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Two Methods for Determining Properly Efficient Solutions with a Minimum Upper Bound for Trade-Offs

Behnam Hozzar[a], Ghasem Tohidi[b], Behrouz Daneshian[b]

[a] *Department of Mathematics, Lahijan Branch, Islamic Azad University, Lahijan, Iran*
[b] *Department of Mathematics, Islamic Azad University, Central Tehran Branch, Tehran, Iran*

Abstract. This paper aims to investigate proper efficiency in multiobjective optimization. We suggest two nonlinear optimization problems to determine upper bound for trade-offs among objective functions. Based on these problems we introduce some properly efficient solutions which are closer to the ideal point. Weighted sum scalarization and Kuhn-Tucker conditions will be used to obtain these nonlinear optimization problems.

1. Introduction and preliminaries

Multiobjective optimization problems originated in decision making problems for example economics, management and other social science, where they are often required decision making based on optimizing several criteria[16]. The Pareto concept of solutions (efficient solutions) are used in multiobjective optimization instead of optimality because in general it lacks a feasible solution to simultaneously minimize all objective functions. Therefore, the Pareto concept of optimality appears where none of the components can be improved without deteriorating at least one of the others. On the other hand, some efficient points exhibit certain abnormal features. They may cause arbitrarily large marginal trade-off [7, 14]. Therefore, various concepts of proper efficiency have been proposed to eliminate such anomalous efficient points [5, 16]. Hence, proper efficiencies notions are used in multiobjective optimization to exclude anomalous efficient solutions. This concept of efficiency is firstly introduced by Kuhn and Tucker [14]. Then, it is studied by Geoffrion [7], Borwein [3], Benson [2], Hartley [10], Henig [11], we refer [12, 20, 21] to more study. This concept is investigated by so many authors, particularly in interactive optimization literature and bounded trade-offs in multiobjective optimization. While solving a multiobjective optimization problem, there is typically a decision maker who is responsible for determining the most preferred Pareto optimal solution based on his preferences. To ensure the decision maker that the selected decisions are the right ones, it is important to understand the trade-offs related to different Pareto optimal solutions [16, 19]. Eskelinen and Miettinen [19] proposed a trade-off analysis approach that can be connected to various multiobjective optimization methods utilizing a certain type of scalarization to produce Pareto optimal solutions. With this approach, the decision maker can conveniently learn about local trade-offs between the conflicting

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Corresponding author: Ghasem Tohidi
Email addresses: behnam.hozzar@gmail.com (Behnam Hozzar), gh_tohidi@iauctb.ac.ir (Ghasem Tohidi)

objectives and judge whether they are acceptable or not. Miettinen and Ruiz [18] proposed NAUTILUS framework. In NAUTILUS, one enables the decision maker to make a free search without having to trade-off by starting from an efficient solution and iteratively approaching the Pareto optimal set by allowing all objective functions to improve [18]. We refer to [6, 16, 17, 19] for studying trade-off among objective function in multiobjective optimization.

One of the most important tools for obtaining solution with bounded trade-offs is weighted sum scalarization that many authors have investigated it in the light of interactive optimization. In this paper, we consider two methods for obtaining properly efficient solution, theoretically and interactively. We would like to consider the decision maker preferences, but not in order to sacrifice theoretically. To this end, we use weighted sum scalarization and Kuhn-Tucker conditions for our aims. In applicable system some solution which are close to ideal point are important because they meet the majority of decision maker preferences. Therefore, the paper investigates proper efficiency so that one can find an upper bound for bounded trade-offs. Also one can find some solutions that upper bound of those are the least value than others. On the other hand, these solutions are the closest solution to ideal point and this is an aim in interactive optimization.

This paper is organized as follows. Section 2 contains some basic definitions and notations that are used throughout the paper. Section 3 is devoted to some note about proper efficiency and trade-off. In section 4, the main result of the paper is stated. In this section, we propose two optimization problems for determining an efficient properly efficient solution in applicable with upper bound and weighted vector for objective functions under convexity. Weighted sum scalarization is used to achieve one of proposed optimization problems and for the other one Kuhn-Tucker optimality conditions is going to be used. The last section is devoted to conclusion and suggestions for future studies.

2. Background

This section describes some basic definitions used in this paper.

Assume that $\mathbb{R}^p_+ = \left\{y : y_i \geqq 0, \ \ \forall i \in \{1, 2, ..., p\}\right\}$ and $\mathbb{R}^p_{++} = \left\{y : y_i > 0, \ \ \forall i \in \{1, 2, ..., p\}\right\}$. We also use the following notations for $y^1, \ y^2 \in \mathbb{R}^p$:

$y^1 \leq y^2 \Leftrightarrow y^2 - y^1 \in \mathbb{R}^p_+$ and $y_1 \neq y_2$

$y^1 < y^2 \Leftrightarrow y^2 - y^1 \in \mathbb{R}^p_{++}$.

Consider the following multiobjective problem

$$\min_{x \in X \subseteq \mathbb{R}^n} f(x) \tag{1}$$

where $f(x) = (f_1(x), ..., f_p(x))$ and $f_i : X \to \mathbb{R}$ is a single value function, for all $i = 1, ..., p$.

Definition 2.1. *[5] The feasible solution $\hat{x}$ is an efficient solution of the problem* (1) *whenever*

$$\left(f(X) - f(\hat{x})\right) \cap -\mathbb{R}^p_+ = \{0\}$$

where $f(X) = \{f(x) : \ x \in X\}$.

Kuhn and Tucker [14] showed that some efficient solutions have undesirable property as follows: let $f_i : X \to \mathbb{R}$ be a differentiable objective function, for all $i = 1, ..., p$ on $X = \{x \in \mathbb{R}^n : \ g_k(x) \leq 0, \ k = 1, ..., m\}$, where $g_k : X \to \mathbb{R}$ is a differentiable function, for all $k = 1, ..., m$, in this case we say the problem (1) is differentiable. Assume that $\hat{x}$ is an efficient solution of the problem (1) and the following system has a nonzero solution $\hat{d} \in \mathbb{R}^p_+$

$$\begin{aligned} &\nabla f_j(\hat{x})^T.d \leq 0, \forall j = 1, ..., p \\ &\nabla f_i(\hat{x})^T.d < 0, \text{for some } i \in \{1, ..., p\} \\ &\nabla g_k(\hat{x})^T.d \leq 0, \forall k = 1, ..., m. \end{aligned} \tag{2}$$

Because $\hat{x}$ is an efficient solution there exist some index $s \in \{1, ..., p\}$ such that $\nabla f_s(\hat{x})^T.\hat{d} = 0$. Let $f_l(\hat{x})^T.\hat{d} = \min_{i\in\{1,...,p\}} f_i(\hat{x})^T.\hat{d} < 0$. Let $\hat{d} = \|x - \hat{x}\|$ for some $x \in X$. Based on Taylor series, we can write

$$f_s(x) - f_s(\hat{x}) = o(\hat{d}), \tag{3}$$

$$f_l(x) - f_l(\hat{x}) = \nabla f_l(\hat{x})^T.\hat{d} + o(\hat{d}), \tag{4}$$

It is clear that if $f_s(x)$ is increased then the objective function f_l will impair. But the growth rate of impairing is strictly greater than one and the growth rate of improving is one because $\nabla f_l(\hat{x})^T.\hat{d} \neq 0$. This undesirable property has been studied primarily by Kuhn and Tucker in [14]. The following definition proposed by Kuhn and Tucker in [14].

Definition 2.2. *Let the problem* (1) *be a differentiable optimization problem with* $X = \{x \in \mathbb{R}^n : \ g_k(x) \leq 0, \ k = 1, ..., m\}$ *and let* $\hat{x}$ *be an efficient solution of the problem* (1). $\hat{x}$ *is a properly efficient solution in sense Kuhn-Tucker if the system* (2) *has no solution.*

Then, Geoffrion [7] generalized the Kuhn-Tucker definition for non-differentiable optimization problem by an approximation of the vector $\nabla f(\hat{x})^T.d \ (\approx f(x) - f(\hat{x}))$ as following:

Definition 2.3. *[7] An efficient solution* $\hat{x} \in X$ *is said to be a properly efficient solution for Problem* (1), *if there exists a positive real number* M *such that for all* $x \in X$ *and* $i \in \{1, ..., p\}$ *with* $f_i(x) < f_i(\hat{x})$ *there exist some* $j \in \{1, ..., p\}$ *with* $f_j(\hat{x}) < f_j(x)$ *and*

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leq M. \tag{5}$$

Definition 2.4. *([15, 16])* $y^I \in \mathbb{R}^p$ *is said to be the ideal point of the problem* (1) *if the component* y^I_i, *for all* $i = 1, ..., p$, *equals to the optimal solution of the following problem*

$$\min f_i(x) \ \ s.t. \ \ x \in X, \tag{6}$$

where f_i *is ith component of* f *from* (1) *and* X *is too.*

Definition 2.5. *([15, 16])* $y^P \in \mathbb{R}^p$ *is said to be the nadir point of the problem* (1) *if the component* y^P_i, *for all* $i = 1, ..., p$, *equals the optimal solution of the following problem*

$$\max f_i(x) \ \ s.t. \ \ x \in X, \tag{7}$$

where f_i *and* X *define as in the definition 2.4.*

Definition 2.6. *[4] Consider the following problem:*

$$\begin{aligned} &\min h(x, y) \\ &\min f(x) \\ &\quad s.t. \ x \in X_1 = \{x \in \mathbb{R}^n : \ h_i(x, y) \leq 0, \ i = 1, ..., s\} \\ &\quad s.t. \ (x, y) \in X_2 = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : \ g_j(x, y) \leq 0, \ j = 1, ..., t\}. \end{aligned} \tag{8}$$

The problem (8) *is said to be a bi-level optimization problem if it is written as follows:*

$$\begin{aligned} &\min h(x, y) \\ &\quad s.t.(x, y) \in X_2 \cap X^*_f, \end{aligned}$$

where X^*_f *is the optimal solution set of following problem:*

$$\begin{aligned} &\min f(x) \\ &\quad s.t. x \in X_1 = \{x \in \mathbb{R}^n : \ h_i(x, y) \leq 0, \ i = 1, ..., s\}. \end{aligned}$$

3. Upper bound for trade-offs

This section is devoted to proper efficiency. Properly efficient solutions are theoretically and applicably an important group of efficient solutions in order to show treatment of objective functions during changes. Therefore, proper efficiency is investigated in term of change in objective functions in this section. At first, definition of trade-off is given as follows.

Definition 3.1. *[16] Let $\hat{x}$ and $\bar{x}$ be two distinct efficient solutions of* (1) *such that they satisfy $f_i(\bar{x}) < f_i(\hat{x})$ for index $i \in \{1, ..., p\}$ and $f_j(\hat{x}) < f_j(\bar{x})$ for index $j \in \{1, ..., p\} \backslash \{i\}$. The following fraction is said to be a trade-off between objective functions f_i and f_j at $\hat{x}$ and $\bar{x}$*

$$\frac{f_i(\hat{x}) - f_i(\bar{x})}{f_j(\bar{x}) - f_j(\hat{x})}.$$

Remark 3.2. *The Geoffrion's definition of proper efficiency, 2.3, is proposed based on finite trade-offs. This definition gives some important information about scalarization and behavior of objective functions. Now, the relation between proper efficiency and scalarization, weighted sum scalarization, is considered. Let $\lambda \in \mathbb{R}^p_+$ be a weighted vector. The following problem is said to be the weighted sum problem of the problem* (1)*:*

$$\begin{aligned} &\min \sum_{k=1}^{p} \lambda_k f_k(x) \\ &s.t. \;\; x \in X. \end{aligned} \tag{9}$$

Geoffrion in [7] showed that any optimal solution of the problem (9) *with positive weighted vector, $\lambda \in \mathbb{R}^p_{++}$, is a properly efficient solution. At first, the relation between the concept of trade-off and weighted sum scalarization for facilitating is considered. Let $\hat{x}$ be a properly efficient solution of* (1) *and let λ be a positive weighted vector for obtaining $\hat{x}$. Geromel and Ferreira in [8] showed that if for any $x \in X$ and for any $i \in \{1, 2, ..., p\}$ with $f_i(x) < f_i(\hat{x})$ there is an index $j \in \{1, 2, ..., p\}$ with $f_j(\hat{x}) < f_j(x)$, then*

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leq \frac{\lambda_j}{\lambda_i}. \tag{10}$$

Therefore, $\frac{\lambda_j}{\lambda_i}$ can be a good substitute for M in Geoffrion's definition of proper efficiency if λ is a positive weighted vector. In this regard, the following set Λ can be defined for determining properly efficient solutions:

$$\Lambda = \{\lambda \in \mathbb{R}^p : \sum_{k=1}^{p} \lambda_k = 1, \text{ and } \lambda_k > 0, \; \forall k = 1, 2, ..., p\}. \tag{11}$$

Consider the formulation (10). It is clear that if $\lambda_i \to 0$ trade-off between f_i and f_j is unbounded. Therefore, by improving of the set Λ, for preventing undesirability of some efficient solution [13], it can be written as follows:

$$\Lambda_\varepsilon = \{\lambda \in \mathbb{R}^p : \sum_{k=1}^{p} \lambda_k = 1, \text{ and } \lambda_k \geq \varepsilon, \; \forall k = 1, 2, ..., p\}, \tag{12}$$

where ε is a given positive real number.
Consider the upper bound $\frac{\lambda_j}{\lambda_i}$ in (10). By the definition of Λ_ε, it is clear that

$$\frac{\lambda_j}{\lambda_i} \leq \frac{1}{\varepsilon}.$$

Therefore $\frac{1}{\varepsilon}$ can be an appropriate upper bound for trade-offs between objective functions.

4. Assessing weighted vector for determining upper bound of trade-offs

4.1. Assessing weighted vector using weighted sum scalarization

This subsection focuses on the relation between proper efficiency and weighted sum scalarization. We will first improve the set Λ_ε which is defined in the previous section. Then, we will propose a scalar problem to determine some proper efficiency so that they meet preferences of DM, both theoretically and interactively.

Consider y^I and y^P the ideal point and nadir point as 2.4 and 2.5, respectively. Geromel and Valente Ferreira in [8] considered the following problem:

$$\begin{aligned} &\min \sum_{k=1}^{p} \lambda_k f_k(x) \\ &\text{s.t. } \ x \in X. \end{aligned} \tag{13}$$

where $\lambda \in \Lambda_\varepsilon$ ($\varepsilon > 0$). They showed that under convexity assumption the optimal solutions of the problem (13) dominate $y^I + \frac{1}{\varepsilon}(y^P - y^I)$. Thus, the section $\mathcal{Y}_\varepsilon := y^I + \frac{1}{\varepsilon}(y^P - y^I) - \mathbb{R}^p_+$ contains all the properly efficient solution produced by weighted vectors in Λ_ε such that $\frac{1}{\varepsilon}$ is an upper bound for all trade-off of these properly efficient solutions [8](see the Figure 1). It is clear that if ε is increased, $\mathcal{Y}_\varepsilon$ will be shrunk.

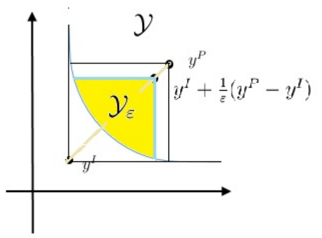

Figure 1: The section $\mathcal{Y}_\varepsilon := y^I + \frac{1}{\varepsilon}(y^P - y^I) - \mathbb{R}^p_+$.

Motivated by this discussion, it is quite natural to investigate some properly efficient solutions such that they are the closest properly efficient solutions to the ideal point with the upper bound $\frac{1}{\varepsilon}$. Hence, this paper proposes the following bi-level optimization problem to produce the closest properly efficient solution to the ideal point:

$$\begin{aligned} &\max \varepsilon \\ &\min \sum_{k=1}^{p} \lambda_k f_k(x) \\ &\text{s.t. } \ x \in X, \\ &\qquad \sum_{k=1}^{p} \lambda_k = 1, \\ &\qquad \lambda_k \geq \varepsilon, \ \ k = 1, ..., p. \\ &\text{s.t. } \varepsilon \geq 0. \end{aligned} \tag{14}$$

In the bi-level optimization problem (14), we combine the optimization problem (13) with the ε increase idea. In other words, if the solution $(x^*, \lambda^*, \varepsilon^*)$ is an optimal solution of (14) with $\varepsilon^* > 0$, then x^* is a proper efficient solution of the problem (1) such that $\frac{1}{\varepsilon^*}$ is an upper bound for trade-offs between objective functions.

Proposition 4.1. *Let the problem* (1) *be a convex optimization problem and let* $(\varepsilon^*, \lambda^*, x^*)$ *be an optimal solution of the problem* (14). *If* $\varepsilon^* = 0$, *then the problem* (1) *does not have any properly efficient solution.*

Proof. Assume, to the contrary, that $\bar{x} \in X$ is a properly efficient solution of the problem (1). Because the problem (1) is a convex optimization problem, there exists a positive weighted vector $\bar{\lambda}$ such that

$$\begin{aligned} \bar{x} = \operatorname{argmin} & \sum_{k=1}^{p} \bar{\lambda}_k f_k(x) \\ \text{s.t.} \ \ & x \in X. \end{aligned}$$

Define $\bar{\varepsilon} := \min_{k=1,\ldots,p} \bar{\lambda}_k$. It is clear, $(\bar{\varepsilon}, \bar{\lambda}, \bar{x})$ is a feasible solution for the problem (14) with $\bar{\varepsilon} > 0$. This is a contradiction with assumption of the proposition. □

Proposition 4.2. *Let the problem* (1) *be a convex optimization problem and let* $\varepsilon^* > 0$ *is an optimal value for the variable* ε *in the problem* (14). *Then* $\frac{1}{\varepsilon^*}$ *is an upper bound for any trade-off of properly efficient solutions in the section* $\mathcal{Y}_{\varepsilon^*}$.

Proof. The proof of the proposition is obvious. □

To clarify the aforementioned, the following example is given.

Example 4.3. *Consider* $\min_{x \in X} (x_1, x_2)$ *where* $X = \{x \in \mathbb{R}^2 : (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1, \ -1 \leq x_1, x_2 \leq 1\}$. *The optimal solution of the problem 14 is* $\varepsilon^* = 0.5$, $\lambda_1^* = \lambda_2^* = 0.5$ *and* $x^* = (1 - \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}) \approx (0.2929, 0.2929)$. *It is clear that* x^* *is the closest point to* y^I. *Also,* $M := \frac{1}{\varepsilon^*} = 2$ *is an upper bound for trade-offs between two objectives. The Figure 2 shows the geometrical interpretation of the example.*

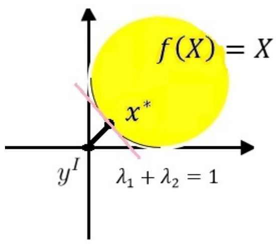

Figure 2: Geometrical interpretation of Example 4.3.

4.2. *Assessing weighted vector with Kuhn-Tucker conditions*

In this subsection, Kuhn-Tucker optimality condition is considered for obtaining some properly efficient solutions which are they meet the DM's preferences. Consider the following vector optimization problem

$$\begin{aligned} & \min f(x) = (f_1(x), \ldots, f_p(x)) \\ & \text{s.t.} \ \ x \in \{x \in \mathbb{R}^n : g_j(x) \leq 0, \ j = 1, \ldots, m\}. \end{aligned} \tag{15}$$

The following theorem states Kuhn-Tucker conditions:

Theorem 4.4. *Assume that the problem* (15) *is a convex optimization problem,* f_i *and* g_j *are differentiable for* $i = 1, \ldots, p$ *and* $j = 1, \ldots, m$, *respectively. If* x^* *is a properly efficient solution and KT constraint qualification is satisfied* *at x^*, then, there are $\lambda^* \in \mathbb{R}^p$ and $\mu^* \in \mathbb{R}^m$ such that*

$$\sum_{k=1}^{p} \lambda_k^* \nabla f_k(x^*) + \sum_{j=1}^{m} \mu_j^* \nabla g_j(x^*) = 0 \tag{16}$$

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0 \tag{17}$$

$$\lambda_k^* > 0, \ \textit{for all} \ k \in \{1, ..., p\} \tag{18}$$

$$\mu_j^* \geq 0, \ \textit{for all} \ j \in \{1, ..., m\}. \tag{19}$$

Proof. The proof of the theorem follows from the theorems 2.50, 3.25 and 3.27 in [5]. □

Now, we consider the following problem

$$\begin{aligned} &\min \sum_{k=1}^{p} \lambda_k f_k(x) \\ &\text{s.t.} \ \ x \in \{x \in \mathbb{R}^n : g_j(x) \leq 0, \ j = 1, ..., m\} \end{aligned} \tag{20}$$

where λ is a positive weighted vector. It is clear that the conditions (16)-(19) are optimality conditions for the problem (20) [1]. Therefore, if there exist a positive weighted vector $\lambda^* \in \mathbb{R}^p$ and a nonnegative vector μ^* such that (λ^*, μ^*) is a solution of the nonlinear system (16)-(19), then the solution x^* is a properly efficient solution. Hence, the goal of this subsection is finding a feasible solution to the problem (15) and positive weighted vector $\lambda^* \in \mathbb{R}^p$ and a nonnegative vector μ^* such that (x^*, λ^*, μ^*) such that it satisfies (16)-(19). regarding this, we propose the following optimization problem:

$$\begin{aligned} \max \ & \varepsilon \\ \text{s.t.} & \sum_{k=1}^{p} \lambda_k \nabla f_k(x) + \sum_{j=1}^{m} \mu_j \nabla g_j(x) = 0 \\ & \sum_{j=1}^{m} \mu_j g_j(x) = 0 \\ & \sum_{k=1}^{p} \lambda_k = 1, \\ & g_j(x) \leq 0, \ \ j = 1, ..., m \\ & \lambda_k \geq \varepsilon, \ \ \text{for all} \ \ k \in \{1, ..., p\} \\ & \mu_j \geq 0, \ \ \text{for all} \ j \in \{1, ..., m\}. \end{aligned} \tag{21}$$

Proposition 4.5. *Under convexity, if optimal value objective function of the problem* (21) *is zero, then there is no properly efficient solution of* (15).

Proof. The proof is obvious. □

Theorem 4.6. *Let the problem* (15) *be convex and let $(\varepsilon^*, \lambda^*, \mu^*, x^*)$ be an optimal solution of the problem* (21). *If $\varepsilon^* > 0$ then x^* is a properly efficient solution of* (15).

Proof. Because $\varepsilon^* > 0$ and (15) is convex, there exists λ^* and μ^* such that x^* satisfies the conditions (16)-(19). Hence, x^* is a properly efficient solution. □

It must be mentioned that in contrast to the problem (21), the problem (14) is a bi-level optimization problem. The problem (21) can be solved by one of the well-known nonlinear optimization constrained algorithms such as Active Set Algorithm , Interior Point Algorithm and so on [9]. The Example 4.3 is resolved for comparing (14) and (21).

Example 4.7. *Consider Example 4.3.* $f_1(x) = x_1$, $f_2(x) = x_2$, $g_1(x) = (x_1-1)^2+(x_2-1)^2-1$, $g_2(x) = x_1-1$, $g_3(x) = x_2-1$, $g_4(x) = -x_1-1$, *and* $g_5(x) = -x_2-1$. $\varepsilon^* = 0.5$, $x_1^* = x_2^* = 1-\frac{\sqrt{2}}{2} \approx 0.29292$, $\lambda_1^* = \lambda_2^* = 0.5$, $\mu_2^* = \frac{1}{2\sqrt{2}} \approx 0.3536$ *and* $\mu_1^* = \mu_3^* = \mu_4^* = \mu_5^* = 0$ *is an optimal solution. Thus* $x^* \approx (0.2929, 0.2929)$ *is a properly efficient solution of* (15).

The following example is related to optimization problems (21) and (14). As is seen two optimization problems (21) and (14) have similar optimal solution. The Active Set Algorithm is used to solve the following example.

Example 4.8. *Consider* $f_1(x) = x_1$, $f_2(x) = x_2$, $f_3(x) = x_1^2-x_1x_2^2+3x_1$, $g_1(x) = x_1+1.5$, $g_2(x) = -x_1-2$, $g_3(x) = -x_2$, *and* $g_4(x) = x_2-1$. *At first, we show that*

$$\min_{x\in X} (f_1(x), f_2(x), f_3(x))$$

is a convex problem on $X = \{x \in \mathbb{R}^2 : g_i(x) \le 0, \text{ for all } i = 1, ..., 4\}$. f_1, f_2 *and* g_i, *for all* $i = 1, ..., 4$ *are linear function then they are convex. Because* $X = \{(x_1, x_2) : -2 \le x_1 \le -1.5,\ 0 \le x_2 \le 1\}$ *and* $\det(H(f_3(x))) \ge 2 > 0$ *where* $\det(\cdot)$ *is the Determinant function and* $H(f_3(x))$ *is the Hessian matrix* f_3 *as follows:*

$$H(f_3(x)) = \begin{pmatrix} 2 & -2x_2 \\ -2x_2 & -2x_1 \end{pmatrix}.$$

It is clear that $(-2, 0, -2.25)$ *is the Ideal point. From solving the problem* (21) *by "FMINCON", MATLAB optimization tools for solving nonlinear problems by nonlinear constrained, one has* $x = (-1.5, 1)$, $\lambda_1 = \lambda_2 = \lambda_3 = 0.3333$, $\varepsilon = 0.3333$ *and* $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$. *Thus,* $(-1.5, 1, -0.75)$ *is a properly efficient solution of the optimization problem* (15). $x = (-1.5, 1)$, $\lambda_1 = \lambda_2 = \lambda_3 = 0.3333$ *and* $\varepsilon = 0.3333$ *so is the optimal solution of the optimization problem* (14).

5. Conclusion

In this paper, we proposed two optimization problems for determining some properly efficient solutions so that they meet the DM's preferences. In multi-objective optimization, decision maker usually would like to use some solutions which are close to the ideal point. We provide these matter under convexity by weighted sum scalarization, in order to find some properly efficient solutions for meeting preferences of decision maker, one of the important subjects for future researches will be investigating some scalarization methods which are related to the proper efficiency, for example The Elastic Constraint Method [5]. Other interesting topic in this field of study will be investigation on obtaining these properly efficient solutions without presence of convexity.

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