Filomat 33:7 (2019), 1967–1977
https://doi.org/10.2298/FIL1907967K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the generalized Apostol-type Frobenius-Genocchi polynomials

Waseem A. Khan[a], Divesh Srivastava[a]

[a]*Department of Mathematics, Faculty of Science, Integral University, Lucknow-226026, India*

Abstract. The main object of this work is to introduce a new class of the generalized Apostol-type Frobenius-Genocchi polynomials and is to investigate some properties and relations of them. We derive implicit summation formulae and symmetric identities by applying the generating functions. In addition a relation in between Array-type polynomials, Apostol-Bernoulli polynomials and generalized Apostol-type Frobenius-Genocchi polynomials is also given.

1. Introduction

Let $\alpha \in \mathbb{Z}, \lambda \in \mathbb{C}, a, b, c \in \mathbb{R}^{+}$, $a \neq b$ and $x \in \mathbb{R}$. The generalized Apostol-Bernoulli, Euler and Genocchi polynomials with the parameters are given by means of the following generating function as follows (see [1-15]):

$$\left(\frac{t}{\lambda b^{t}-a^{t}}\right)^{\alpha} c^{xt}=\sum_{n=0}^{\infty} B_{n}^{(\alpha)}(x;\lambda;a,b,c)\frac{t^{n}}{n!}, \mid t\ln\frac{b}{a}\mid< 2\pi, \tag{1.1}$$

$$\left(\frac{2}{\lambda b^{t}+a^{t}}\right)^{\alpha} c^{xt}=\sum_{n=0}^{\infty} E_{n}^{(\alpha)}(x;\lambda;a,b,c)\frac{t^{n}}{n!}, \mid t\ln\frac{b}{a}\mid< \pi, \tag{1.2}$$

and

$$\left(\frac{2t}{\lambda b^{t}+a^{t}}\right)^{\alpha} c^{xt}=\sum_{n=0}^{\infty} G_{n}^{(\alpha)}(x;\lambda;a,b,c)\frac{t^{n}}{n!}, \mid t\ln\frac{b}{a}\mid< \pi. \tag{1.3}$$

Obviously, we have
$B_{n}^{(\alpha)}(x;\lambda;1,e,e)=B_{n}(x;\lambda)$, $E_{n}^{(\alpha)}(x;\lambda;1,e,e)=E_{n}(x;\lambda)$, and $G_{n}^{(\alpha)}(x;\lambda;1,e,e)=G_{n}(x;\lambda)$.

Recently, Kurt et al. [1] and Simsek [11, 12] introduced the Apostol type Frobenius-Euler polynomials as follows.

Let $\alpha \in \mathbb{Z}, \lambda \in \mathbb{C}, a, b, c \in \mathbb{R}^{+}$, $a \neq b, x \in \mathbb{R}$. The generalized Apostol type Frobenius-Euler polynomials are defined by means of the following generating function:

$$\left(\frac{a^{t}-u}{\lambda b^{t}-u}\right)^{\alpha} c^{xt}=\sum_{n=0}^{\infty} H_{n}^{(\alpha)}(x;u,a,b,c,\lambda)\frac{t^{n}}{n!}. \tag{1.4}$$

2010 *Mathematics Subject Classification*. 11B68, 05A10, 05A15, 33C45, 26B99
Keywords. (Frobenius-Genocchi polynomials, Apostol-type Genocchi polynomials)
Received: 19 June 2018; Accepted: 28 Sepetember 2019
Communicated by Dragan S. Djordjević
Email addresses: waseem08_khan@rediffmail.com (Waseem A. Khan), divesh2712@gmail.com (Divesh Srivastava)

For $x = 0$ and $\alpha = 1$ in (1.4), we get

$$\frac{a^t - u}{\lambda b^t - u} = \sum_{n=0}^{\infty} H_n(u, a, b; \lambda)\frac{t^n}{n!}, \tag{1.5}$$

where $H_n(u, a, b; \lambda)$ denotes the generalized Apostol type Frobenius-Euler numbers (see [9], [12], [14]).

On setting $a = 1$, $b = e$, $\lambda = 1$ in (1.4), the result reduces to

$$\left(\frac{1-u}{e^t - u}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x; u)\frac{t^n}{n!}, \tag{1.6}$$

where $H_n^{(\alpha)}(x; u)$ is called classical Frobenius-Euler polynomial of order α (see [1], [12], [15]).

Observe that $H_n^{(1)}(x, u) = H_n(x, u)$, which denotes the Frobenius-Euler polynomials and $H_n^{(\alpha)}(0; u) = H_n^{(\alpha)}(u)$, which denotes the Frobenius -Euler numbers of order α. $H_n^{(1)}(x; -1) = E_n(x)$, which denotes the Euler polynomials (see [5], [7], [12]).

Recently, Yaşar and Özarslan [15] introduced the Frobenius-Genocchi polynomials by means of the following generating function:

$$\frac{(1-\lambda)t}{e^t - \lambda} e^{xt} = \sum_{n=0}^{\infty} G_n^F(x; \lambda)\frac{t^n}{n!}. \tag{1.7}$$

On setting $\lambda = -1$ in (1.7), we get

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, | t |< \pi, \tag{1.8}$$

where $G_n(x)$ are called the Genocchi polynomials (see [5]).

In (2013), Simsek [11] introduced the λ-stirling type number of second kind $S(n, v; a, b; \lambda)$ by means of the following generating function:

$$\sum_{n=0}^{\infty} \mathcal{S}(n, v; a, b; \lambda)\frac{t^n}{n!} = \frac{(\lambda b^t - a^t)^v}{v!} \tag{1.9}$$

and the generalized array type polynomials are defined by Simsek [11, p.6, Eq. (3.1)] as:

$$\sum_{n=0}^{\infty} \mathcal{S}_v^n(x; a, b; \lambda)\frac{t^n}{n!} = \frac{(\lambda b^t - a^t)^v}{v!} b^{xt}. \tag{1.10}$$

Kurt and Simsek [1] introduced the polynomial $Y_n(x; \lambda; a)$, which is given by the following generating function:

$$\frac{t}{\lambda a^t - 1} a^{xt} = \sum_{n=0}^{\infty} Y_n(x; \lambda; a)\frac{t^n}{n!}, \quad (a \geq 1). \tag{1.11}$$

For $x = 0$ in (1.11), we get

$$Y_n(0; \lambda; a) = Y_n(\lambda; a), (see[15]). \tag{1.12}$$

Again if we set $x = 0$ and $a = 1$, in (1.11), we obtain

$$Y_n(\lambda; 1) = \frac{t}{\lambda - 1}. \tag{1.13}$$

This paper is organized as follows. We give a brief review of generalized Apostol-type Frobenius-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;u,a,b,c;\lambda)$ and their properties. Some explicit and implicit summation formulae and general symmetric identities are derived by using different analytical means and applying generating functions. Also, we established relation between λ-type Stirling polynomials, Apostol-Bernoulli polynomials and generalized Apostol Frobenius-Genocchi polynomials.

2. Definition and properties of the generalized Apostol-type Frobenius-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;u;a,b,c;\lambda)$

In this section, we introduce the generalized Apostol-type Frobenius-Genocchi polynomials and investigate some basic properties.

Definition 2.1. The generalized Apostol-type Frobenius-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;u;a.b.c;\lambda)$ of order α are defined by means of the following generating function:

$$\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;u;a,b,c;\lambda)\frac{t^n}{n!}, \tag{2.1}$$

where $(\alpha \in \mathbb{Z}, \lambda \in \mathbb{C}, a,b,c \in \mathbb{R}^+,\ a \neq b,\ x \in \mathbb{R})$.

Remark 2.1. For $x = 0$ and $\alpha = 1$, (2.1) reduces to

$$\left(\frac{(a^t-u)t}{\lambda b^t-u}\right) = \sum_{n=0}^{\infty} \mathcal{G}_n(u;a,b;\lambda)\frac{t^n}{n!}, \tag{2.2}$$

where $\mathcal{G}_n(u;a.b.c;\lambda)$ denotes the Apostol-type Frobenius-Genocchi numbers.

Remark 2.2. If we set $a = 1, b = c = e, u = -1$, (2.1) immediately reduces to the Apostol-type Genocchi polynomials (see [4], [12], [15]).

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda), |t| < \pi. \tag{2.3}$$

We have the following properties of (2.1), which are stated in terms of theorems as:

Theorem 2.1 The following recurrence relation holds true:

$$\begin{aligned}(2u-1)\sum_{r=0}^{n}\binom{n}{r}&\mathcal{G}_r(x;u;a,b,c;\lambda)\,\mathcal{G}_{n-r}(y;1-u;a,b,c;\lambda)\\
= n(u-1)\mathcal{G}_{n-1}(x+y;u;a,b,c;\lambda) &+ nu\,\mathcal{G}_{n-1}(x+y;1-u,a,b,c;\lambda)\\
&+\sum_{r=0}^{n}\binom{n}{r}(\ln a)^{n-r}\mathcal{G}_r(x+y;u;a,b,c;\lambda)\\
&-\sum_{r=0}^{n}\binom{n}{r}(\ln a)^{n-r}\mathcal{G}_r(x+y;1-u,a,b,c;\lambda).\end{aligned} \tag{2.4}$$

Proof. In order to prove (2.4), we set

$$(2u-1)\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)c^{xt}\left(\frac{(a^t-(1-u))t}{\lambda b^t-(1-u)}\right)c^{yt}$$

$$= t^2(a^t - u)(a^t - (1-u))c^{(x+y)t}\left[\frac{1}{\lambda b^t - u} - \frac{1}{\lambda b^t - (1-u)}\right]. \tag{2.5}$$

Employing the result (2.2), equation (2.5) reduces as

$$(2u-1)\sum_{r=0}^{\infty}\mathcal{G}_r(x;u;a,b,c;\lambda)\frac{t^r}{r!}\sum_{n=0}^{\infty}\mathcal{G}_n(y;1-u;a,b,c;\lambda)\frac{t^n}{n!} = (a^t - (1-u)t)$$

$$\times\sum_{r=0}^{\infty}\mathcal{G}_r(x+y;u,a,b,c;\lambda)\frac{t^r}{r!} - (a^t-u)t\sum_{r=0}^{\infty}\mathcal{G}_r(x+y;1-u;a,b,c;\lambda)\frac{t^r}{r!}. \tag{2.6}$$

Using [13, p. 100, Eq. 2], we get

$$(2u-1)\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\mathcal{G}_r(x;u;a,b,c;\lambda)\,\mathcal{G}_{n-r}(y;1-u;a,b,c;\lambda)\frac{t^n}{n!}$$

$$= (a^t-(1-u)t)\sum_{r=0}^{\infty}\mathcal{G}_r(x+y;u,a,b,c;\lambda)\frac{t^r}{r!} - (a^t-u)t$$

$$\times\sum_{r=0}^{\infty}\mathcal{G}_r(x+y;1-u;a,b,c;\lambda)\frac{t^r}{r!}. \tag{2.7}$$

$$(2u-1)\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\mathcal{G}_r(x;u;a,b,c;\lambda)\,\mathcal{G}_{n-r}(y;1-u;a,b,c;\lambda)\frac{t^n}{n!}$$

$$= (u-1)\sum_{r=0}^{\infty}\mathcal{G}_r(x+y;u,a,b,c;\lambda)\frac{t^{r+1}}{r!}$$

$$+u\sum_{r=0}^{\infty}\mathcal{G}_r(x+y;1-u,a,b,c;\lambda)\frac{t^{r+1}}{r!}$$

$$+\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}(\ln a)^{n-r}\mathcal{G}_r(x+y;u;a,b,c;\lambda)\frac{t^n}{n!}$$

$$-\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}(\ln a)^{n-r}\mathcal{G}_r(x+y;1-u;a,b,c;\lambda)\frac{t^n}{n!}. \tag{2.8}$$

On comparing the coefficients of t^n in both sides, we arrive at the desired result (2.4). □

Theorem 2.2. The following relation holds true:

$$\left(\mathcal{G}_{n+1}(x;u;a,b,b;\lambda) - \ln(b)^{x^2}\mathcal{G}_n(x;u;a,b,b;\lambda)\right)$$

$$= \ln(a)^{\frac{1}{u}}\sum_{k=0}^{n+1}\binom{n+1}{k}Y_{n+1-k}\left(1;\frac{1}{u};a\right)\mathcal{G}_k(x;u;a,b,b;\lambda)$$

$$+\ln(b)^{\frac{\lambda}{u}}\sum_{k=0}^{n+2}\binom{n+2}{k}Y_{n+2-k}\left(\frac{1}{u};b\right)\mathcal{G}_k^{(2)}(x;u;a,b,b;\lambda). \tag{2.9}$$

Proof. In order to prove (2.9), we set $c = b$ and $\alpha = 1$ in equation (2.1) and then taking it's derivative with respect to t, we have

$$\sum_{n=0}^{\infty}\mathcal{G}_{n+1}(x;u;a,b,b;\lambda)\frac{t^n}{n!}=\left[\frac{(\lambda b^t-u)(a^t-u)+ta^t\ln(a)-(a^t-u)t\lambda b^t\ln(b)}{(\lambda b^t-u)^2}\right]b^{xt}$$
$$+\frac{(a^t-u)t}{\lambda b^t-u}b^{xt}\ln(b)^{x^2}. \tag{2.10}$$

On arranging the above equation and making use of (1.11) and (2.1), we get

$$\sum_{n=0}^{\infty}\mathcal{G}_{n+1}(x;u;a,b,b;\lambda)\frac{t^n}{n!}=\frac{1}{t}\sum_{n=0}^{\infty}\mathcal{G}_{n}(x;u;a,b,b;\lambda)\frac{t^n}{n!}+\frac{1}{t}\frac{\ln(a)}{u}$$
$$\times\sum_{n=0}^{\infty}Y_n(1;\frac{1}{u};a)\frac{t^n}{n!}\sum_{k=0}^{\infty}\mathcal{G}_{k}(x;u;a,b,b;\lambda)\frac{t^k}{k!}-\frac{1}{t^2}\lambda b^t\ \ln(b)$$
$$\times\sum_{n=0}^{\infty}Y_n(1;\frac{1}{u};b)\frac{t^n}{n!}\sum_{k=0}^{\infty}\mathcal{G}_{k}^{(2)}(x;u;a,b,b;\lambda)\frac{t^k}{k!}$$
$$+\ln(b)^{x^2}\sum_{n=0}^{\infty}\mathcal{G}_{n}(x;u;a,b,b;\lambda)\frac{t^n}{n!}. \tag{2.11}$$

Making use of Lemma [13, p.100, Eq.2], above equation reduces as

$$\sum_{n=0}^{\infty}\mathcal{G}_{n+1}(x;u;a,b,b;\lambda)\frac{t^n}{n!}=\frac{1}{t}\sum_{n=0}^{\infty}\mathcal{G}_{n}(x;u;a,b,b;\lambda)\frac{t^n}{n!}+\frac{1}{t}\frac{\ln(a)}{u}$$
$$\times\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}Y_{n-k}(1;\frac{1}{u};a)\frac{t^n}{n!}\mathcal{G}_{k}(x;u;a,b,b;\lambda)$$
$$-\frac{1}{t^2}\lambda b^t\ \ln(b)\ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n}{k}Y_{n-k}(1;\frac{1}{u};b)$$
$$\times\mathcal{G}_{k}^{(2)}(x;u;a,b,b;\lambda)\frac{t^n}{n!}+\ln(b)^{x^2}\sum_{n=0}^{\infty}\mathcal{G}_{n}(x;u;a,b,b;\lambda)\frac{t^n}{n!}. \tag{2.12}$$

On equating the coefficients of t^n in both sides of the above equation, we arrive at the required result (2.9). □

Theorem 2.3. The following relationship holds true

$$\mathcal{G}_{n}^{(-m)}(u;a,b,c;\lambda)=\sum_{k=0}^{n}\mathcal{G}_{k}^{(-\alpha)}(-x;u;a,b,c;\lambda)\mathcal{G}_{(n-k)}^{(\alpha-m)}(x;u;a.b.c;\lambda), \tag{2.13}$$

Proof. In order to prove (2.13), replacing x by $-x$ and α by $-\alpha$ in (2.1), to get

$$\sum_{n=0}^{\infty}\mathcal{G}_{n}^{(-\alpha)}(-x;u;a,b,c;\lambda)\frac{t^n}{n!}=\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{(-\alpha)}c^{-xt}. \tag{2.14}$$

Making use of the above equation, we can write

$$\sum_{k=0}^{\infty}\mathcal{G}_{k}^{(-\alpha)}(-x;u;a,b,c;\lambda)\frac{t^k}{k!}\ \sum_{n=0}^{\infty}\mathcal{G}_{n}^{(\alpha-m)}(x;u;a,b,c;\lambda)\frac{t^n}{n!}=\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{-m}. \tag{2.15}$$

$$\sum_{k=0}^{\infty}\mathcal{G}_k^{(-\alpha)}(-x;u;a,b,c;\lambda)\frac{t^k}{k!}\sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha-m)}(x;u;a,b,c;\lambda)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\mathcal{G}_n^{(-m)}(u;a,b,c;\lambda)\frac{t^n}{n!}. \tag{2.16}$$

Using Lemma [13, p.100, Eq.2], and comparing the coefficients of t^n from the resulting equation, we acquire the result (2.13). □

Theorem 2.4. The following relationships hold true:

$$\mathcal{G}_n^{(\alpha)}(x;u,a,b,c;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_k^{(\alpha)}(u;a,b,c;\lambda)(x\ln c)^{(n-k)}. \tag{2.17}$$

$$\mathcal{G}_n^{(\alpha+\beta)}(x+y;u,a,b,c;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_k^{(\alpha)}(x;u;a,b,c;\lambda)\,\mathcal{G}_{n-k}^{(\beta)}(y;u;a,b,c;\lambda). \tag{2.18}$$

$$((x+y)\ln c)^n=\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_{n-k}^{(\alpha)}(y;u;a,b,c;\lambda)\,\mathcal{G}_k^{(-\alpha)}(x;u;a,b,c;\lambda). \tag{2.19}$$

$$\mathcal{G}_n^{(-\alpha)}(2x;u^2;a^2,b^2,c^2;\lambda^2)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_k^{(-\alpha)}(x;u;a,b,c;\lambda)$$

$$\times\mathcal{H}_{n-k}^{(-\alpha)}(x;-u;a,b,c;\lambda). \tag{2.20}$$

Proof. By using (1.4) and (2.1), we can easily find the results (2.17)-(2.20). We omit the proof. □

3. Implicit Summation Formulae Involving Generalized Apostol-type Frobenius-Genocchi Polynomials

Here in this section, we provide some implicit formulae for generalized Apostol-type Frobenius-Genocchi polynomials.

Theorem 3.1. The following implicit formula for the generalized Apostol-type Frobenius-Genocchi polynomials holds true:

$$\mathcal{G}_{k+l}^{(\alpha)}(z;u;a,b,c;\lambda)=\sum_{n,m=0}^{k,l}\binom{l}{m}\binom{k}{n}(\ln c)^{(m+n)}(z-x)^{m+n}\mathcal{G}_{k-n,l-m}^{(\alpha)}(x;u;a,b,c;\lambda). \tag{3.1}$$

Proof. Replacing t by $(t+w)$ in (2.1) and rewriting equation (2.1) as

$$\left(\frac{(a^{(t+w)}-u)(t+w)}{\lambda b^{t+w}-u}\right)^{\alpha}=c^{-x(t+w)}\sum_{k,l=0}^{\infty}\mathcal{G}_{k+l}^{(\alpha)}(x;u;a,b,c;\lambda)\frac{t^k}{k!}\frac{w^l}{l!}. \tag{3.2}$$

Replacing x by z, and equating the obtained equation with the above equation, we have

$$c^{(z-x)(t+w)}\sum_{k,l=0}^{\infty}\mathcal{G}_{k+l}^{(\alpha)}(x;u;a,b,c;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}=\sum_{k,l=0}^{\infty}\mathcal{G}_{k+l}^{(\alpha)}(z;u;a,b,c;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}. \tag{3.3}$$

Expanding the exponent part in the above equation, we have

$$\sum_{N=0}^{\infty}\frac{[(z-x)(t+w)]^N}{N!}\sum_{k,l=0}^{\infty}\mathcal{G}_{k+l}^{(\alpha)}(x;u;a,b,c;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}$$

$$= \sum_{k,l=0}^{\infty} \mathcal{G}_{k+l}^{(\alpha)}(z;u;a,b,c;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}. \tag{3.4}$$

Using the result [13, p.52(2)], we have

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{m,n=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{x^m}{m!}$$

in the left-hand side, we get

$$\sum_{n,m=0}^{\infty} \frac{(\ln c)^{n+m}(z-x)^{(n+m)}t^n w^m}{n!m!} \sum_{k,l=0}^{\infty} \mathcal{G}_{k+l}^{(\alpha)}(x;u;a,b,c;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}$$

$$= \sum_{k,l=0}^{\infty} \mathcal{G}_{k+l}^{(\alpha)}(z;u;a,b,c;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}. \tag{3.5}$$

Replacing k by $k-n$ and l by $l-m$ in the above equation and equating the coefficients of t^k and w^l from the resulting equation, we obtain the required result (3.1). □

Corollary 3.1. For $l = 0$ in Theorem 3.1, we have the following relation

$$\mathcal{G}_{k+l}^{(\alpha)}(z;u;a,b,c;\lambda) = \sum_{n=0}^{k} \binom{k}{n} (\ln c)^{n+m}(z-x)^n \mathcal{G}_{k+l}^{(\alpha)}(x;u;a,b,c;\lambda). \tag{3.6}$$

Theorem 3.2. The following relation holds true:

$$\mathcal{G}_{n}^{(\alpha)}(x+1;u;a,b,c;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \ln(c)^{n-k} \mathcal{G}_{k}^{(\alpha)}(x;u;a,b;\lambda). \tag{3.7}$$

Proof. Replacing x by $x+1$ in equation (2.1), we get

$$\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{\alpha} c^{(x+1)t} = \sum_{n=0}^{\infty} \mathcal{G}_{n}^{(\alpha)}(x+1;u;a,b,c;\lambda)\frac{t^n}{n!}. \tag{3.8}$$

$$\sum_{k=0}^{\infty} \mathcal{G}_{k}^{(\alpha)}(x;u;a,b,c;\lambda)\frac{t^k}{k!} \sum_{n=0}^{\infty} (\ln c)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{G}_{n}^{(\alpha)}(x+1;u;a,b,c;\lambda)\frac{t^n}{n!}. \tag{3.9}$$

On replacing k by $k-n$ and equating the coefficients of t^n in the resulting equation, we obtain the desired result (3.7). □

Theorem 3.3. The following relation holds true:

$$\mathcal{G}_{n}^{(\alpha+1)}(x;u;a,b,c;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{G}_{n-m}(u;a,b;\lambda)\mathcal{G}_{m}^{(\alpha)}(x;u;a,b;\lambda). \tag{3.10}$$

Proof. Replacing α by $(\alpha+1)$ in equation (2.1), we have

$$\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{\alpha+1} c^{xt} = \left(\frac{(a^t-u)t}{\lambda b^t-u}\right) \left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{\alpha} c^{xt}$$

$$= \sum_{n=0}^{\infty} \mathcal{G}_n(u;a,b;\lambda)\frac{t^n}{n!} \sum_{m=0}^{\infty} \mathcal{G}_m^{(\alpha)}(x;u;a,b,c;\lambda)\frac{t^m}{m!}. \tag{3.11}$$

On setting n by $n-m$ in the above equation and equating the coefficients of t^n, we obtain the required result. □

Theorem 3.4. The following implicit summation formula holds true:

$$\sum_{m=0}^{n}(-1)^{\alpha}\binom{n}{m}(\ln(ab))^m(\alpha)^m\mathcal{G}_{n-m}^{(\alpha)}(x;u;a,b,c;\lambda) = (-1)^n\mathcal{G}_n^{(\alpha)}(x;u;a,b,c;\lambda), \tag{3.12}$$

Proof. First, we replace t by $-t$ in (2.1) and then we subtract the obtained equation with (2.1), we get

$$\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{\alpha}[c^{xt}-(ab)^{\alpha t}(-1)^{\alpha}c^{-xt}] = \sum_{n=0}^{\infty}[1-(-1)^n]\,\mathcal{G}_n^{(\alpha)}(x;u;a,b,c;\lambda)\frac{t^n}{n!}. \tag{3.13}$$

By using (2.1) and Lemma [13, p.100, Eq.2] we get

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha)}(x;u;a,b,c;\lambda)\frac{t^n}{n!} - (-1)^{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(\alpha)^m(\ln ab)^m\mathcal{G}_{n-m}^{(\alpha)}(x;u;a,b,c;\lambda)\frac{t^n}{(n-m)!}$$

$$= \sum_{n=0}^{\infty}[1-(-1)^n]\,\mathcal{G}_n^{(\alpha)}(x;u;a,b,c;\lambda)\frac{t^n}{n!}. \tag{3.14}$$

On equating the coefficients of t^n in the both sides of above equation, yield result (3.12). □

4. Symmetric identities for the generalized Apostol-type Frobenius-Genocchi polynomials

In this section, we establish symmetric identities for the generalized Apostol type Frobenius-Genocchi polynomials by applying the generating function (2.1). The results extends some known identities of Khan et al. [3-5] and Pathan and Khan [8-10].

Theorem 4.1. The following identity holds true:

$$\sum_{k=0}^{n}\binom{n}{k}b^k a^{n-k}\mathcal{G}_{n-k}^{(\alpha)}(bx,by;u;A,B,c;\lambda)\mathcal{G}_k^{(\alpha)}(ax,ay;u;A,B,c;\lambda)$$

$$= \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\mathcal{G}_{n-k}^{(\alpha)}(ax,ay;u;A,B,c;\lambda)\mathcal{G}_k^{(\alpha)}(bx,by;u;A,B,c;\lambda). \tag{4.1}$$

Proof. Let

$$H(t) = \left[\left(\frac{(A^{at}-u)at}{\lambda B^{at}-u}\right)\left(\frac{(A^{bt}-u)bt}{\lambda B^{bt}-u}\right)\right]^{\alpha} c^{ab(x+y)t}. \tag{4.2}$$

The above expression is symmetric in a and b, we can write $H(t)$ into two ways as:

$$H(t) = \sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha)}(bx,by;u;A,B,c;\lambda)\frac{(at)^n}{n!}\sum_{k=0}^{\infty}\mathcal{G}_k^{(\alpha)}(ax,ay;u;A,B,c;\lambda)\frac{(bt)^k}{k!}$$

$$H(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}b^k a^{n-k}\mathcal{G}_{n-k}^{(\alpha)}(bx,by;u;A,B,c;\lambda)\mathcal{G}_k^{(\alpha)}(ax,ay;u;A,B,c;\lambda)\frac{t^n}{n!}. \tag{4.3}$$

On the other hand, we have

$$H(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\mathcal{G}_{n-k}^{(\alpha)}(ax,ay;u;A,B,c;\lambda)\mathcal{G}_{k}^{(\alpha)}(bx,by;u;A,B,c;\lambda)\frac{t^n}{n!}. \tag{4.4}$$

On equating the coefficients of t^n from equations (4.3) and (4.4), we arrive at the desired result. □

Corollary 4.1. For $\alpha=1$, Theorem 4.1 reduces to:

$$\sum_{k=0}^{n}\binom{n}{k}b^k a^{n-k}\mathcal{G}_{n-k}(bx;u;A,B,c;\lambda)\mathcal{G}_{k}(ax;u;A,B,c;\lambda)$$

$$=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\mathcal{G}_{n-k}(ax;u;A,B,c;\lambda)\mathcal{G}_{k}(bx;u;A,B,c;\lambda). \tag{4.5}$$

Theorem 4.2. The following identity holds true:

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{(i+j)}b^k a^{n-k}\mathcal{G}_{n-k}^{(\alpha)}(bx+\frac{b}{a}i+j;u;A,B,c;\lambda)\mathcal{G}_{k}^{(\alpha)}(ay;u;A,B,c;\lambda)$$

$$=\sum_{k=0}^{n}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{(i+j)}\binom{n}{k}a^k b^{n-k}\mathcal{G}_{n-k}^{(\alpha)}(ax+\frac{a}{b}i+j;u;A,B,c;\lambda)$$

$$\times\mathcal{G}_{k}^{(\alpha)}(by;u;A,B,c;\lambda). \tag{4.6}$$

Proof. Consider

$$I(t)=\left[\left(\frac{(A^{at}-u)at}{\lambda B^{at}-u}\right)\left(\frac{(A^{bt}-u)bt}{\lambda B^{bt}-u}\right)\right]^{\alpha}\frac{1+\lambda(-1)^{a+1}c^{abt}}{(\lambda c^{at}+1)(\lambda c^{bt}+1)}c^{ab(x+y)t}$$

$$=\left(\frac{(A^{at}-u)at}{\lambda B^{at}-u}\right)^{\alpha}c^{abxt}\sum_{i=0}^{a-1}(-\lambda)^i c^{ibt}\left(\frac{(A^{bt}-u)bt}{\lambda B^{bt}-u}\right)^{\alpha}c^{abyt}(-\lambda)^j c^{jat}. \tag{4.7}$$

$$I(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j}a^{n-k}b^k\mathcal{G}_{n-k}^{(\alpha)}(bx+\frac{b}{a}i+j;u;A,B,c;\lambda)$$

$$\times\mathcal{G}_{k}^{(\alpha)}(ay;u;A,B,c;\lambda)\frac{t^n}{n!}. \tag{4.8}$$

On the other hand, we have

$$I(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{i+j}b^{n-k}a^k\mathcal{G}_{n-k}^{(\alpha)}(ax+\frac{a}{b}i+j;u;A,B,c;\lambda)$$

$$\times\mathcal{G}_{k}^{(\alpha)}(by;u;A,B,c;\lambda)\frac{t^n}{n!}. \tag{4.9}$$

On equating the coefficients of t^n from last two equations (4.8) and (4.9), we acquire at the desired result (4.6). □

5. Relation between λ-type Striling polynomials, Apostol-Bernoulli polynomial and generalized Apostol-type Frobenius-Genocchi polynomial

This section deals with some relationships in between Array-type polynomials, Apostol-Bernoulli polynomial and generalized Apostol-type Frobenius-Genocchi polynomial.

Theorem 5.1. The following relationship holds true:

$$\mathcal{G}_{n-2\nu}^{(-\nu)}(x;u;a,b,b;\lambda)=\frac{(\nu)!}{(-n)_{2\nu}}\sum_{k=0}^{n}\binom{n}{k}S_{\nu}^{n}\left(x;1,b;\frac{\lambda}{u}\right)Y_{n-k}^{(\nu)}\left(\frac{1}{u};a\right). \tag{5.1}$$

Proof. On replacing c by b and α by $-\nu$ in equation (2.1), we get

$$\sum_{n=0}^{\infty}\mathcal{G}_{n}^{(-\nu)}(x;u;a,b,b;\lambda)\frac{t^n}{n!}=\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{(-\nu)}b^{xt}. \tag{5.2}$$

$$\sum_{n=0}^{\infty}\mathcal{G}_{n}^{(-\nu)}(x;u;a,b,b;\lambda)\frac{t^n}{n!}=\nu!\frac{\left(\frac{\lambda}{u}b^t-1\right)^{\nu}b^{xt}}{(\nu!)\left(\frac{a^t}{u}-1\right)^{\nu}t^{\nu}}\frac{t^{\nu}}{t^{\nu}}. \tag{5.3}$$

Using equations (1.10) and (1.11), the above equation reduces to

$$\sum_{n=0}^{\infty}\mathcal{G}_{n}^{(-\nu)}(x;u;a,b,b;\lambda)\frac{t^{n+2\nu}}{n!}=\nu!\sum_{k=0}^{\infty}S_{\nu}^{k}(x;1,b;\frac{\lambda}{u})\frac{t^k}{k!}\sum_{m=0}^{\infty}Y_{m}^{(\nu)}\left(\frac{1}{u};a\right)\frac{t^m}{m!}. \tag{5.4}$$

Replacing m by $m-k$ in the above equation, we get

$$\sum_{n=0}^{\infty}\mathcal{G}_{n}^{(-\nu)}(x;u;a,b,b;\lambda)\frac{t^{n+2\nu}}{n!}=\nu!\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}S_{\nu}^{k}(x;1,b;\frac{\lambda}{u})Y_{n-k}^{(\nu)}\left(\frac{1}{u};a\right)\frac{t^n}{n!}. \tag{5.5}$$

On equating the coefficients of t^n, we arrive at the required result. □

Theorem 5.2. The following relationship holds true:

$$\mathcal{G}_{n-2\nu}^{(-\nu)}(x;u;a,b,b;\lambda)=\frac{(\nu)!}{(-n)_{2\nu}}\sum_{k=0}^{n}\binom{n}{k}\mathcal{S}\left(k,\nu,1,b;\frac{\lambda}{u}\right)\mathcal{B}_{n-k}^{(\nu)}\left(x,,\frac{1}{u},1,a,b\right). \tag{5.6}$$

Proof. Making replacement of c with b and α with $-\nu$ in equation (2.1), we get

$$\sum_{n=0}^{\infty}\mathcal{G}_{n}^{(-\nu)}(x;u;a,b,b;\lambda)\frac{t^n}{n!}=\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{(-\nu)}b^{xt}. \tag{5.7}$$

$$\sum_{n=0}^{\infty}\mathcal{G}_{n}^{(-\nu)}(x;u;a,b,b;\lambda)\frac{t^n}{n!}=(\nu!)\frac{\left(\frac{\lambda}{u}b^t-1\right)^{\nu}b^{xt}}{(\nu!)\left(\frac{a^t}{u}-1\right)^{\nu}t^{\nu}}\frac{t^{\nu}}{t^{\nu}}. \tag{5.8}$$

Using equations (1.10) and (1.1), the above equation reduces to

$$\sum_{n=0}^{\infty}\mathcal{G}_{n}^{(-\nu)}(x;u;a,b,b;\lambda)\frac{t^{n+2\nu}}{n!}=(\nu!)\sum_{k=0}^{\infty}\mathcal{S}\left(k,\nu,1,b;\frac{\lambda}{u}\right)\frac{t^k}{k!}$$

$$\times\sum_{n=0}^{\infty}\mathcal{B}_{n}^{(\nu)}\left(x,\frac{1}{u},1,a,b\right)\frac{t^n}{n!}. \tag{5.9}$$

Using Lemma [13, p.100, Eq.2], we get

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(-\nu)}(x;u;a,b,b;\lambda)\frac{t^{n+2\nu}}{n!} = \nu! \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \mathcal{S}(k,\nu,1,b,\frac{\lambda}{u})$$

$$\times \mathcal{B}_{n-k}^{(\nu)}\left(x, \frac{1}{u}, 1, a, b\right)\frac{t^n}{n!}. \tag{5.10}$$

On equating the coefficients of t^n, we arrive at the required result. □

Acknowledge

The authors are thankful to Integral University, Lucknow, India for acknowledging the paper with the manuscript number "IU/R&D/2018-MCN-000399".

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