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Convergence Analysis of the Generalized Euler-Maclaurin Quadrature Rule for Solving Weakly Singular Integral Equations

Grzegorz Rządkowski[a], **Emran Tohidi**[b]

[a]*Department of Finance and Risk Management, Warsaw University of Technology, Warsaw, Poland*
[b]*Department of Mathematics, Kosar University of Bojnord, P. O. Box 9415615458, Bojnord, Iran*

Abstract. In the present paper we use the generalized Euler-Maclaurin summation formula to study the convergence and to solve weakly singular Fredholm and Volterra integral equations. Since these equations have different nature, the proposed convergence analysis for each equation has a different structure. Moreover, as an application of this summation formula, we consider the numerical solution of the fractional ordinary differential equations (FODEs) by transforming FODEs into the associated weakly singular Volterra integral equations of the first kind. Some numerical illustrations are designed to depict the accuracy and versatility of the idea.

1. Introduction

Weakly singular integral equations appear in important applications of some real world mathematical models. As some applications of such these equations, one can refer to the potential problems, radiative equilibrium, fracture mechanics and Dirichlet problems [4, 11]. Because of the limitations of several traditional analytical methods for solving these equations, some new numerical methods have been proposed. For instance, Du et al [7] proposed the reproducing kernel method for solving Fredholm integro-differential equations (FIDEs) with weakly singularity. Lifanov et al [12] also suggested some new numerical methods are for solving singular integral equations. Moreover, the discrete Galerkin method, for solving FIDEs with weakly singular kernels, was provided by Pedas and Tamme [17]. However, although many other attempts have been considered for solving weakly singular integral equations, but only a few explore both numerical discussions and theoretical analysis such as [14, 15, 23].

In this paper we provide, by using the terminology of the generalized Euler-Maclaurin summation formula, the convergence analysis of the trapezoidal product quadrature rule for solving weakly singular Fredholm integral equations (FIEs) in the form of

$$u(t) = f(t) + \int_a^b K(t,x)u(x)dx, \qquad t \in [a,b], \tag{1}$$

with the assumption that the integral $\int_a^b K(t,x)dx$ exist. Such a problem has been considered in [5] with respect to numerics (via the generalized Euler-Maclaurin summation formula) but without any theoretical

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Email addresses: grzegorz.rzadkowski@pw.edu.pl (Grzegorz Rządkowski), emrantohidi@gmail.com (Emran Tohidi)

discussion. Therefore we will focus on the convergence analysis of the mentioned quadrature rule for solving (1) by using the terminology of the generalized Euler-Maclaurin summation formula. Moreover, we will study the rate of convergence of the weakly singular Volterra integral equations (VIEs) in the form of

$$u(t) = f(t) + \int_a^t K(t,x)u(x)dx, \qquad t \in [a,b]. \tag{2}$$

Also, since any fractional ordinary differential equation (FODE) can be considered as a weakly singular VIE of the first kind, we will apply the above-mentioned quadrature rule for approximating the solution of FODEs in the form of

$$D_*^\alpha u(t) = g(t) + L(u(t)), \qquad 0 < \alpha < 1, \qquad u(0) = u_0, \tag{3}$$

where $D_*^\alpha u(t)$ denotes the left Caputo fractional derivative for $u(t)$ and is defined as [24]

$$D_*^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} u^{(n)}(x)dx = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-x)^{-\alpha} u'(x)dx, \tag{4}$$

where $n = [\alpha] + 1 = 1$ and $L(u(t)) = \theta_1 u(t) + \theta_2$ is a linear known function in terms of $u(t)$ (i.e., θ_1 and θ_2 are known).

This paper is organized as follows. In Section 2 some preliminaries about the generalized Euler-Maclaurin quadrature rule will be reviewed. In Section 3, some useful lemmas associated with this quadrature rule together with the convergence analysis for equation (1) are provided. Convergence analysis associated with equation (2) is given in Section 4. An application of this summation formula for solving FODEs is considered in Section 5. Some numerical examples, for illustrating the accuracy and versatility of the presented idea, are given in Section 6. Conclusions are formulated in Section 7.

2. Preliminaries

For clarity of presentation, we will review some preliminaries about Bernoulli functions and the generalized Euler-Maclaurin summation formula in the following lines.

We assume that $F_0(x)$ is a given integrable function in the interval $[a,b]$. Let points $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ divide the interval into n subintervals. Throughout the paper, we will assume that the partition fulfills the following condition

$$x_i - x_{i-1} \le \frac{C}{n}, \tag{5}$$

where $i = 1, 2, 3, \ldots, n$ and $C \ge b - a$ is a constant independent on n.

The first order Bernoulli function $F_{1(k)}(x)$ (see [18], [19]), associated with $F_0(x)$ in the interval $[x_{k-1}, x_k]$ is defined as an anti-derivative of $F_0(x)$ such that

$$\int_{x_{k-1}}^{x_k} F_{1(k)}(x)dx = 0. \tag{6}$$

To calculate $F_{1(k)}(x)$ we should subtract, from an anti-derivative of $F_0(x)$, its mean value in the interval $[x_{k-1}, x_k]$. Thus we can express $F_{1(k)}(x)$ in the following form

$$F_{1(k)}(x) = \int_{x_{k-1}}^{x} F_0(t)dt - \frac{1}{x_k - x_{k-1}} \int_{x_{k-1}}^{x_k} (x_k - t)F_0(t)dt. \tag{7}$$

At the boundary of $[x_{k-1}, x_k]$ we have

$$F_{1(k)}(x_{k-1}) = -\frac{1}{x_k - x_{k-1}} \int_{x_{k-1}}^{x_k} (x_k - t)F_0(t)dt, \tag{8}$$

$$F_{1(k)}(x_k) = \frac{1}{x_k - x_{k-1}} \int_{x_{k-1}}^{x_k} (t - x_{k-1})F_0(t)dt. \tag{9}$$

From (8) and (9) or from (7) it follows immediately that

$$F_{1(k)}(x_k) - F_{1(k)}(x_{k-1}) = \int_{x_{k-1}}^{x_k} F_0(t)dt. \tag{10}$$

In a similar way we define, in the interval $[x_{k-1}, x_k]$, the second order Bernoulli function $F_{2(k)}(x)$ such that

$$F'_{2(k)}(x) = F_{1(k)}(x) \quad \text{and} \quad \int_{x_{k-1}}^{x_k} F_{2(k)}(x)dx = 0. \tag{11}$$

Using definition (11) and performing similar calculations we arrive at

$$F_{2(k)}(x) = \int_{x_{k-1}}^{x} (x-t)F_0(t)dt - \frac{1}{2(x_k - x_{k-1})}\int_{x_{k-1}}^{x_k} (x_k - t)(2x - x_{k-1} - t)F_0(t)dt. \tag{12}$$

By (12), the function $F_{2(k)}(x)$ takes equal values at the boundary of the interval $[x_{k-1}, x_k]$

$$F_{2(k)}(x_k) = F_{2(k)}(x_{k-1}) = \frac{1}{2(x_k - x_{k-1})}\int_{x_{k-1}}^{x_k} (x_k - t)(t - x_{k-1})F_0(t)dt. \tag{13}$$

Similarly by induction one can define, in the interval $[x_{k-1}, x_k]$, higher order Bernoulli functions. If $F_{m-1(k)}(x)$ is given, then we define $F_{m(k)}(x)$ by

$$F'_{m(k)}(x) = F_{m-1(k)}(x) \quad \text{and} \quad \int_{x_{k-1}}^{x_k} F_{m(k)}(x)dx = 0.$$

The following formula holds [18]

$$F_{m(k)}(x) = \frac{(x_k - x_{k-1})^{m-1}}{m!}\left[B_m\left(\frac{x - x_{k-1}}{x_k - x_{k-1}}\right)\int_{x_{k-1}}^{x_k} F_0(t)dt - \int_{x_{k-1}}^{x_k} F_0(t)B_m^*\left(\frac{x-t}{x_k - x_{k-1}}\right)dt\right], \tag{14}$$

where $B_m(x)$ is the mth Bernoulli polynomial defined in interval $[0, 1]$ and $B_m^*(x)$ is the periodic continuation of $B_m(x)$ i.e., $B_m^*(x) = B_m(x)$ for $x \in [0, 1]$ and $B_m^*(x+1) = B_m^*(x)$ if $x \in R$. Formula (14) has been used by Krylov [10], in the case of the whole interval $[a, b]$, for reducing the value of the approximation error.

From (14) it follows that for $m \geq 2$ the mth Bernoulli function takes equal values at endpoints of the interval $[x_{k-1}, x_k]$

$$F_{m(k)}(x_k) = F_{m(k)}(x_{k-1}) = \frac{(x_k - x_{k-1})^{m-1}}{m!}\int_{x_{k-1}}^{x_k} F_0(t)\left(B_m - B_m\left(\frac{x_k - t}{x_k - x_{k-1}}\right)\right)dt, \tag{15}$$

where $B_m = B_m(0)$ is the mth Bernoulli number.

It is easy to see that, starting from $F_0(x) = 1$, $x \in [0, 1]$, up to constant factors, the conventional Bernoulli polynomials can be defined.

Let us consider integral

$$\int_a^b F_0(x)g(x)dx = \sum_{k=1}^{n}\int_{x_{k-1}}^{x_k} F_0(x)g(x)dx, \tag{16}$$

where $g(x)$ is m times continuously differentiable in the interval $[a, b]$. Integrating in (16), m times by parts in each interval $[x_{k-1}, x_k]$, then summing up over $k = 1, 2, \ldots, n$ and using (15) we obtain the following form of the generalized Euler-Maclaurin summation formula of the mth order

$$\int_a^b F_0(x)g(x)dx = F_{1(n)}(b)g(b) - F_{1(1)}(a)g(a) + \sum_{k=1}^{n-1}(F_{1(k)}(x_k) - F_{1(k+1)}(x_k))g(x_k)$$
$$+ \sum_{j=2}^{m}(-1)^{j-1}\sum_{k=1}^{n} F_{j(k)}(x_k)(g^{(j-1)}(x_k) - g^{(j-1)}(x_{k-1})) + (-1)^m \sum_{k=1}^{n}\int_{x_{k-1}}^{x_k} F_{m(k)}(x)g^{(m)}(x)dx. \tag{17}$$

The idea of the generalized Euler-Maclaurin summation formula (17) goes back to Kronecker [9] in 1885, where the Bernoulli functions are expressed by the Fourier series.

If $m = 2$ then (17) gives

$$\int_a^b F_0(x)g(x)dx = F_{1(n)}(b)g(b) - F_{1(1)}(a)g(a) + \sum_{k=1}^{n-1}(F_{1(k)}(x_k) - F_{1(k+1)}(x_k))g(x_k)$$
$$- \sum_{k=1}^{n} F_{2(k)}(x_k)(g'(x_k) - g'(x_{k-1})) + \sum_{k=1}^{n}\int_{x_{k-1}}^{x_k} F_{2(k)}(x)g''(x)dx. \tag{18}$$

The expression

$$Q_n(F_0, g) := F_{1(n)}(b)g(b) - F_{1(1)}(a)g(a) + \sum_{k=1}^{n-1}(F_{1(k)}(x_k) - F_{1(k+1)}(x_k))g(x_k), \tag{19}$$

in (18) is a quadrature for integrals $\int_D F_0(x)g(x)dx$ over $D = [a, b]$.

Let us observe that quadrature (19) is in fact the trapezoidal product integration rule. In order to see this, we will express it in an integral form. For each term of the quadrature, using formulas (8) and (9) we get

$$\begin{aligned} F_{1(k)}(x)g(x)|_{x_{k-1}}^{x_k} &= F_{1(k)}(x_k)g(x_k) - F_{1(k)}(x_{k-1})g(x_{k-1}) \\ &= \frac{g(x_k)}{x_k - x_{k-1}}\int_{x_{k-1}}^{x_k}(t - x_{k-1})F_0(t)dt + \frac{g(x_{k-1})}{x_k - x_{k-1}}\int_{x_{k-1}}^{x_k}(x_k - t)F_0(t)dt \\ &= \int_{x_{k-1}}^{x_k}\frac{(t - x_{k-1})g(x_k) + (x_k - t)g(x_{k-1})}{x_k - x_{k-1}}F_0(t)dt. \end{aligned} \tag{20}$$

We see that under the integral sign in (20) the function $F_0(t)$ is multiplied by a linear function joining points $(x_{k-1}, g(x_{k-1}))$ and $(x_k, g(x_k))$. This proves the similarity of our quadrature with the trapezoidal product integration method. Some other applications of the generalized Euler-Maclaurin summation formula, for different meshes, are given in [20].

In next sections the following lemma will be needed and for convenience of the reader we put it here.

Lemma 2.1. *Assume that a function $h(x)$ is continuous in the closed interval $[c, d]$ and has a continuous and positive second derivative in the open interval (c, d). If moreover*

$$\int_c^d h(x)dx = 0$$

and $h(c) = h(d)$, then

$$\max_{x\in[c,d]} |h(x)| = h(c) = h(d). \tag{21}$$

Proof. See [19].

3. Weakly singular Fredholm integral equations

In [19] we have proved, that if $F_0(x)$ is a continuous and positive function and also $g(x)$ is twice continuously differentiable in D then $Q_n(F_0, g)$ is convergent to the integral with the error $O(1/n^2)$. We will prove now that the quadrature is convergent in the case when $F_0(x)$ is absolutely integrable and $g(x)$ is continuous in D.

Lemma 3.1. *If $F_0(x)$ is absolutely integrable in $[a, b]$ then for any function $u(x)$ being continuous in $[a, b]$ the quadrature $Q_n(F_0, u)$ is convergent to $\int_a^b F_0(x)u(x)dx$.*

Proof. Since $u(x)$ is uniformly continuous in the closed interval $[a, b]$ then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|t_2 - t_1| < \delta$ then $|u(t_2) - u(t_1)| < \varepsilon$. Suppose the mesh fulfills the condition that $(x_k - x_{k-1}) < \delta$ for any $k = 1, 2, \ldots, n$. Then in any subinterval $[x_{k-1}, x_k]$, using (8) and (9), we have:

$$\begin{aligned}
&\Big| \int_{x_{k-1}}^{x_k} F_0(x)u(x)dx - F_{1(k)}(x)u(x)\big|_{x_{k-1}}^{x_k} \Big| \\
&\quad = \Big| \int_{x_{k-1}}^{x_k} F_0(x)\Big[\frac{x - x_{k-1}}{x_k - x_{k-1}}(u(x) - u(x_k)) + \frac{x_k - x}{x_k - x_{k-1}}(u(x) - u(x_{k-1}))\Big]dx\Big| \\
&\quad \le \int_{x_{k-1}}^{x_k} |F_0(x)|\Big[\frac{x - x_{k-1}}{x_k - x_{k-1}}|u(x) - u(x_k)| + \frac{x_k - x}{x_k - x_{k-1}}|u(x) - u(x_{k-1})|\Big]dx \le \varepsilon \int_{x_{k-1}}^{x_k} |F_0(x)|dx.
\end{aligned}$$

Summing up the above inequalities we get

$$\Big| \int_a^b F_0(x)u(x)dx - Q_n(F_0, u)\Big| \le \varepsilon \int_a^b |F_0(x)|dx. \tag{22}$$

□

Before presenting the next lemma, we denote by $F^*_{1(k)}(x)$ the first Bernoulli function associated with $|F_0(x)|$ in the interval $[x_{k-1}, x_k]$, $(k = 1, 2, \ldots, n)$.

Lemma 3.2. *Suppose that a function $F_0(x)$ is absolutely integrable in the interval $[x_{k-1}, x_k]$. Then we have:*

$$|F_{1(k)}(x_{k-1})| \le -F^*_{1(k)}(x_{k-1}), \quad |F_{1(k)}(x_k)| \le F^*_{1(k)}(x_k). \tag{23}$$

Proof. The inequalities (23) are the immediate consequence of formulas (8) and (9):

$$\begin{aligned}
|F_{1(k)}(x_{k-1})| &\le \frac{1}{x_k - x_{k-1}} \int_{x_{k-1}}^{x_k} (x_k - x)|F_0(x)|dx = -F^*_{1(k)}(x_{k-1}), \\
|F_{1(k)}(x_k)| &\le \frac{1}{x_k - x_{k-1}} \int_{x_{k-1}}^{x_k} (x - x_{k-1})|F_0(x)|dx = F^*_{1(k)}(x_k).
\end{aligned}$$

□

In the next Corollary, we consider the case of two adjacent intervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$ and a function $F_0(x)$, absolutely integrable in the interval $[x_{k-1}, x_{k+1}]$.

Corollary 3.3. *For the common point x_k of the two intervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$, the following estimation holds:*

$$|F_{1(k)}(x_k) - F_{1(k+1)}(x_k)| \le F^*_{1(k)}(x_k) - F^*_{1(k+1)}(x_k). \tag{24}$$

Proof. From Lemma 3.2 it follows that

$$|F_{1(k)}(x_k)| \le F^*_{1(k)}(x_k), \quad |F_{1(k+1)}(x_k)| \le -F^*_{1(k+1)}(x_k).$$

Thus

$$|F_{1(k)}(x_k) - F_{1(k+1)}(x_k)| \le |F_{1(k)}(x_k)| + |F_{1(k+1)}(x_k)| \le F^*_{1(k)}(x_k) - F^*_{1(k+1)}(x_k).$$

□

Corollary 3.4. *The sum of the weights of quadrature (19) is uniformly bounded with the following estimation*

$$|F_{1(n)}(x_n)| + |F_{1(1)}(x_0)| + \sum_{k=1}^{n-1} |F_{1(k)}(x_k) - F_{1(k+1)}(x_k)| \leq \int_a^b |F_0(t)|dt. \tag{25}$$

Proof. From Lemma 3.2 and Corollary 3.3 we get the following estimation for the weights of the quadrature (19):

$$|F_{1(n)}(x_n) - F_{1(1)}(x_0) + \sum_{k=1}^{n-1}(F_{1(k)}(x_k) - F_{1(k+1)}(x_k))| \leq |F_{1(n)}(x_n)| + |F_{1(1)}(x_0)| + \sum_{k=1}^{n-1} |F_{1(k)}(x_k) - F_{1(k+1)}(x_k)|$$
$$\leq F^*_{1(n)}(x_n) - F^*_{1(1)}(x_0) + \sum_{k=1}^{n-1}(F^*_{1(k)}(x_k) - F^*_{1(k+1)}(x_k)) = \sum_{k=1}^{n}(F^*_{1(k)}(x_k) - F^*_{1(k)}(x_{k-1}))$$
$$= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |F_0(t)|dt = \int_a^b |F_0(t)|dt.$$

□

At this stage, we consider again the weakly singular FIE (1) with this assumption that the integral $\int_a^b K(t,x)dx$ does exist, possibly as an improper integral, and is continuous as a function of $t \in D = [a,b]$. It should be recalled that equation (1) has been considered in [5] just in the numerical aspect and we do not recall the discretization process and just focus on the convergence analysis concepts.

The integral $(Ku)(t) := \int_a^b K(t,x)u(x)dx$ defines the operator $K: \ u \to Ku$, $(K \in L(C(D), C(D)))$. We will approximate the integral $\int_a^b K(t,x)u(x)dx$ for a fixed $t \in [a,b]$ by quadrature (19) for $F_0(x) = K(t,x)$. In this particular case we denote the quadrature by $Q_n = Q_n(K(t,x), u(x))$ and by $F_{1(k)}(t,x)$ the first order Bernoulli function associated with $K(t,x)$ in the kth subinterval $[x_{k-1}, x_k]$. The quadrature Q_n defines also the operator $K_n \in L(C(D), C(D))$, $K_n = (K_n u)(t) = Q_n(K(t,\cdot), u(\cdot))$.

Corollary 3.4 implies the stability of the quadrature Q_n, $\|Q_n(t)\| \leq \int_a^b |K(t,x)|dx$ and shows the uniform boundedness of the operators $K_n(u)(t)$:

$$\|K_n\| \leq \sup_{t\in D} \int_a^b |K(t,x)|dx,$$

where $\|\cdot\|$ denotes the l_1 norm.

The following Lemma 3.5 shows the equicontinuity property of the set of operators $\{K_n u : \ n \in N\}$.

Lemma 3.5. *The following estimation holds for $t_1, t_2 \in D$*

$$|(K_n u)(t_2) - (K_n u)(t_1)| \leq 2\int_a^b |K(t_2,x) - K(t_1,x)|dx \cdot \|u(x)\|_\infty. \tag{26}$$

Proof. Using formula (9) we obtain for any $k = 1, 2, \ldots, n$

$$F_{1(k)}(t_2, x_k) - F_{1(k)}(t_1, x_k) = \int_{x_{k-1}}^{x_k} \frac{x - x_{k-1}}{x_k - x_{k-1}}(K(t_2,x) - K(t_1,x))dx,$$

which gives the following estimation

$$|F_{1(k)}(t_2, x_k) - F_{1(k)}(t_1, x_k)| \leq \int_{x_{k-1}}^{x_k} |K(t_2,x) - K(t_1,x)|dx. \tag{27}$$

Similar estimation is provided by formula (8):

$$|F_{1(k)}(t_2, x_{k-1}) - F_{1(k)}(t_1, x_{k-1})| \leq \int_{x_{k-1}}^{x_k} |K(t_2, x) - K(t_1, x)| dx. \tag{28}$$

Then (27) and (28) lead to (26):

$$\begin{aligned}
&|K_n(u)(t_2) - K_n(u)(t_1)| = |\left(-F_{1(1)}(t_2, x_0) + F_{1(1)}(t_1, x_0)\right) u(x_0) \\
&+ \sum_{k=1}^{n-1} \left(F_{1(k)}(t_2, x_k) - F_{1(k)}(t_1, x_k) - F_{1(k+1)}(t_2, x_k) + F_{1(k+1)}(t_1, x_k)\right) u(x_k) + \left(F_{1(n)}(t_2, x_n) - F_{1(n)}(t_1, x_n)\right) u(x_n)| \\
&\leq \sum_{k=1}^{n} | - F_{1(k)}(t_2, x_{k-1}) + F_{1(k)}(t_1, x_{k-1})| \cdot |u(x_{k-1})| + \sum_{k=1}^{n} |F_{1(k)}(t_2, x_k) - F_{1(k)}(t_1, x_k)| \cdot |u(x_k)| \\
&\leq 2 \int_a^b |K(t_2, x) - K(t_1, x)| dx \cdot \|u(x)\|_\infty.
\end{aligned}$$

□

At point $t = x_i$ we get

$$\begin{aligned}
u(x_i) &= f(x_i) + \sum_{j=1}^{n} F_{1(j)}(x_i, x) u(x)|_{x_{j-1}}^{x_j} + r_n(x_i) = f(x_i) + F_{1(n)}(x_i, x_n) u(x_n) \\
&- F_{1(1)}(x_i, x_0) u(x_0) + \sum_{j=1}^{n-1} (F_{1(j)}(x_i, x_j) - F_{1(j+1)}(x_i, x_j)) u(x_j) + r_n(x_i),
\end{aligned} \tag{29}$$

where

$$r_n(x_i) = \int_a^b K(x_i, x) u(x) dx - \sum_{j=1}^{n} F_{1(j)}(x_i, x) u(x)|_{x_{j-1}}^{x_j}.$$

Let $(u_n(x_0), u_n(x_1), \ldots, u_n(x_n))$ be an approximate solution of the following system of linear equations ($i = 0, 1, 2, \ldots, n$):

$$u_n(x_i) = f(x_i) + F_{1(n)}(x_i, x_n) u_n(x_n) - F_{1(1)}(x_i, x_0) u_n(x_0) + \sum_{j=1}^{n-1} (F_{1(j)}(x_i, x_j) - F_{1(j+1)}(x_i, x_j)) u_n(x_j). \tag{30}$$

If the values $(u_n(x_0), u_n(x_1), \ldots, u_n(x_n))$ are known then we can approximate the value $u(t)$ at any $t \in D$ by $u_n(t)$ given by the formula:

$$u_n(t) = f(t) + F_{1(n)}(t, x_n) u_n(x_n) - F_{1(1)}(t, x_0) u_n(x_0) + \sum_{j=1}^{n-1} (F_{1(j)}(t, x_j) - F_{1(j+1)}(t, x_j)) u_n(x_j).$$

Subtracting (29) from (30) we have

$$\begin{aligned}
u_n(x_i) - u(x_i) =& F_{1(n)}(x_i, x_n)(u_n(x_n) - u(x_n)) - F_{1(1)}(x_i, x_0)(u_n(x_0) - u(x_0)) \\
&+ \sum_{j=1}^{n-1} (F_{1(j)}(x_i, x_j) - F_{1(j+1)}(x_i, x_j))(u_n(x_j) - u(x_j)) - r_n(x_i).
\end{aligned}$$

Lemma 3.5 shows moreover that the set of operators $\{K_n\}$ is collectively compact, i.e., the closure of the set $\{K_n u : n \in N, \|u\|_\infty \leq 1\}$ is compact (see Hackbusch [8], p. 140). We can use now the results of Anselone–Moore [2] , Anselone [1] and Brakhage [6] (see Hackbusch [8], Theorem 4.7.11, p. 135). From them it follows that, if 1 is a regular value of the operator K, then there exists, for all $n \geq n_0$ (for a sufficiently large n_0), the operator $(I - K_n)^{-1}$ and the following error estimate holds

$$\|u - u_n\|_\infty \leq \|(I - K_n)^{-1}\| \, \|(K - K_n)u\|_\infty, \tag{31}$$

where $\|(I - K_n)^{-1}\|$ is the operator norm. Thus inequality (31) proves the requested convergence in the case of the weakly singular Fredholm integral equations.

In the case of the kernel function of the form $K(t,x) = \frac{1}{|t-x|^\alpha}$ or $K(t,x) = \frac{1}{(b-x)^\alpha}$, assuming that $u(x)$ is of class $C^2[a,b]$ one can perform the same calculations as we have done in the next section (see estimation (47) and earlier steps) for the weakly singular Volterra integral equation. From these it follows that the rate of convergence in this case is $O(1/n^2)$.

In the next section, we will study the convergence rate of the generalized Euler-Maclaurin summation formula for solving weakly singular Volterra integral equations in the form of (2).

4. Weakly singular Volterra integral equations

We again consider equation (2). For clarity of presentation and in the sequel, we assume that the kernel function has the form of $K(t,x) = (t-x)^{-\alpha}$, where $0 < \alpha < 1$. Convergence analysis with other kernels may be done by a similar approach. For applying the quadrature rule (19), we should construct the first order Bernoulli functions associated with the above-mentioned kernel $K(t,x)$. Therefore by (7) we have for $x \in [x_{i-1}, x_i]$ and $1 \leq k \leq n$:

$$\begin{aligned} F_{1(i)}(x_k, x) &= \int_{x_{i-1}}^{x} K(x_k, \xi)d\xi - \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} (x_i - \xi)K(x_k, \xi)d\xi \\ &= \int_{x_{i-1}}^{x} (x_k - \xi)^{-\alpha} d\xi - \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} (x_i - \xi)(x_k - \xi)^{-\alpha} d\xi, \end{aligned} \tag{32}$$

and by (8), (9) the values of $F_{1(i)}(x_k, x)$, at the boundary of $[x_{i-1}, x_i]$, are as follows

$$F_{1(i)}(x_k, x_{i-1}) = -\frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} (x_i - x)(x_k - x)^{-\alpha} dx \tag{33}$$

$$F_{1(i)}(x_k, x_i) = \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} (x - x_{i-1})(x_k - x)^{-\alpha} dx. \tag{34}$$

Now, collocating equation (2) at the nodes $t = x_k, \ k = 1, 2, \ldots, n$, yields

$$\begin{aligned} u(x_1) &= f(x_1) + \int_a^{x_1} K(x_1, x)u(x)dx, \\ u(x_2) &= f(x_2) + \int_a^{x_2} K(x_2, x)u(x)dx, \\ &\vdots \\ u(x_n) &= f(x_n) + \int_a^{x_n} K(x_n, x)u(x)dx. \end{aligned} \tag{35}$$

At this stage we will apply the quadrature rule (19) for approximating the integral terms $\int_a^{x_i} K(x_i,x)u(x)dx$ for $i = 1,2,\ldots,n$, in equations (35). Thus

$$\begin{aligned}
u(x_1) &\approx f(x_1) + F_{1(1)}(x_1,x_1)u(x_1) - F_{1(1)}(x_1,a)u(a),\\
u(x_2) &\approx f(x_2) + F_{1(2)}(x_2,x_2)u(x_2) + \Big(F_{1(1)}(x_2,x_1) - F_{1(2)}(x_2,x_1)\Big)u(x_1) - F_{1(1)}(x_2,a)u(a),\\
&\vdots\\
u(x_n) &\approx f(x_n) + F_{1(n)}(x_n,x_n)u(x_n) + \sum_{i=1}^{n-1}\Big(F_{1(i)}(x_n,x_i) - F_{1(i+1)}(x_n,x_i)\Big)u(x_i) - F_{1(1)}(x_n,a)u(a).
\end{aligned} \tag{36}$$

It should be noted that $u(a) = f(a)$. We assume that u_i is an approximation of $u(x_i)$ for $i = 1,2,\ldots,n$, and hence one can rewrite equations (36) in the following form

$$\begin{aligned}
u_1 &= \Big(f(x_1) - F_{1(1)}(x_1,a)f(a)\Big) + F_{1(1)}(x_1,x_1)u_1,\\
u_2 &= \Big(f(x_2) - F_{1(1)}(x_2,a)f(a)\Big) + \Big(F_{1(1)}(x_2,x_1) - F_{1(2)}(x_2,x_1)\Big)u_1 + F_{1(2)}(x_2,x_2)u_2,\\
&\vdots\\
u_n &= \Big(f(x_n) - F_{1(1)}(x_n,a)f(a)\Big) + \sum_{i=1}^{n-1}\Big(F_{1(i)}(x_n,x_i) - F_{1(i+1)}(x_n,x_i)\Big)u_i + F_{1(n)}(x_n,x_n)u_n.
\end{aligned} \tag{37}$$

(38)

System (37) can be rewritten in the matrix form $U = F + AU$, where $U = [u_1\ u_2\ \cdots\ u_n]^T$, $F = [f(x_1) - F_{1(1)}(x_1,a)f(a)\quad f(x_2) - F_{1(1)}(x_2,a)f(a)\quad \cdots\quad f(x_n) - F_{1(1)}(x_n,a)f(a)]^T$ and

$$A = \begin{bmatrix}
F_{1(1)}(x_1,x_1) & 0 & \cdots & 0\\
F_{1(1)}(x_2,x_1) - F_{1(2)}(x_2,x_1) & F_{1(2)}(x_2,x_2) & \cdots & 0\\
\vdots & \ddots & \ddots & \vdots\\
F_{1(1)}(x_n,x_1) - F_{1(2)}(x_n,x_1) & \cdots & \cdots & F_{1(n)}(x_n,x_n)
\end{bmatrix}_{n\times n}. \tag{39}$$

The above-mentioned algebraic system can be solved by some suitable iterative methods and so we have an approximation for the solution of (2). We now turn to the convergence analysis subject associated with the above discretization.

Theorem 1. Suppose that $K(t,x) = (t-x)^{-\alpha}$, where α is a given real constant in the interval (0,1) and also $f(t)$ a given function in equation (2). If for approximating the solution of (2), we use the generalized Euler-Maclaurin summation formula (19) and if the solution of the algebraic system (37) is the approximate solution of (2), then the convergence rate is 2. In other words, $|u(x_k) - u_k| \le c/n^2$ for a constant c independent of n.

Proof. By using (33) and (34), for $i = 1,2,\cdots,k-1$ $(k \le n)$ we get

$$\begin{aligned}
C_{ki} :=& F_{1(i)}(x_k,x_i) - F_{1(i+1)}(x_k,x_i) = \int_{x_{i-1}}^{x_i}\frac{(x-x_{i-1})(x_k-x)^{-\alpha}}{x_i - x_{i-1}}dx + \int_{x_i}^{x_{i+1}}\frac{(x_{i+1}-x)(x_k-x)^{-\alpha}}{x_{i+1}-x_i}dx\\
&\le (x_{i+1} - x_{i-1})(x_k - x_i)^{-\alpha}.
\end{aligned} \tag{40}$$

The inequality in (40) holds because the integrand of the first integral is increasing for $x \in [x_{i-1}, x_i]$ and the integrand of the second integral is decreasing as a function of $x \in [x_i, x_{i+1}]$. Therefore the maximum of both the integrands is attained at point $x = x_i$ and equals $(x_k - x_i)^{-\alpha}$.

Moreover, since

$$F_{1(k)}(x_k,x_k) = \frac{1}{x_k - x_{k-1}}\int_{x_{k-1}}^{x_k}(x-x_{k-1})(x_k-x)^{-\alpha}dx = \frac{(x_k-x_{k-1})^{1-\alpha}}{(2-\alpha)(1-\alpha)},$$

then we choose sufficiently large number n that, under condition (5), $F_{1(k)}(x_k, x_k) < 1/2$ i.e.,

$$n > C\left(\frac{2}{(2-\alpha)(1-\alpha)}\right)^{1/(1-\alpha)}. \tag{41}$$

Assuming that $u(x)$ is of class $C^2[a, b]$, we will now estimate the difference

$$A_{ki} := \int_{x_{i-1}}^{x_i} K(x_k, x)u(x)dx - F_{1(i)}(x_k, x)u(x)|_{x_{i-1}}^{x_i} = -F_{2(i)}(x_k, x)u'(x)|_{x_{i-1}}^{x_i} + \int_{x_{i-1}}^{x_i} F_{2(i)}(x_k, x)u''(x)dx, \tag{42}$$

for $i = 1, 2, \ldots, k$. Formula (13) gives

$$F_{2(i)}(x_k, x_i) = F_{2(i)}(x_k, x_{i-1}) = \frac{1}{2(x_i - x_{i-1})}\int_{x_{i-1}}^{x_i} (x_i - x)(x - x_{i-1})(x_k - x)^{-\alpha}dx > 0,$$

and since $0 \le (x_i - x)(x - x_{i-1}) \le (x_i - x_{i-1})^2/4$ for $x \in [x_{i-1}, x_i]$ then

$$F_{2(i)}(x_k, x_i) = F_{2(i)}(x_k, x_{i-1}) \le \frac{(x_i - x_{i-1})}{8}\int_{x_{i-1}}^{x_i} (x_k - x)^{-\alpha}dx. \tag{43}$$

Thus inequality (43) implies the following estimation for the first term on the right hand side of the equation (42)

$$\begin{aligned}&|-F_{2(i)}(x_k, x)u'(x)|_{x_{i-1}}^{x_i}| = |F_{2(i)}(x_k, x_i)u'(x_i) - F_{2(i)}(x_k, x_{i-1})u'(x_{i-1})| = F_{2(i)}(x_k, x_i)|u'(x_i) - u'(x_{i-1})|\\ &\le \max_{x_{i-1}\le x\le x_i} |u''(x)|(x_i - x_{i-1})F_{2(i)}(x_k, x_i) \le \max_{x_{i-1}\le x\le x_i} |u''(x)|\frac{(x_i - x_{i-1})^2}{8}\int_{x_{i-1}}^{x_i} (x_k - x)^{-\alpha}dx.\end{aligned} \tag{44}$$

Similarly we can proceed with the second term on the right hand side of the equation (42). From Lemma 2.1 we know that function $F_{2(i)}(x_k, x)$ defined in the interval $[x_{i-1}, x_i]$ takes its maximal absolute value at points x_{i-1} and x_i i.e., $\max_{x_{i-1}\le x\le x_i} |F_{2(i)}(x_k, x)| = F_{2(i)}(x_k, x_{i-1}) = F_{2(i)}(x_k, x_i) > 0$. Therefore using (43) we have

$$|\int_{x_{i-1}}^{x_i} F_{2(i)}(x_k, x)u''(x)dx| \le \int_{x_{i-1}}^{x_i} |F_{2(i)}(x_k, x)||u''(x)|dx \le \max_{x_{i-1}\le x\le x_i} |u''(x)|\frac{(x_i - x_{i-1})^2}{8}\int_{x_{i-1}}^{x_i} (x_k - x)^{-\alpha}dx. \tag{45}$$

Using (44) and (45) we obtain the requested estimate for the difference in (42)

$$|A_{ki}| \le \max_{x_{i-1}\le x\le x_i} |u''(x)|\frac{(x_i - x_{i-1})^2}{4}\int_{x_{i-1}}^{x_i} (x_k - x)^{-\alpha}dx. \tag{46}$$

Therefore, to estimate the value of

$$A_k := \sum_{i=1}^{k} A_{ki} = \int_a^{x_k} K(x_k, x)u(x)dx - \sum_{i=1}^{k} F_{1(i)}(x_k, x)u(x)|_{x_{i-1}}^{x_i},$$

we sum up inequalities (46) for $i = 1, 2, \ldots, k$ and use condition (5) for the partition. We arrive at

$$|A_k| \le \sum_{i=1}^{k} |A_{ki}| \le \max_{a\le x\le x_k} |u''(x)| \cdot \frac{C^2}{4n^2}\int_a^{x_k} (x_k - x)^{-\alpha}dx = \max_{a\le x\le x_k} |u''(x)|\frac{C^2}{4n^2(1-\alpha)}(x_k - a)^{1-\alpha}. \tag{47}$$

Trivially for the error at the point $t = x_1$, we have

$$\begin{aligned}u_1 &= f(x_1) + \left(F_{1(1)}(x_1, x_1)u_1 - F_{1(1)}(x_1, x_0)u(a)\right),\\ u(x_1) &= f(x_1) + \int_a^{x_1} K(x_1, x)u(x)dx.\end{aligned}$$

Subtracting the first equation from the second one yields

$$u(x_1) - u_1 = \int_a^{x_1} K(x_1, x)u(x)dx - \left(F_{1(1)}(x_1, x_1)u_1 - F_{1(1)}(x_1, x_0)u(a)\right). \tag{48}$$

We rewrite equation (48) in the form of

$$u(x_1) - u_1 = \left[\int_a^{x_1} K(x_1, x)u(x)dx - \left(F_{1(1)}(x_1, x_1)u(x_1) - F_{1(1)}(x_1, x_0)\overbrace{u(a)}^{u(x_0)}\right)\right] + F_{1(1)}(x_1, x_1)(u(x_1) - u_1)$$

or denoting $e_k = u(x_k) - u_k$ $(k = 1, 2, \ldots, n)$ and using our notation

$$e_1 = A_1 + F_{1(1)}(x_1, x_1)e_1$$

For the error at the point $t = x_2$, the following equations hold

$$u_2 = f(x_2) + F_{1(2)}(x_2, x_2)u_2 + \left(F_{1(1)}(x_2, x_1) - F_{1(2)}(x_2, x_1)\right)u_1 - F_{1(1)}(x_2, x_0)u(a)$$

$$u(x_2) = f(x_2) + \int_a^{x_2} K(x_2, x)u(x)dx.$$

In other words

$$\begin{aligned} u(x_2) - u_2 =& \left[\int_a^{x_2} K(x_2, x)u(x)dx - F_{1(2)}(x_2, x_2)u(x_2) - (F_{1(1)}(x_2, x_1) - F_{1(2)}(x_2, x_1))u(x_1) + F_{1(1)}(x_2, x_0)u(0)\right] \\ &+ (F_{1(1)}(x_2, x_1) - F_{1(2)}(x_2, x_1))(u(x_1) - u_1) + F_{1(2)}(x_2, x_2)(u(x_2) - u_2), \end{aligned}$$

which can be written as

$$e_2 = A_2 + C_{21}e_1 + F_{1(2)}(x_2, x_2)e_2.$$

Analogously for the error at the point $t = x_k$, $(k = 1, 2, \ldots, n)$ we get the following equation

$$e_k = A_k + \sum_{i=1}^{k-1} C_{ki}e_i + F_{1(k)}(x_k, x_k)e_k. \tag{49}$$

Similar system of equations has been regarded by Lü Tao and Huang Yong [21] (see their Theorem 1, p. 59) in the case of the Navot [16] and Lyness [13] quadrature formula. The authors proved a generalization of the Gronwall inequality, namely, in our notation, if in the system of inequalities

$$|e_k| \le A + \sum_{i=1}^{k-1} B_{ki}|e_i|, \quad k = 1, 2, \ldots, n$$

the coefficients $B_{ki} = const(x_k - x_i)^{-\alpha}/n$, then for any k, $|e_k| \le AH$, where a constant H does not depend on n. For the above mentioned quadrature formula and the same kernel function, Tao and Yong obtained the rate of convergence $|e_k| = O(n^{-2+\alpha})$.

Using (47) and taking an n according to (41) we see that in our case of the system (49) we could put

$$A = \max_{1\le k\le n} \frac{|A_k|}{1 - F_{1(k)}(x_k, x_k)} \le 2 \max_{1\le k\le n} |A_k| \le \max_{a\le x\le b} |u''(x)| \frac{C^2}{2n^2(1-\alpha)}(b-a)^{1-\alpha}$$

and by (40)

$$B_{ki} = \frac{C_{ki}}{1 - F_{1(k)}(x_k, x_k)} \le 2C_{ki} \le \frac{4C}{n}(x_k - x_i)^{-\alpha}.$$

Thus from the proof of Theorem 1 by Tao and Yong [21] it follows that for any $k = 1, 2, \ldots, n$ the error in (49) is $|e_k| \le cn^{-2}$ for a constant c independent of n. □

5. Fractional differential equations

In this section, we will obtain a numerical solution for the FODE (3) via the Euler-Maclaurin quadrature rule. For this purpose, we should first transform (3) into the associated weakly singular VIE of the first kind. FODE (3) can be rewritten as

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-x)^{-\alpha}u'(x)dx = g(t)+\theta_1 u(t)+\theta_2, \quad 0\le\alpha\le 1, \quad u(0)=u_0. \tag{50}$$

Now, we assume that $y(t)=u'(t)$. Integrating from this relation in the interval $[0,t]$ yields

$$u(t)=\int_0^t y(x)dx+u_0. \tag{51}$$

By using (51), one can rewrite (50) in the following form

$$\int_0^t \hat{K}(t,x)y(x)dx = f(t), \tag{52}$$

where

$$f(t)=g(t)+\theta_1 u_0+\theta_2, \quad \hat{K}(t,x)=\left(\frac{1}{\Gamma(1-\alpha)}(t-x)^{-\alpha}-\theta_1\right). \tag{53}$$

Equation (52) is a weakly singular VIE of the first kind and can be solved by a similar approach that is provided in previous section.

6. Numerical examples

In this part of the paper, we will provide four test problems to show the efficiency of the considered numerical scheme and all of these examples are designed on a laptop PC using programs written in MATLAB 2015b. In this regard, we have reported in tables the values of the maximum of the absolute error function $e_n(x)=\left|u(x)-u_n(x)\right|$, which is denoted by e_n, (or its logarithm in figures) at the considered uniform or nonuniform meshes of the given interval (e.g., $[0,1]$). It should be noted that in first test problem we consider a system of weakly singular FIE in which our results are more accurate with respect to the trapezoidal rule [3]. Also, since the trapezoidal rule can not solve these singular equations in the interval [0,1], the associated results of this classical scheme have been done in the interval [0.05,0.95]. In the second test problem, we consider a weakly singular VIE and in the third example we provide the numerical solutions of a FODE. Moreover, in the fourth test problem we introduce two nonuniform meshes and compare them with the uniform one.

The first order Bernoulli function $F_{1(k)}(t,x)$ in the second example for the kernel function $K(t,x)=\frac{1}{\sqrt{t-x}}$ is given by

$$F_{1(k)}(t,x)=2(\sqrt{t-x_{k-1}}-\sqrt{t-x})-\frac{6x_k\sqrt{t-x_{k-1}}-4t\sqrt{t-x_{k-1}}-2x_{k-1}\sqrt{t-x_{k-1}}+4t\sqrt{t-x_k}-4x_k\sqrt{t-x_k}}{3(x_k-x_{k-1})}, \tag{54}$$

where $1\le k\le n$.

To get the first order Bernoulli function for the first and fourth examples, where the kernel is $K(t,x)=\frac{1}{\sqrt{1-x}}$, we put $t=1$ in (54). Similarly, to calculate it for the third example for the kernel $K(t,x)=\frac{1}{\Gamma(0.5)\sqrt{t-x}}+1$ we divide (54) by $\Gamma(0.5)=1.7724538509055$ and add the first order Bernoulli function of the constant function 1, which equals $x-\frac{x_{k-1}+x_k}{2}$.

Table 1: Numerical results of the first example

n	e_n of PM for $u_1(t)$	e_n of PM for $u_2(t)$	e_n of TR for $u_1(t)$	e_n of TR for $u_2(t)$
2	2.2179e-01	1.6274e-01	6.8510e-01	3.1458e-01
8	1.5863e-03	8.1953e-04	6.5012e-01	3.1003e-01
16	4.0994e-04	2.2146e-04	6.3125e-01	2.4198e-01
64	2.6590e-05	8.1776e-06	5.8547e-01	2.3000e-01
128	6.7171e-06	1.9624e-06	5.7219e-01	2.1034e-01

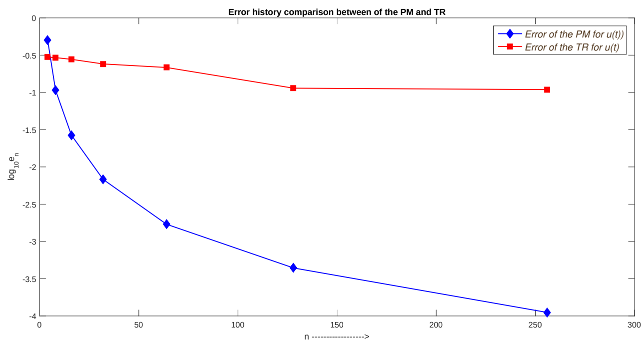

Figure 1: Error history comparison between of the PM and TR for different values of n

Example 1. As the first test problem, we consider the following system of weakly singular FIE with the exact solutions $u_1(t) = t$ and $u_2(t) = e^t$

$$u_1(t) = t - 5.3934 + \int_0^1 \frac{u_1(x)}{\sqrt{1-x}}dx + \int_0^1 \frac{u_2(x)}{\sqrt{1-x}}dx,$$
$$u_2(t) = e^t - 5.3934 + \int_0^1 \frac{u_1(x)}{\sqrt{1-x}}dx + \int_0^1 \frac{u_2(x)}{\sqrt{1-x}}dx.$$

We apply the considered numerical method for solving the above system of weakly singular FIEs. The associated results of the presented method (PM) together with the results of the trapezoidal rule (TR) are provided in Table 1. Superior results of our proposed scheme with respect to the TR can be seen in Table 1.

Example 2. As the second test problem, we consider the following weakly singular VIE with the exact solution $u(t) = t^2$

$$u(t) = t^2 - \frac{16}{15}t^{5/2} + \int_0^t \frac{u(x)}{\sqrt{t-x}}dx.$$

Again, we implement the generalized Euler-Maclaurin quadrature rule for solving the above weakly singular VIE. The associated results of the PM together with the results of the TR are provided in Figure 1. More accurate numerical results of our suggested technique with respect to the TR confirms the efficiency of the generalized Euler-Maclaurin summation formula.

Example 3. As the third test problem we consider the following FODE with the exact solution $u(t) = t^4 + 1$

$$D_*^\alpha u(t) = t^4 + 1 + \frac{24}{\Gamma(4.5)}t^{3.5} - u(t), \quad u(0) = 1.$$

Similar to the previous section, we first assume that $y(t) = u'(t)$ and transform the above FODE into the associated weakly singular VIE of the first kind and implement the generalized Euler-Maclaurin quadrature

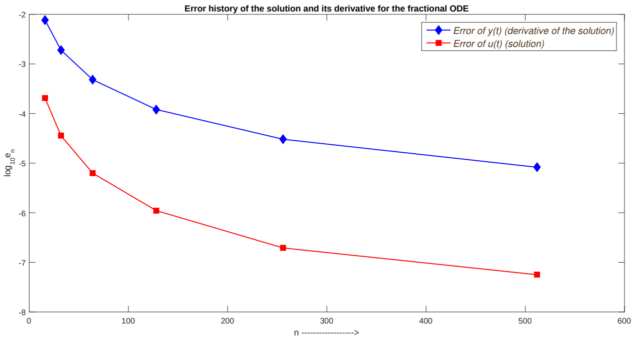

Figure 2: Error history for different values of n for the third example (fractional ODE)

Table 2: Numerical results of the fourth example

n	e_n of UM for $u(t)$	e_n of NM1 for $u(t)$	e_n of NM2 for $u(t)$
4	3.4860e-02	3.2979e-02	2.6986e-02
8	9.2494e-03	8.3115e-03	6.8542e-03
16	2.4012e-03	2.0819e-03	1.7264e-03
32	6.1550e-04	5.2074e-04	4.3302e-04
64	1.5650e-04	1.3020e-04	1.0840e-04
128	3.9587e-05	3.2551e-05	2.7116e-05
256	9,9778e-06	8,1380e-06	6,7806e-06

rule for obtaining the numerical solution. It should be noted that, the solution of the weakly singular VIE of the first kind is $y(t)$ and after approximating this variable one can use a simple quadrature rule such as TR for approximating $u(t) = \int_0^t y(x)dx + u_0 = \int_0^t y(x)dx + 1$. The obtained numerical results for this equation are depicted in Figure 2.

Example 4. As the fourth test problem, we consider the following weakly singular FIE with the exact solution $u(t) = t^3$

$$u(t) = t^3 - \frac{32}{35} + \int_0^1 \frac{u(x)}{\sqrt{1-x}}dx.$$

We implement the generalized Euler-Maclaurin quadrature rule for solving the above weakly singular FIE for three different meshes. The first one is the uniform mesh (UM). In the second, nonuniform mesh (NM1), the nodes are chosen in such a manner that the areas under the graph of the kernel function are equal, i.e., $\int_{x_{k-1}}^{x_k} 1/\sqrt{1-x}dx = \int_0^1 1/\sqrt{1-x}dx/n$ for $k = 1, 2, 3, \ldots, n$, then $x_k = k(2n-k)/n^2$ and the constant of condition (5) for the partition is C = 2. The next nonuniform mesh (NM2) differs from NM1 in taking the square root of the kernel function, i.e., $\int_{x_{k-1}}^{x_k} 1/\sqrt[4]{1-x}dx = \int_0^1 1/\sqrt[4]{1-x}dx/n$ for $k = 1, 2, 3, \ldots, n$, then $x_k = 1-(1-k/n)^{4/3}$ with the constant $C = 4/3$ of condition (5). The associated results of the three methods are provided in Table 2. It seems to be interesting that, in this case, NM2 gives the lowest values of the error.

7. Conclusions

Convergence analysis of the generalized Euler-Maclaurin quadrature rule is provided for solving weakly singular Fredholm and Volterra integral equations. It is proved that the rate of convergence is $O(n^{-2})$ for both of the equations. Also numerical solution of fractional ordinary differential equations (FODEs) is provided as an application of the considered summation formula. The numerical results for the above-mentioned equations confirm the theoretical predictions. We also showed that the generalized Euler-Maclaurin summation formula is a natural theoretical approach to the above issues.

Conflict of Interests

The authors declare that they do not have any conflict of interest in their submitted manuscript.

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