonal projection P_h represents the famous integral operator known as Bergman (harmonic) projection (for more details see [4]).
The kernel of the operator P_h is given by

$$K(z,\xi) = \frac{(1-|z|^2|\xi|^2)^2}{\pi|1-\bar{z}\xi|^4} - \frac{2}{\pi}\frac{|z|^2|\xi|^2}{|1-\bar{z}\xi|^2}, \ z,\xi \in D.$$

Singular numbers. If A is a compact operator on some Hilbert space H, then by $s_n(A)$ we denote the sequence of eigenvalues of the operator $(A^*A)^{1/2}$ in decreasing order (including multiplicity). The numbers $s_n(A)$ are known as the singular numbers (values) of the operator A. By S_∞ we will denote the space of all compact operators on the Hilbert space H.

An operator $T : H \to H$ is said to belong to the Schatten class S_p if $\|T\|_p^p = \sum_{n=1}^\infty s_n^p(T) < \infty$. The "weak S_p ideal" denoted by $S_{p,\infty}$ is defined to be a space of all compact operators T such that

$$\|T\|_{p,\infty} = \sup_n \left(n^{\frac{1}{p}}s_n(T)\right) < \infty.$$

For $1 < p < \infty$, $S_{p,\infty}$ is a complete metric space.

Asymptotic behaviour of $s_n(L)$. Since the operator $L : L^2(D) \to L^2(D)$ is Hilbert-Schmidt, the question of the asymptotic behaviour of its singular number was a topic explored by many authors.

Let us recall some famous results. For instance, in [2] and [3] the following asymptotic relations were established:

$$s_n(L) \asymp \frac{1}{n}, \ s_n(LP_a) \asymp \frac{1}{n^2}, \ s_n(P_aLP_a) \asymp \frac{1}{n^2}.$$

Here, $a_n \asymp b_n$ means $\inf_{n\in\mathbb{N}}\frac{a_n}{b_n} > 0$ and $\sup_{n\in\mathbb{N}}\frac{a_n}{b_n} < \infty$.

Latter, Dostanić proved in [6] and [7] the next general results

$$\lim_{n\to+\infty} ns_n(L) = \frac{|\Omega|}{4\pi}, \ \lim_{n\to+\infty} n^2 s_n(P_aL) = \left(\frac{|\partial\Omega|}{4\pi}\right)^2,$$

where Ω is a simply connected domain in complex plane and $|\Omega|$ and $|\partial\Omega|$ are the area and the length of the boundary of Ω.

The natural continuation of the previous investigations is the consideration of the singular numbers asymptotic for the product of the harmonic Bergman projection and the logarithmic potential type operator. In this manner, we state the following main result of this paper.

Theorem 1.1. *Let $P_h : L^2(D) \to L^2_h(D)$ be the Bergman harmonic projection and $L : L^2(D) \to L^2(D)$ the operator of logarithmic potential type. Then*

$$\lim_{n\to+\infty} n^2 s_n(P_h L) = \sqrt{\frac{4\pi^2}{3} - 1}. \tag{1.2}$$

Remark 1.2. *Here, we should point out the relation between $s_n(P_h L)$ and $s_n(LP_h)$. Namely, the facts that the logarithmic potential operator L is a self-adjoint operator and that the harmonic Bergman kernel $K(z,w)$ satisfies the relation $K(z,w) = \overline{K(w,z)}$ imply*

$$s_n(P_h L) = s_n((P_h L)^*) = s_n(LP_h).$$

So,

$$\lim_{n\to+\infty} n^2 s_n(LP_h) = \sqrt{\frac{4\pi^2}{3} - 1}.$$

Remark 1.3. *It is interesting to notice the relation of asymptotic behaviour between $s_n(P_a L)$ and $s_n(P_h L)$ for the special case when $\Omega = D$. We will denote by $P_{ha} : L^2_h(D) \to L^2_a(D)$ the natural projection from the L^2 space of harmonic functions onto the analytic subspace.*

Keeping in mind the known inequality for any bounded operator B and a compact operator A defined on the Hilbert space H,

$$s_n(BA) \le \|B\| s_n(A), \;\; s_n(AB) \le \|B\| s_n(A),$$

we get $s_n(P_a L) = s_n(P_{ha} P_h L) \le s_n(P_h L)$.

The last inequality corresponds to the obtained main result. Namely, the result by Dostanić result mentioned for $\Omega = D$ implies $\lim_{n\to+\infty} n^2 s_n(P_a L) = \frac{1}{4}$, while $\lim_{n\to+\infty} n^2 s_n(LP_h) = \sqrt{\frac{4\pi^2}{3} - 1} \approx 3,487$.

Similarly,

$$\limsup_{n\to+\infty} n^2 s_n(P_h L P_h) \le \sqrt{\frac{4\pi^2}{3} - 1}.$$

Theorem 1.1 confirms the studied phenomenon of the ”faster” decrease of the singular numbers of the operator L multiplied by Bergman’s projection P_h and generalizes the corresponding results from [3].

The next corollary is an easy consequence of Theorem 1.1.

Corollary 1.4. *$P_h L \in S_{1/2,\infty}$, and $S_{1/2,\infty}$ is the smallest ideal containing $P_h L$.*

2. Preliminaries

At the beginning let us recall that any compact operator T on a Hilbert space H admits a Schmidt expansion.

Namely, the polar decomposition of the operator $T = UA$ and the fact that A is a self-adjoint operator give a uniformly convergent representation

$$A = \sum_{j=1}^{r(A)} \lambda_j(A) \left\langle \cdot, \phi_j \right\rangle \phi_j, \tag{2.1}$$

where ϕ_j is an orthonormal system of eigenvectors of the operator A which is complete in $R(A)$ (the range of the operator A) and $r(A)$ is the dimension of $R(A)$ such that

$$A\phi_j = \lambda_j(A)\phi_j, \quad j = 1, .., r(A).$$

By $\langle \cdot, \cdot \rangle$ we mean the inner-product pairing related to H. It is clear that $s_j(T) = \lambda_j(A)$.

Applying the unitary operator U to both sides of (2.1), we obtain

$$T = \sum_{j=1}^{r(T)} s_j(T) \left\langle \cdot, \phi_j \right\rangle U\phi_j. \tag{2.2}$$

Since the set $\{U\phi_j\}$ is still orthonormal, we have that any compact operator T can be represented in the following manner (Schmidt expansion):

$$T = \sum_{j=1}^{r(T)} s_j(T) \left\langle \cdot, \phi_j \right\rangle \psi_j, \tag{2.3}$$

for a certain orthonormal systems $\{\phi_j\}$ and $\{\psi_j\}$. The series (2.3) converges in the uniform norm.

In the sequel, we will work with the functions

$$\psi_n(z) = z^n \sum_{s=1}^{+\infty} s(s+n)|z|^{2(s-1)} \left(\frac{1}{2(n+s+1)} - \frac{2|z|^2}{2(s+n+2)} + \frac{|z|^4}{2(s+n+3)} \right), \tag{2.4}$$

and

$$\varphi_n(z) = z^n \sum_{s=0}^{+\infty} \frac{|z|^{2(s+1)}}{s+n+3}, \;\; z \in D. \tag{2.5}$$

Estimating the L^2 norms of the functions $\psi_n(z)$ and $\varphi_n(z)$ will play a significant part in our final results.

The functions $\psi_n(z)$ and $\varphi_n(z)$ were introduced in [10]. More precisely, the functions $\tilde{\varphi}_n, \tilde{\psi}_n$ (denoted by φ and ψ respectively in [10]) were defined in a similar manner as

$$\tilde{\psi}_n(z) = z^n \sum_{m>n} m(m-n) \left(\frac{|z|^{2(m-n)}}{m+2} - \frac{2|z|^{2(m-n+1)}}{m+3} + \frac{|z|^{2(m-n+2)}}{m+4} \right),$$

and

$$\tilde{\varphi}_n(z) = z^n \sum_{m \geq n} \frac{|z|^{2(m-n+1)}}{m+3}.$$

Obviously, $\tilde{\varphi}_n = \varphi_n$.

In [10] (Lemma 3.1) it was established that,

$$||\tilde{\varphi}_n||_{L^2(D)} \sim \frac{\pi}{\sqrt{6(n+3)}}, \;\; n \to +\infty,$$

and

$$\frac{1}{4\sqrt{6}} \leq \lim_{n \to +\infty} \sqrt{n} ||\tilde{\psi}_n||_{L^2(D)} \leq \frac{1}{2\sqrt{6}}.$$

Here the notation $a_n \sim b_n$ means

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 1 \;\; (0 < c_1 \leq \frac{a_n}{b_n} \leq c_2 < \infty).$$

The next lemma is a refinement of Lemma 3.1. For the sake of completeness we will repeat some steps of the proof of mentioned lemma.

Lemma 2.1. *For the functions defined by* (2.4) *and* (2.5) *the following asymptotic relations hold:*

$$\|\varphi_n\|_{L^2(D)} \sim \frac{\pi^{3/2}}{\sqrt{3(n+3)}}, \;\; n \to +\infty, \tag{2.6}$$

$$\|\psi_n\|_{L^2(D)} \sim \frac{1}{2}\sqrt{\frac{\pi}{n}}, \;\; n \to +\infty, \tag{2.7}$$

and

$$\langle \varphi_n, \psi_n \rangle \sim \frac{\pi}{2n}, \;\; n \to +\infty, \tag{2.8}$$

where $\langle \varphi_n, \psi_n \rangle = \int_D \varphi_n(z)\overline{\psi_n(z)}dA(z)$ *denotes the usual inner product in* $L^2(D)$.

Proof. Formula (1.8) follows immediately from [10], i.e.,

$$\|\varphi_n\|_{L^2(D)} \sim \frac{\pi^{3/2}}{\sqrt{3(n+3)}}, \;\; n \to +\infty.$$

By using polar coordinates $z = |z|e^{i\theta}$ and Parseval's formula one obtains

$$\begin{aligned}
\|\psi_n\|^2_{L^2(D)} &= \int_D |z|^{2n} \left(\sum_{s\geq 1} s(s+n)|z|^{2(s-1)} f_s(|z|) \right)^2 dA(z) \\
&= \int_D |z|^{2n} \left(\sum_{s\geq 1} s(s+n)|z|^{2(s-1)} f_s(|z|) \right)^2 dA(z) \\
&= \sum_{s=1}^{\infty}\sum_{k=1}^{s} k(k+n)(s-k)(s-k+n) \int_D |z|^{2(s+n-2)} f_k(|z|) f_{s-k}(|z|) dA(z) \\
&= \sum_{s=1}^{\infty}\sum_{k=1}^{s} k(s-k) \int_D |z|^{2(s+n-2)}((k+n)f_k(|z|))((s-k+n)f_{s-k}(|z|)) dA(z)
\end{aligned} \tag{2.9}$$

where by $f_s(|z|), s \in \mathbb{N}$ we denote the function $f_s(|z|) = \frac{1}{2(s+n+1)} - \frac{2|z|^2}{2(s+n+2)} + \frac{|z|^4}{2(s+n+3)}$.

By direct calculation one obtains

$$\begin{aligned}
&\int_D \left(\frac{1}{2}(1-|z|^2)^2 - (s+n)f_s(|z|) \right) dA(z) \\
&\quad = \frac{\pi}{(1+n+s)(2+n+s)(3+n+s)} > 0, \quad s \in \mathbb{N}.
\end{aligned}$$

Then by using elementary transformations and the mean value theorem for definite integrals we have

$$\begin{aligned}
&\int_D |z|^{2(s+n-2)}((k+n)f_k(|z|))((s-k+n)f_{s-k}(|z|)) dA(z) \\
&\quad \leq \frac{1}{4}\int_D |z|^{2(s+n-2)}(1-|z|^2)^4 dA(z), \quad 1 \leq k \leq s.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\psi_n\|^2_{L^2(D)} &\leq \frac{1}{4}\sum_{s=1}^{\infty}\sum_{k=1}^{s} k(s-k) \int_D |z|^{2(s+n-2)}(1-|z|^2)^4 dA(z) \\
&= 6\pi \sum_{s=1}^{\infty}\sum_{k=1}^{s} \frac{k(s-k)}{(-1+n+s)(n+s)(1+n+s)(2+n+s)(3+n+s)} \\
&= \pi \sum_{s=1}^{\infty} \frac{(s-1)s(1+s)}{(-1+n+s)(n+s)(1+n+s)(2+n+s)(3+n+s)} = \frac{\pi}{4(1+n)}.
\end{aligned}$$

The last identity can be obtained by the transformation of the general term

$$
\begin{aligned}
&\frac{(s-1)s(1+s)}{(-1+n+s)(n+s)(1+n+s)(2+n+s)(3+n+s)}\\
&=\frac{n^3-n}{6(s+n)}+\frac{-n^3-9n^2-26n-24}{24(s+n+3)}+\frac{-n^3+3n^2-2n}{24(s+n-1)}\\
&+\frac{-n^3-3n^2-2n}{4(s+n+1)}+\frac{n^3+6n^2+11n+6}{6(n+s+2)}
\end{aligned}
$$

and by proving (induction) the formula

$$
\begin{aligned}
&\sum_{s=1}^{N}\frac{(s-1)s(1+s)}{(-1+n+s)(n+s)(1+n+s)(2+n+s)(3+n+s)}\\
&=\frac{N(N-1)(N+1)(N+2)}{4(n+1)(N+n)(N+n+1)(N+n+2)(N+n+3)}.
\end{aligned}
$$

It is easy to verify the inequality $(s+n)f_s(|z|)\geq\frac{s+n}{2(s+n+1)}(1-|z|^2)^2, z\in D$. Therefore,

$$
\begin{aligned}
\|\psi_n\|^2_{L^2(D)}&\geq\frac{1}{4}\sum_{s=1}^{\infty}\sum_{k=1}^{s}\frac{k(s-k)(k+n)(s-k+n)}{(k+n+1)(s-k+n+1)}\int_D|z|^{2(s+n-2)}(1-|z|^2)^4dA(z)\\
&=\frac{1}{4}\sum_{s=1}^{\infty}\sum_{k=1}^{s}k(s-k)\int_D|z|^{2(s+n-2)}(1-|z|^2)^4dA(z)\\
&+\frac{1}{4}\sum_{s=1}^{\infty}\sum_{k=1}^{s}\frac{k(s-k)}{(k+n+1)(s-k+n+1)}\int_D|z|^{2(s+n-2)}(1-|z|^2)^4dA(z)\\
&-\frac{1}{2}\sum_{s=1}^{\infty}\sum_{k=1}^{s}\frac{k(s-k)}{k+n+1}\int_D|z|^{2(s+n-2)}(1-|z|^2)^4dA(z)\\
&=\frac{\pi}{4(1+n)}+I_1(n)-I_2(n),
\end{aligned}
$$

where

$$
\begin{aligned}
I_1&=\frac{1}{4}\sum_{s=1}^{\infty}\sum_{k=1}^{s}\frac{k(s-k)}{(k+n+1)(s-k+n+1)}\int_D|z|^{2(s+n-2)}(1-|z|^2)^4dA(z)\\
&\geq\frac{1}{4(n+2)^2}\sum_{s=1}^{\infty}\int_D|z|^{2(s+n-2)}(1-|z|^2)^4dA(z)=\frac{6\pi}{4(n+2)^3n(1+n)(3+n)}
\end{aligned}
$$

and

$$
\begin{aligned}
&I_1(n)\leq I_2(n)\\
&\quad=\frac{1}{2}\sum_{s=1}^{\infty}\sum_{k=1}^{s}\frac{k(s-k)}{k+n+1}\int_D|z|^{2(s+n-2)}(1-|z|^2)^4dA(z)\\
&\quad\leq\frac{1}{2}\sum_{s=1}^{\infty}\sum_{k=1}^{s}(s-k)\int_D|z|^{2(s+n-2)}(1-|z|^2)^4dA(z)=\frac{\pi}{(1+n)(2+n)}.
\end{aligned}
$$

Finally, we get

$$\lim_{n\to+\infty}n\|\psi_n\|^2_{L^2(D)}=\frac{\pi}{4}.$$

Further, by using the same type of inequalities as before we obtain

$$\begin{aligned}
\int_D \psi_n(z)\overline{\varphi_n(z)}dA(z) &= \int_D |z|^{2(s+n-1)} \sum_{s=1}^{\infty}\sum_{k=1}^{s} \frac{(s-k)(s-k+n)}{k+n+2} f_{s-k}(|z|)dA(z)\\
&\leq \frac{1}{2}\sum_{s=1}^{\infty}\sum_{k=1}^{s} \frac{s-k}{k+n+2}\int_D |z|^{2(s+n-1)}(1-|z|^2)^2 dA(z)\\
&= \pi \sum_{s=1}^{\infty} \frac{\sum_{k=1}^{s}\frac{s-k}{k+n+2}}{(n+s)(1+n+s)(2+n+s)}\\
&= \pi \sum_{s=1}^{\infty} \frac{\sum_{k=1}^{s}\frac{1}{k+n+2}}{(n+s)(1+n+s)} - \pi\sum_{s=1}^{\infty}\frac{s}{(n+s)(1+n+s)(s+n+2)}\\
&\leq \sum_{s=1}^{\infty}\frac{\pi\ln(\frac{s+n+2}{n+2})}{(n+s)(1+n+s)} - \frac{\pi}{2(n+1)}\\
&\leq A(n) + \int_0^{+\infty}\frac{\pi\ln(\frac{x+n+2}{n+2})}{(n+x)(1+n+x)}dx - \frac{\pi}{2(n+1)},
\end{aligned}$$

where $A(n) = O(n^{-2}), n \to +\infty$. Therefore,

$$\begin{aligned}
\limsup_{n\to+\infty}\langle\varphi_n,\psi_n\rangle &\leq \lim_{n\to+\infty} n\int_0^{+\infty}\frac{\pi\ln(\frac{x+n+2}{n+2})}{(n+x)(1+n+x)}dx - \frac{\pi}{2}\\
&= \pi\int_0^{+\infty}\frac{\ln(1+x)}{(x+1)^2}dx - \frac{\pi}{2} = \frac{\pi}{2}.
\end{aligned} \tag{2.10}$$

On the other hand,

$$\begin{aligned}
\int_D \psi_n(z)\overline{\varphi_n(z)}dA(z) &= \int_D |z|^{2(s+n-1)} \sum_{s=1}^{\infty}\sum_{k=1}^{s} \frac{(s-k)(s-k+n)}{k+n+2} f_{s-k}(|z|)dA(z)\\
&\geq \frac{1}{2}\int_D |z|^{2(s+n-1)}\sum_{s=1}^{\infty}\sum_{k=1}^{s}\frac{(s-k)(s-k+n)}{(k+n+2)(s-k+n+1)}(1-|z|^2)^2 dA(z)\\
&= \frac{1}{2}\sum_{s=1}^{\infty}\sum_{k=1}^{s}\frac{(s+n+2)\left(1-\frac{1}{s+2n+3}\right)}{k+n+2}\int_D |z|^{2(s+n-1)}(1-|z|^2)^2 dA(z)\\
&+ \frac{1}{2}\sum_{s=1}^{\infty}\frac{n+1}{s+2n+3}\sum_{k=1}^{s}\frac{1}{s-k+n+1}\int_D |z|^{2(s+n-1)}(1-|z|^2)^2 dA(z)\\
&- \sum_{s=1}^{\infty}\frac{\pi s}{(n+s)(1+n+s)(2+n+s)} \geq \pi\sum_{s=1}^{\infty}\frac{\left(1-\frac{1}{s+2n+3}\right)\ln\left(\frac{s+n+2}{n+3}\right)}{(n+s)(1+n+s)}\\
&+ \pi\sum_{s=1}^{\infty}\frac{(n+1)\ln\left(\frac{s+n+1}{n+2}\right)}{(s+2n+3)(n+s)(1+n+s)(2+n+s)} - \frac{\pi}{2(1+n)}.
\end{aligned}$$

It is not hard to show that

$$\sum_{s=1}^{\infty}\frac{(n+1)\ln\left(\frac{s+n+1}{n+2}\right)}{(s+2n+3)(n+s)(1+n+s)(2+n+s)} = o(n^{-1}), \quad n\to\infty$$

and

$$\sum_{s=1}^{\infty}\frac{\ln\left(\frac{s+n+2}{n+3}\right)}{(s+2n+3)(n+s)(1+n+s)} = o(n^{-1}), \quad n\to\infty.$$

The last conclusion implies

$$\liminf_{n\to+\infty} n\,\langle\varphi_n,\psi_n\rangle \geq \lim_{n\to+\infty} n\int_1^{+\infty}\frac{\pi\ln\left(\frac{x+n+2}{n+3}\right)}{(n+x)(1+n+x)}dx-\frac{\pi}{2}=\frac{\pi}{2}, \tag{2.11}$$

which together with (2.10) yields (2.8). □

Remark 2.2. *Taking into account* (2.6) *and* (2.7) *and applying the Cauchy-Schwarz inequality we get*

$$\limsup_{n\to+\infty} n\,\langle\varphi_n,\psi_n\rangle \leq \lim_{n\to+\infty} n\|\varphi_n\|_{L^2(D)}\|\psi_n\|_{L^2(D)}=\frac{\pi^2}{2\sqrt{3}},$$

a less precisely asymptotic upper bound for $\langle\varphi_n,\psi_n\rangle$ *compared with the obtained result.*

3. The proof of the main result

We begin this section by first resolving the principle problem of finding an explicit formula for the kernel of the operator $P_hL : L^2(D)\to L^2(D)$. Recall that the operator P_hL acts as the integral operator

$$P_hLf(z)=\int_D H(z,\xi)f(\xi)dA(\xi),$$

where

$$H(z,\xi)=-\frac{1}{2\pi}\int_D K(z,\omega)\ln|\xi-\omega|dA(\omega).$$

Lemma 3.1. *The kernel of the operator* P_hL *is*

$$H(z,\xi)=A(z,\xi)+B(z,\xi)+C(z,\xi),\quad \textit{where}$$

$$\begin{aligned}
A(z,\xi)&=-\frac{1}{2\pi}|\xi|^2(\ln|\xi|+\ln|1-z\bar{\xi}|),\\
B(z,\xi)&=\frac{1}{\pi}\sum_{k=1}^{\infty}k^2|z|^{2(k-1)}\left[|\xi|^{2k}\ln|\xi|A_1(k,|z||\xi|)+A_2(k,|z|)-|\xi|^{2k}A_2(k,|\xi||z|)\right]\\
&\quad+\frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{|z|^{2k+2}}{(k+2)^2}\left[|\xi|^{2k+4}-2(k+2)|\xi|^{2k+4}\ln|\xi|-1\right],\\
C(z,\xi)&=\frac{1}{\pi}\sum_{k=1}^{+\infty}Re\left(\frac{\bar{z}^k\xi^k}{k}\right)\sum_{s=1}^{\infty}s(s+k)|z|^{2(s-1)}\left[A_1(s,|z|)-|\xi|^{2s}A_1(s,|z||\xi|)\right]\\
&\quad+\frac{1}{\pi|z|^2}\left(\ln|1-\bar{z}\xi|\left(|z|^2(1-|\xi|^2)-\ln\left(\frac{1-|z|^2}{1-|\xi|^2|z|^2}\right)\right)\right).
\end{aligned} \tag{3.1}$$

Here,

$$A_1(k,x)=\frac{1}{2k}-\frac{2x^2}{2k+2}+\frac{x^4}{2k+4},\quad x\in[0,1],$$

$$A_2(k,x)=\frac{1}{(2k)^2}-\frac{2x^2}{(2k+2)^2}+\frac{x^4}{(2k+4)^2},\quad x\in[0,1].$$

Proof. As we have stated above,

$$H(z,\xi)=-\frac{1}{2\pi}\int_D\ln|\xi-\omega|K(z,\omega)dA(\omega).$$

Let $\xi = \rho e^{i\theta}$. Then by using polar coordinates $\omega = re^{it}$ we have

$$\begin{aligned}
&-\int_D \ln|\xi-\omega|K(z,\omega)dA(\omega)\\
&\quad = -\int_0^{2\pi} dt \int_0^{\rho} \ln|\rho e^{i\theta} - re^{it}|K(z,re^{it})rdr\\
&\quad - \int_0^{2\pi} dt \int_\rho^{1} \ln|\rho e^{i\theta} - re^{it}|K(z,re^{it})rdr\\
&\quad = -\int_0^{2\pi} dt \int_0^{\rho} \left[\ln\rho + \ln\left|1-\frac{r}{\rho}e^{i(t-\theta)}\right|\right]K(z,re^{it})rdr\\
&\quad - \int_0^{2\pi} dt \int_\rho^{1} \left[\ln r + \ln\left|1-\frac{\rho}{r}e^{i(\theta-t)}\right|\right]K(z,re^{it})rdr.
\end{aligned} \tag{3.2}$$

Let us introduce the notations

$$I_1 = -\int_0^{2\pi} dt \int_0^{\rho} \left[\ln\rho + \ln\left|1-\frac{r}{\rho}e^{i(t-\theta)}\right|\right]K(z,re^{it})rdr,$$

and

$$I_2 = -\int_0^{2\pi} dt \int_\rho^{1} \left[\ln r + \ln\left|1-\frac{\rho}{r}e^{i(\theta-t)}\right|\right]K(z,re^{it})rdr.$$

Then

$$\begin{aligned}
I_1 &= -\ln\rho\int_0^{2\pi} dt\int_0^{\rho} K(z,re^{it})rdr - \int_0^{2\pi} dt \int_0^{\rho} \ln\left|1-\frac{r}{\rho}e^{i(t-\theta)}\right|K(z,re^{it})rdr\\
&= -\ln\rho\int_{|\omega|<\rho} K(z,\omega)dA(\omega) - \int_{|\omega|<\rho}\ln\left|1-\frac{\omega}{\xi}\right|K(z,\omega)dA(\omega)\\
&= -\rho^2\ln\rho - \rho^2\int_{|\omega'|<1}\ln|1-\omega'|K(z,\rho e^{i\theta}\omega')dA(\omega')\\
&= -\rho^2\ln\rho - \rho^2\int_{|\omega'|<1}\ln|1-\omega'|K(z\rho e^{-i\theta},\omega')dA(\omega')\\
&= -\rho^2\ln\rho - \rho^2\ln|1-z\rho e^{-i\theta}| = -|\xi|^2(\ln|\xi| + \ln|1-z\bar{\xi}|),
\end{aligned} \tag{3.3}$$

where $\omega = \xi\omega'$.
Further,

$$\begin{aligned}
I_2 = &-\int_0^{2\pi} dt\int_\rho^1 r\ln rK(z,re^{it})dr - \frac{1}{2}\int_0^{2\pi} dt\int_\rho^1 \ln(1-\frac{\rho}{r}e^{i(\theta-t)})K(z,re^{it})rdr\\
&-\frac{1}{2}\int_0^{2\pi} dt\int_\rho^1 \ln(1-\frac{\rho}{r}e^{i(t-\theta)})K(z,re^{it})rdr.
\end{aligned} \tag{3.4}$$

The first integral in (3.4) can be represented as

$$\begin{aligned}
&-\int_0^{2\pi} dt\int_\rho^1 r\ln rK(z,re^{it})dr\\
&\quad = -\frac{1}{\pi}\int_0^{2\pi} dt\int_\rho^1 r\ln r\frac{(1-|z|^2r^2)^2}{|1-\bar{z}re^{it}|^4}dr\\
&\quad + \frac{2}{\pi}\int_0^{2\pi} dt\int_\rho^1 r\ln r\frac{|z|^2r^2}{|1-\bar{z}re^{it}|^2}dr.
\end{aligned} \tag{3.5}$$

Since

$$\frac{1}{|1-\bar{z}re^{it}|^4} = \left|\sum_{k=1}^{\infty} k\bar{z}^{k-1}r^{k-1}e^{i(k-1)t}\right|^2,$$

we have

$$\begin{aligned}
&-\frac{1}{\pi}\int_0^{2\pi} dt\int_\rho^1 r\ln r\frac{(1-|z|^2r^2)^2}{|1-\bar{z}re^{it}|^4}dr\\
&=-\frac{1}{\pi}\sum_{k,m=1}^{\infty} kmz^{m-1}\bar{z}^{k-1}\int_\rho^1 r^{k+m-1}(1-|z|^2r^2)^2\ln r dr\int_0^{2\pi} e^{i(k-m)t}dt\\
&=-2\sum_{k=1}^{\infty} k^2|z|^{2(k-1)}\int_\rho^1 r^{2k-1}(1-|z|^2r^2)^2\ln r dr\\
&=2\sum_{k=1}^{\infty} k^2|z|^{2(k-1)}\left[|\xi|^{2k}\ln|\xi|A_1(k,|z||\xi|)+A_2(k,|z|)-|\xi|^{2k}A_2(k,|\xi||z|)\right].
\end{aligned} \tag{3.6}$$

Here,

$$A_1(k,|z||\xi|) = \frac{1}{2k}-\frac{2|z|^2|\xi|^2}{2k+2}+\frac{|z|^4|\xi|^4}{2k+4},$$

$$A_2(k,|z|) = \frac{1}{(2k)^2}-\frac{2|z|^2}{(2k+2)^2}+\frac{|z|^4}{(2k+4)^2}.$$

$$\begin{aligned}
&\frac{2}{\pi}\int_0^{2\pi} dt\int_\rho^1 r\frac{|z|^2r^2}{|1-\bar{z}re^{it}|^2}\ln r dr\\
&=4\sum_{k=0}^{\infty}|z|^{2k+2}\int_\rho^1 r^{2k+3}\ln r dr\\
&=\sum_{k=0}^{\infty}\frac{|z|^{2k+2}}{(k+2)^2}\left[|\xi|^{2k+4}-2(k+2)|\xi|^{2k+4}\ln|\xi|-1\right].
\end{aligned} \tag{3.7}$$

Further,

$$\begin{aligned}
&-\frac{1}{2\pi}\int_0^{2\pi} dt\int_\rho^1 \ln\left(1-\frac{\rho}{r}e^{i(\theta-t)}\right)\frac{(1-|z|^2r^2)^2}{|1-\bar{z}re^{it}|^4}rdr\\
&=\frac{1}{2\pi}\int_0^{2\pi} dt\int_\rho^1\sum_{k=1}^{+\infty}\frac{(\frac{\rho}{r})^k}{k}e^{ik(\theta-t)}\frac{(1-|z|^2r^2)^2}{|1-\bar{z}re^{it}|^4}rdr\\
&=\frac{1}{2\pi}\sum_{k=1}^{+\infty}\frac{\rho^ke^{ik\theta}}{k}\sum_{s,m=1}^{\infty} sm\bar{z}^{m-1}z^{s-1}\int_0^{2\pi} e^{-ikt}e^{(m-s)it}dt\int_\rho^1 r^{s+m-k-1}(1-|z|^2r^2)^2dr\\
&=\sum_{k=1}^{+\infty}\frac{\rho^ke^{ik\theta}}{k}\bar{z}^k\sum_{s=1}^{\infty} s(s+k)|z|^{2(s-1)}\int_\rho^1 r^{2s-1}(1-|z|^2r^2)^2dr\\
&=\sum_{k=1}^{+\infty}\frac{\bar{z}^k\xi^k}{k}\sum_{s=1}^{\infty} s(s+k)|z|^{2(s-1)}\left[A_1(s,|z|)-|\xi|^{2s}A_1(s,|z||\xi|)\right]
\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{\pi}\int_0^{2\pi}dt\int_\rho^1 \ln\Big(1-\frac{\rho}{r}e^{i(\theta-t)}\Big)\frac{|z|^2r^2}{|1-\bar{z}re^{it}|^2}rdr\\
&=-\frac{1}{\pi}\int_0^{2\pi}dt\int_\rho^1\sum_{k=1}^{+\infty}\frac{(\frac{\rho}{r})^k}{k}e^{ik(\theta-t)}\frac{|z|^2r^2}{|1-\bar{z}re^{it}|^2}rdr\\
&=-\frac{1}{\pi}\sum_{k=1}^{+\infty}\frac{\rho^ke^{ik\theta}}{k}|z|^2\sum_{s,m=0}^{\infty}\bar{z}^mz^s\int_0^{2\pi}e^{-ikt}e^{(m-s)it}dt\int_\rho^1 r^{s+m-k+3}dr\\
&=\sum_{k=1}^{+\infty}\frac{\bar{z}^k\xi^k}{k}\sum_{s=0}^{\infty}\frac{|z|^{2s+2}}{s+2}(|\xi|^{2s+4}-1)\\
&=\frac{1}{|z|^2}\left(\ln(1-\bar{z}\xi)\left(|z|^2(1-|\xi|^2)-\ln\left(\frac{1-|z|^2}{1-|\xi|^2|z|^2}\right)\right)\right).
\end{aligned}
$$

Since

$$\overline{\int_{|t|>|\xi|}\ln\Big(1-\frac{\xi}{t}\Big)K(z,t)dA(t)}=\int_{|t|>|\xi|}\ln\Big(1-\frac{\bar{\xi}}{\bar{t}}\Big)K(z,t)dA(t).$$

the remaining integrals can be computed in a similar manner. □

Proof. [Proof of the Theorem 1.1] Let us consider the orthogonal sets $\eta_n(\xi)=\xi^n$ and $\overline{\eta_n}(\xi)=\bar{\xi}^n, n\in\mathbb{N}$ in $L^2(D)$.

The fact that any harmonic function h in D can be represented as a sum $h=f+\bar{g}$, where the function f is analytic and $\bar{g}$ is anti-analytic in the unit disc, implies the completeness of the sequences $\{\eta_n(\xi)\}_n\cup\{\bar{\eta}_n(\xi)\}_n, n\in\mathbb{N}$ in $L^2_h(D)$ and consequently in the range set $R(P_hL)$.

First of all, let us note that for $n\geq 0$ we have

$$
\begin{aligned}
P_hL(\eta_n)(z)&=\int_D A(z,\xi)\eta_n(\xi)dA(\xi)+\int_D C(z,\xi)\eta_n(\xi)dA(\xi)\\
&=\frac{z^n}{4n(n+2)}-\frac{z^n}{2n(n+1)}\sum_{s=0}^{+\infty}\frac{|z|^{2s+2}}{s+n+3}\\
&-\frac{z^n}{4n}\sum_{s=1}^{+\infty}s(s+n)|z|^{2(s-1)}\left[A_1(s,|z|)-A_3(s,n,|z|)\right]\\
&=\frac{z^n}{4n(n+2)}-\frac{z^n}{2n(n+1)}\sum_{s=0}^{+\infty}\frac{|z|^{2s+2}}{s+n+3}\\
&-\frac{z^n}{2n(n+1)}\sum_{s=1}^{+\infty}s(s+n)|z|^{2(s-1)}A_1(s+n+1,|z|)\\
&=\vartheta_n(z)-\frac{1}{2n(n+1)}\varphi_n(z)-\frac{1}{2n(n+1)}\psi_n(z),
\end{aligned}
\tag{3.8}
$$

where

$$A_3(s,n,|z|)=\left(\frac{1}{2s(1+n+s)}-\frac{2|z|^2}{2(s+1)(s+n+2)}+\frac{|z|^4}{2(2+s)(n+s+3)}\right)$$

and $\vartheta_n(z) = \frac{z^n}{4n(n+2)}$, and similarly

$$\begin{aligned} P_hL(\bar{\eta}_n)(z) &= \int_D A(z,\xi)\bar{\eta}_n(\xi)dA(\xi) + \int_D C(z,\xi)\bar{\eta}_n(\xi)dA(\xi) \\ &= \frac{\bar{z}^n}{4n(n+2)} - \frac{\bar{z}^n}{2n(n+1)}\sum_{s=0}^{+\infty}\frac{|z|^{2s+2}}{s+n+3} \\ &- \frac{\bar{z}^n}{4n}\sum_{s=1}^{+\infty} s(s+n)|z|^{2(s-1)}\left[A_1(s,|z|) - A_3(s,n,|z|)\right]. \end{aligned} \tag{3.9}$$

The mentioned representation for the harmonic function P_hLf implies

$$P_hLf(z) = \sum_{n\geq 0}\langle P_hLf, \eta_n\rangle\,\eta_n(z) + \sum_{n\geq 0}\langle P_hLf, \bar{\eta}_n\rangle\,\bar{\eta}_n(z).$$

On the other hand, for $f \in L^2(D)$ let $\{f_n\}$ be a sequence of continuous functions with compact support which converge to f in $L^2(D)$.

Then Fubini's theorem gives

$$\begin{aligned} &\lim_{m\to+\infty}\langle P_hLf_m, \eta_n\rangle \\ &= \lim_{m\to+\infty}\int_D\int_D (A(z,\xi) + C(z,\xi))\, f_m(\xi)dA(\xi)\bar{\eta}_n(z)dA(z) \\ &= \lim_{m\to+\infty}\int_D\int_D (A(z,\xi) + C(z,\xi))\,\bar{\eta}_n(z)dA(z)f_m(\xi)dA(\xi) \\ &= \left\langle P_hL\bar{\eta}_n, \bar{f}\right\rangle. \end{aligned}$$

Further, for a function $g \in L^2(D)$ we denote by $g^{**}(z) = \|g\|^{-1}_{L^2(D)}g(z)$, the "normalized" function g. Thus,

$$\begin{aligned} P_hLf(z) &= \sum_{n\geq 0}\frac{1}{\|\eta_n\|^2_{L^2(D)}}\langle P_hLf, \eta_n\rangle\,\eta_n(z) + \sum_{n\geq 0}\frac{1}{\|\eta_n\|^2_{L^2(D)}}\langle P_hLf, \bar{\eta}_n\rangle\,\bar{\eta}_n(z) \\ &= \sum_{n\geq 0}\frac{1}{\|\eta_n\|^2_{L^2(D)}}\left\langle P_hL\bar{\eta}_n, \bar{f}\right\rangle\eta_n(z) + \sum_{n\geq 0}\frac{1}{\|\eta_n\|^2_{L^2(D)}}\left\langle P_hL\eta_n, \bar{f}\right\rangle\bar{\eta}_n(z) \\ &= \sum_{n\geq 0}\frac{\|P_hL\eta_n\|_{L^2(D)}}{\|\eta_n\|_{L^2(D)}}\left\langle ((P_hL\bar{\eta}_n))^{**}, \bar{f}\right\rangle\eta_n^{**}(z) \\ &+ \sum_{n\geq 0}\frac{\|P_hL\eta_n\|_{L^2(D)}}{\|\eta_n\|_{L^2(D)}}\left\langle ((P_hL\eta_n))^{**}, \bar{f}\right\rangle\bar{\eta}_n^{**}(z). \end{aligned}$$

If we denote by $V : L^2(D) \to L^2(D)$ the isometry $Vf(z) = \overline{f(z)}$, then

$$P_hLVf = \sum_{n\geq 0}\frac{\|P_hL\eta_n\|_{L^2(D)}}{\|\eta_n\|_{L^2(D)}}\left(\langle((P_hL\bar{\eta}_n))^{**}, f\rangle\,\eta_n^{**}(z) + \langle((P_hL\eta_n))^{**}, f\rangle\,\bar{\eta}_n^{**}(z)\right),$$

and $s_n(P_hLV) = s_n(P_hL)$.

Let us point out that the action of the adjoint operator $(P_hLV)^*$ is then given by

$$\begin{aligned} (P_hLV)^*(\cdot) &= LP_h(\cdot) \\ &= \sum_{n\geq 0}\frac{\|P_hL\eta_n\|_{L^2(D)}}{\|\eta_n\|_{L^2(D)}}\left(\langle(\eta_n^{**}, \cdot\rangle\,(P_hL\bar{\eta}_n))^{**} + \langle(\bar{\eta}_n^{**}, \cdot\rangle\,(P_hL\eta_n))^{**}\right). \end{aligned}$$

It is clear that by the appropriate numeration of the indices n the fractions $\frac{\|P_hL\eta_n\|_{L^2(D)}}{\|\eta_n\|_{L^2(D)}}$ are the singular numbers for the operator P_hL.

We have

$$\begin{aligned}
&\frac{\|PL\eta_n\|^2_{L^2(D)}}{\|\eta_n\|^2_{L^2(D)}}\\
&= \frac{n+1}{\pi}\left\|\vartheta_n(z) - \frac{1}{2n(n+1)}\varphi_n(z) - \frac{1}{2n(n+1)}\psi_n(z)\right\|^2_{L^2(D)}\\
&= \frac{n+1}{\pi}\|\vartheta_n\|^2_{L^2(D)} + \frac{1}{4\pi n^2(n+1)}\|\varphi_n\|^2_{L^2(D)} + \frac{1}{4\pi n^2(n+1)}\|\psi_n\|^2_{L^2(D)}\\
&- \frac{1}{2n\pi}\left\langle \vartheta_n, \varphi_n + \psi_n\right\rangle + \frac{1}{4\pi n^2(n+1)}\left\langle \varphi_n, \psi_n\right\rangle.
\end{aligned}$$

By a direct calculation one obtains,

$$\|\vartheta_n\|_{L^2(D)} = \frac{\sqrt{\pi}}{4n(n+2)\sqrt{n+1}}, \tag{3.10}$$

and

$$\left\langle \vartheta_n, \varphi_n + \psi_n\right\rangle = \frac{5\pi}{8n(n+2)^2}. \tag{3.11}$$

According to the Lemma (2.1) and (3.10), (3.11) we have

$$\lim_{n\to+\infty}\frac{n^4\|PL\eta_n\|^2_{L^2(D)}}{\|\eta_n\|^2_{L^2(D)}} = \frac{\pi^2}{12} - \frac{1}{16}. \tag{3.12}$$

Finally, setting that $s_{2n-1}(P_hL) = s_{2n}(P_hL) = \frac{\|PL\eta_n\|_{L^2(D)}}{\|\eta_n\|_{L^2(D)}}, n \geq 1$ implies

$$s_n(P_hL) \sim \frac{\sqrt{\frac{4\pi^2}{3} - 1}}{n^2},\ \ n \to +\infty.$$

$\square$

Remark 3.2. *From the proof of the Theorem 1.1 it is clear that* $\frac{\|PL\eta_n\|^2_{L^2(D)}}{\|\eta_n\|^2_{L^2(D)}}$, $n \in \mathbb{N}$ *is a double eigenvalue of the operator* $(LP_h)^*(LP_h) = P_hL^2P_h$ *and that the corresponding eigenspace is spanned by* η_n *and* $\bar{\eta}_n$.

Remark 3.3. *In case we consider a general bounded domain* $\Omega \subset \mathbb{C}$ *and* $P_hL : L^2(\Omega) \to L^2(\Omega)$, *it stays an open problem to determine the exact asymptotic behavior of* $s_n(P_hL)$. *Specifically, in [11] a simply connected domain* Ω *(with analytic boundary* $\partial\Omega$*) was introduced and a two-side asymptotic estimate was given for the singular numbers of Cauchy's operator restricted to harmonic function space (Theorem 1.1). According to the mentioned work, the natural way of approaching the problem would be transferring the above procedure through the Riemann mapping theorem on the unit disc.*

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