The Characterization and the Product of Quasi-Ehresmann Transversals

Xiangjun Kong\textsuperscript{a}, Pei Wang\textsuperscript{b}

\textsuperscript{a} School of Mathematical Sciences and School of Statistics, Qufu Normal University, 273165, P. R. China
\textsuperscript{b} School of Software Engineering, Qufu Normal University, Qufu, 273165, P. R. China

Abstract. Wang (Filomat 29(5), 985-1005, 2015) introduced and investigated quasi-Ehresmann transversals of semi-abundant semigroups satisfy conditions (CR) and (CL) as the generalizations of orthodox transversals of regular semigroups in the semi-abundant case. In this paper, we give two characterizations for a generalized quasi-Ehresmann transversal to be a quasi-Ehresmann transversal. These results further demonstrate that quasi-Ehresmann transversals are the “real” generalizations of orthodox transversals in the semi-abundant case. Moreover, we obtain the main result that the product of any two quasi-ideal quasi-Ehresmann transversals of a semi-abundant semigroup $S$ which satisfy the certain conditions is a quasi-ideal quasi-Ehresmann transversal of $S$.

1. Introduction

The concept of inverse transversals of regular semigroups was introduced by Blyth-McFadden [1]. Since then, inverse transversals have attracted much attention and a series of important results have been obtained and generalized (see [1-5,11,13-21,23-26]). If $S$ is a regular semigroup, then an inverse transversal of $S$ is an inverse subsemigroup $S^o$ which meets $V(a)$ precisely once for each $a \in S$ (that is, $|V(a) \cap S^o| = 1$), where $V(a) = \{ x \in S | axa = a$ and $xax = x \}$ denotes the set of inverses of $a$. Since orthodox semigroups can be considered as generalizations of inverse semigroups, Chen [2] generalized inverse transversals to orthodox transversals in the class of regular semigroups and gave a construction theorem for regular semigroups with quasi-ideal orthodox transversals. Chen-Guo [4] obtained some important properties associated with orthodox transversals in the general case. Most recently, Kong, Meng, Zhao [13,15,16,17,21] investigated orthodox transversals and obtained some interesting results. Especially, Kong-Meng [17] acquired the characterization for a generalized orthodox transversal to be an orthodox transversal and present a concrete description of the maximum idempotent separating congruence on regular semigroups with orthodox transversals. If the concept of transversals could be introduced in the $E$-inversive semigroups, then the congruences [6,7] on them will be characterized more neatly.

The concept of adequate transversals was introduced for abundant semigroups by El-Qallali [5] as an analogue of inverse transversals, and followed by Chen, Guo, Shum, Kong and Wang etc. [3,11,14,18,19]. In [19], the authors shown that the product of any two quasi-ideal adequate transversals of an abundant...
semigroup $S$ which satisfy the regularity condition is a quasi-ideal adequate transversal of $S$.

Semi-abundant semigroups satisfy conditions (CR) and (CL) were introduced by Fountain-Gomes-Gould [8] as generalized regular semigroups and studied by many authors [8,9,10,22,26]. Gomes-Gould [9] studied some classes of semi-abundant semigroups satisfy conditions (CR) and (CL) by fundamental approaches and Lawson [22] considered some kinds of semi-abundant semigroups satisfy conditions (CR) and (CL) by category approaches, and Gould [10] gave a survey of investigations of special semi-abundant semigroups satisfy conditions (CR) and (CL), namely restriction semigroups and Ehresmann semigroups. Wang [26] introduced the concept of quasi-Ehresmann transversals of semi-abundant semigroups satisfy conditions (CR) and (CL), as a generalization of the concept of orthodox transversals of regular semigroups, and gave some properties associated with quasi-Ehresmann transversals.

In this paper, we continue along the line of [8,17,19,26] by studying quasi-Ehresmann transversals of semi-abundant semigroups which satisfy conditions (CR) and (CL). In this paper, we give two characterizations for a generalized quasi-Ehresmann transversal to be a quasi-Ehresmann transversal which further demonstrate that quasi-Ehresmann transversals are the “real” generalizations of orthodox transversals in the semi-abundant case. The main purpose of this paper is to show that the product of any two quasi-ideal quasi-Ehresmann transversals is a quasi-ideal quasi-Ehresmann transversal of $S$. The corresponding results associate with orthodox transversals and adequate transversals are generalized and enriched.

2. Preliminaries

Let $S$ and $S^o$ be semigroups. Throughout this paper, if no confusion, the set of idempotents of $S$ and $S^o$ are denoted by $E$ and $E^o$, respectively. For short, the set $V(a) \cap S^o$ is denoted by $V_S^o(a)$. If $E$ generates a regular semiband, that is, $(E)$ is a regular subsemigroup of $S$, then $S$ is said to satisfy the regularity condition.

$S^o$ is called a quasi-ideal of $S$, if $S^oSS^o \subseteq S^o$. We list some basic results as follows which are frequently used in this paper.

**Definition 2.1**[2] Let $S$ be a regular semigroup with an orthodox subsemigroup of $S^o$. Then $S^o$ is said to be an orthodox transversal of $S$, if the following two conditions are satisfied:

1. $(\forall a \in S) \ V_S^o(a) \neq \emptyset$;
2. For any $a, b \in S$, if $[a, b] \cap S^o \neq \emptyset$, then $V_S^o(a)V_S^o(b) \subseteq V_S^o(ab)$.

**Lemma 2.1**[17] Let $S$ be a regular semigroup and $S^o$ a subsemigroup of $S$ with $V_S^o(a) \neq \emptyset$ for each $a \in S$. Then $S^o$ is an orthodox transversal of $S$ if and only if

$$(\forall a, b \in S) \ [V_S^o(a) \cap V_S^o(b) \neq \emptyset \Rightarrow V_S^o(a) = V_S^o(b)].$$

The so-called Miller-Clifford theorem will be frequently used in this paper.

**Lemma 2.2**[12] (1) Let $e$ and $f$ be $D$-equivalent idempotents of a semigroup $S$. Then each element $a$ of $R_e \cap L_f$ has a unique inverse $a'$ in $R_f \cap L_e$, such that $aa' = e$ and $a'a = f$;

(2) Let $a, b \in S$. Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.

Let $S$ be a semigroup and $a, b \in S$. By $aR^b$ we mean that $xa = ya$ if and only if $xb = yb$ for all $x, y \in S$. The relation $L^*$ can be defined dually. $R^*$ is a left congruence and $L^*$ is a right congruence on $S$. A semigroup $S$ is called abundant if each $L^*$-class and each $R^*$-class of $S$ contains at least one idempotent. An abundant semigroup $S$ is called quasi-adequate if its idempotents form a band. A band $B$ is called a rectangular band if it satisfies the identity $abc = ac$ for all $a, b, c \in B$. An adequate semigroup is an abundant semigroup in which the idempotents commute.

Let $S$ be an abundant semigroup and $U$ an abundant subsemigroup of $S$. $U$ is called a $*$-subsemigroup of $S$, if for any $a \in U$, there exist idempotents $e \in L_e^*(S) \cap U$ and $f \in R_f^*(S) \cap U$.

**Definition 2.2**[5] Let $S$ be an abundant semigroup and $S^o$ a $*$-adequate subsemigroup of $S$. $S^o$ is called an adequate transversal of $S$, if for each $x \in S$ there exist idempotents $e, f \in S$ and a unique element $\overline{x} \in S^o$ such that $x = e\overline{x}f$, where $\overline{eL^*f}$ and $\overline{fR^*e}$. 
Let $S$ be a semigroup and $a, b \in S$. That $a \overline{R} b$ means that $ea = a$ if and only if $eb = b$ for all $e \in E$. The relation $\overline{L}$ can be defined dually. Denote $\overline{H} = \overline{L} \cap \overline{R}$. In general, $\overline{L}$ is not a right congruence and $\overline{R}$ is not a left congruence. Obviously, $L \subseteq \overline{L}$ and $R \subseteq \overline{R}$. If $a, b \in \text{Reg}S$, the set of regular elements of $S$, then $a \overline{R} b$ ($a \overline{L} b$) if and only if $a \overline{R}b$ ($a \overline{L} b$). On the relation $\overline{R}$ on a semigroup $S$, we have the following useful result.

**Lemma 2.3** Let $S$ be a semigroup and $a \in S, e \in E$. Then the following statements are equivalent:

1. $eRa$;
2. $ea = a$ and for all $f \in E, fa = a$ implies $fe = e$.

Now, we state the following fundamental concept of our paper. Semi-abundant semigroups satisfy conditions (CR) and (CL) were introduced by Fountain-Gomes-Gould\cite{8}.

**Definition 2.3** A semigroup $S$ is called semi-abundant if each $\overline{L}$-class and each $\overline{R}$-class of $S$ contains idempotents. In particular, if $\overline{L}$ is a right congruence and $\overline{R}$ is a left congruence on a semi-abundant semigroup $S$, then we say that $S$ satisfies conditions (CR) and (CL).

A semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) is quasi-Ehresmann if its idempotents form a subsemigroup of $S$. Certainly, regular semigroups are semi-abundant semigroups satisfy conditions (CR) and (CL), and orthodox semigroups are quasi-Ehresmann semigroups. It is easy to see a semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) is quasi-Ehresmann if and only if $\text{Reg}S$ is an orthodox subsemigroup of $S$. Let $S$ be a semi-abundant semigroup satisfies conditions (CR) and (CL). For $K \in \{L, R\}$ and $a \in S$, the $K$-class of $S$ containing $a$ is denoted by $K_a$.

A semi-abundant subsemigroup $U$ of a semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) is called a quasi-Ehresmann transversal of $S$. For any $x \in S$, denote

$$\Omega_{\overline{R}}(x) = \{ (e, \overline{x}, f) \in E \times S' \times E : x = e\overline{f}x', f\overline{R}x' \text{ for some } x', x' \in E'\},$$

and $\Gamma_x = \{ f : (e, \overline{x}, f) \in \Omega_{\overline{R}}(x) \}$, $I(x) = \{ e : (e, \overline{x}, f) \in \Omega_{\overline{R}}(x) \}$, $\Lambda(x) = \{ f : (e, \overline{x}, f) \in \Omega_{\overline{R}}(x) \}$, $I = \bigcup_{x \in S} I(x)$, $\Lambda = \bigcup_{x \in S} \Lambda(x)$.

**Lemma 2.4**\cite{26} Let $S$ be a semi-abundant semigroup satisfies conditions (CR) and (CL) and $S'$ a quasi-Ehresmann subsemigroup of $S$. Then $I = \{ e \in E : (\exists e' \in E') e\overline{L}e' \}$ and $\Lambda = \{ f \in E : (\exists f^+ \in E^+) f\overline{R}f^+ \}$.

**Definition 2.4**\cite{26} Let $S$ be a semi-abundant semigroup satisfies conditions (CR) and (CL) and $S'$ a quasi-Ehresmann subsemigroup of $S$. Then $S'$ is called a quasi-Ehresmann transversal of $S$ if the following three conditions hold:

1. $\Gamma_x \neq \emptyset$ for all $x \in S$;
2. $is \in I$ and $si \in \text{Reg}S$ implies $si \in E$ for all $i \in I$ and $s \in E'$;
3. $s\lambda \in \Lambda$ and $s\lambda \in \text{Reg}S$ implies $s\lambda \in E$ for all $\lambda \in \Lambda$ and $s \in E'$.

### 3. Two characterizations of quasi-Ehresmann transversals

Let $S$ be a semi-abundant semigroup satisfies conditions (CR) and (CL) with the set of idempotents $E$ and $S'$ a quasi-Ehresmann subsemigroup of $S$ with the set of idempotents $E'$. $S'$ is called a generalized quasi-Ehresmann transversal of $S$ if $\Gamma_x \neq \emptyset$ for all $x \in S$.

In the following, we shall give two characterizations for a generalized quasi-Ehresmann transversal to
be a quasi-Ehresmann transversal which further demonstrate that quasi-Ehresmann transversals are the “real” generalizations of orthodox transversals in the semi-abundant case.

**Theorem 3.1** Let $S$ be a semi-abundant semigroup satisfies conditions (CR) and (CL) with a generalized quasi-Ehresmann transversal $S^o$. Then $S^o$ is a quasi-Ehresmann transversal of $S$ if and only if

\[(V(a, b \in \text{Reg}(S)), [V_S(a) \cap V_S(b) \neq \emptyset \Rightarrow V_S(a) = V_S(b)]).
\]

**Proof.** (Sufficiency) Let $f \in E^o, e \in 1$ with $eLe^e \in E^o$. By means of $S^o$ is quasi-Ehresmann and $efLef \in E^o$ we have

\[fe^e \cdot ef \cdot fe^e = f \cdot e^e \cdot f \cdot fe^e = fe^e  \\
ef \cdot fe^e \cdot ef = e \cdot ff \cdot ef^e \cdot f = ef \cdot ef = ef.
\]

Thus $f \in V_S(fe^e) \cap V_S(ef)$. By the condition, we obtain $V_S(fe^e) = V_S(ef)$. From $S^o$ is quasi-Ehresmann, we deduce that $E^o$ is a band and so the semilattice $Y$ of rectangular bands $E_a, (a \in Y)$. Since $ef$ and $fe^e$ are in the same rectangular band, and so are inverses of each other. Hence $ef \in V_S(ef)$ and so

\[ef = (ef)(ef)(ef) = (ef)^2
\]

That is $ef$ is idempotent and we have in fact proved $IE^o \subseteq E$.

If $fe$ is regular, take $x \in V_S(fe)$ and $x^e \in V_S(x)$. Then $ef$ is idempotent and $ef \in V(ef)$ with $ef \subseteq eLe^e \in S^o$. Let $(e^exf)^r \in E^o$ with $(e^exf)^r \subseteq Le^e \times f$ since $e$ is a left congruence. Then $ef \subseteq L(e^exf)^r \in E^o$ and so $(e^exf)^r \in V_S(exf) \cap V_S((e^exf)^r)$. From the assumption and $S^o$ is quasi-Ehresmann, we have $V_S(ef) = V_S((e^exf)^r)$ and hence $V_S(exf) \subseteq E^o$. Meanwhile we deduce that the regular elements of $S^o$ form an orthodox subsemigroup of $S^o$, and so $fx^e \in V_S(e^exf)$ since $e^e, f \in E^o$. Hence

\[fx^e \cdot ef \cdot fx^e = fx^e \cdot e^e \cdot fx^e = fx^e
\]

and

\[fx^e \cdot ef \cdot fx^e = e \cdot ef \cdot fx^e \cdot e^e = e \cdot e^e = ef
\]

since $eLe^e$ with $e, e^e$ are idempotent. So, $fx^e \in V_S(exf)$. Similarly, one can prove that $e^e \cdot fx^e \in V_S(exf)$. Thus $fx^e \in E^o$ and $V_S(fx^e) \subseteq E^o$ and consequently $x \in V^o$. Therefore $e^e \in E^o$ and

\[fe = fe \cdot e^e \cdot f = fe \cdot ef \cdot e^e \cdot fe = fe \cdot e^e \cdot xe
\]

Premultiplying and postmultiplying by $x$, we obtain

\[x = xfe = xfe \cdot fe^e \cdot xfe = xfe^e x
\]

Thus $fe^e \subseteq Lx$, for $fe^e \in E^o$, from which we deduce that $fx^e = fe^e \cdot xf \times Lxf$. By means of $fe^e \in V(ef)$ and $xf \in E^o$, we have $fe^e \subseteq V_S(exf)$. It is obvious that $xf \subseteq V(ef)$ and so $fe^e \subseteq V_S(ef)$. Therefore $fe^e \subseteq V_S(ef)$. Since $fe \subseteq V_S(ef)$ and so $fe^e \subseteq V_S(ef)$, then it is idempotent. Dually, we can prove that $E^o \subseteq V^o$. For all $\lambda \in \Lambda, f \in EF$, if $\lambda f$ is regular, then it is idempotent.

(Necessity) By [26, Theorem 3.6 (4)], the condition is necessary. \[\square\]

**Theorem 3.2** Let $S$ be a semi-abundant semigroup satisfies conditions (CR) and (CL) with a generalized quasi-Ehresmann transversal $S^o$. Then $S^o$ is a quasi-Ehresmann transversal if and only if for any regular elements $a \in S$, $b \in S^o$, if $ba$ is regular, then $V_S(a)V_S(b) \subseteq V_S(ba)$, and if $ab$ is regular, then $V_S(b)V_S(a) \subseteq V_S(ab)$.
Proof. (Necessity) For any regular elements \( a \in S \), \( b \in S' \), take \( a^r \in V_S(a) \), \( b^r \in V_S(b) \), if \( S' \) is a quasi-Ehresmann transversal, then by the definition, \( aa'b'b \in IE' \subseteq E \). If \( ba \) is regular, take \( (ba)^r \in V_S(ba) \), then
\[
(a^r(ba)^r)b((ba)^r) = b^r(b^r a)(ba)'(ba)b = b^r(ba)(ba)'(ba)b = b^r (ba)d = b^r b a a.
\]
Thus \( b^r b a a \) is regular and so \( b^r b a a \in E'I \subseteq E \). Therefore
\[
ad^r b' \cdot ba \cdot d^r b' = a'(aa'b'b)(ad^r b'b)b' = a' \cdot ad^r b' \cdot b' = d^r b'
\]
and so \( V_S(a)V_S(b) \subseteq V_S(ba) \). Similarly, if \( ab \) is regular, then \( V_S(b)V_S(a) \subseteq V_S(ab) \).

(Sufficiency) For any regular elements \( t_1, t_2 \in S' \), if \( V(t_1) \cap V(t_2) \neq \emptyset \), take \( t \in V(t_1) \cap V(t_2) \) and \( t^r \in V_S(t_1) \). From \( t, L, t, R \), by Lemma 2.2, \( t^r \in t_1^r, t_2^r \), and \( t_1^r t_2^r = V_S(t_1 t_2) \) since by the assumption \( t_1^r t_2 \in V_S(t_1 t_2) \). Similarly, \( t_1^r t_2^r \) with \( t^r t_1 t_2 \in E \). Thus
\[
t_1^r t_2^r = t_1^r t_2^r = t_1^r t_2^r = t_1^r t_2^r = (t_1^r t_2) t_1^r t_2 = (t_1^r t_2) t_1^r t_2 = t_1^r t_2.
\]
Hence \( t^r \in V_S(t_1) \), that is, \( V_S(t_1) \cup V_S(t_2) \neq \emptyset \). Therefore \( V_S(t_1) \cap V_S(t_2) = V_S(t_2) \) since the regular elements of \( S' \) form an orthodox subsemigroup of \( S \).

For any \( e \in S \), \( V_S(e) \cap E' \neq \emptyset \), take \( f \in V_S(e) \cap E' \). Then for any \( e^r \in V_S(e) \), we have \( e \in V(f) \cap V(e^r) \) and so by the above result, \( V_S(f) = V_S(e^r) \). Consequently, \( e^r \) is an inverse of \( f \) in \( S' \) and \( e^r \in E' \) since \( S' \) is quasi-Ehresmann. That is, if \( V_S(e) \cap E' \neq \emptyset \), then \( V_S(e) \subseteq E' \).

Let \( e, f \in I \) with \( e \Lambda f \). Take \( h \in E' \) such that \( h \Lambda L e \Lambda f \), then \( h \in V_S(e) \cap V_S(f) \). For any \( g \in V_S(e) \), by the above result we have \( g \in E' \). It is easy to see that \( ghg \in V_S(gf \bar{g}) \) and \( ghg \in V_S(g \bar{g}) = V_S(g) \). Then \( g \bar{f} g \) and \( g \bar{h} g \) have a common inverse \( ghg \). Consequently \( ghg \cdot g \bar{f} g \cdot ghg = ghg \) and thus \( g \bar{f} g = g \). Since \( gh \Lambda L e \Lambda f \), by Lemma 2.2, \( f \bar{g} \Lambda R f \) and so \( f \bar{g} f = f \). Thus \( g \in V_S(f) \) and so \( V_S(e) \subseteq V_S(f) \). Similarly, we have the reverse inclusion and hence \( V_S(e) = V_S(f) \). Dually, if \( e, f \in I \) with \( e \Lambda R f \), then \( V_S(e) = V_S(f) \).

It is easy to see that for any \( a \in E \), we have \( V_S(a) = V_S(a^r) \). Similarly, \( V_S(a') \) is a quasi-Ehresmann transversal. Therefore \( V_S(a) = V_S(b) \) and so by Theorem 3.1 \( S' \) is a quasi-Ehresmann transversal.

Obviously, a regular semigroup with an orthodox transversal is a semi-abundant semigroup satisfies conditions (CR) and (CL) with a generalized quasi-Ehresmann transversal. Comparing Lemma 2.1 with Theorem 3.1, and Definition 2.1 with Theorem 3.2, it is illustrated by these two points of view that the transversal is a quasi-Ehresmann transversal. Thus, quasi-Ehresmann transversals are the generalization of orthodox transversals in the semi-abundant case.

By means of the properties of adequate transversal\(^3\), one can easily observe that an abundant semigroup with an adequate transversal is a semi-abundant semigroup satisfies conditions (CR) and (CL) with a quasi-Ehresmann transversal.

In the following, we will investigate when a quasi-Ehresmann transversal is an orthodox transversal and when a quasi-Ehresmann transversal is an adequate transversal, respectively. We have the following results.

**Theorem 3.3** Let \( S' \) be a quasi-Ehresmann transversal of the semi-abundant semigroup \( S \) satisfies conditions (CR) and (CL). Then

(i) \( S' \) is an orthodox transversal of \( S \) if and only if \( S \) is a regular semigroup.

(ii) if \( S \) and \( S' \) are abundant, then \( S' \) is an adequate transversal of \( S \) if and only if \( S' \) is an adequate semigroup.
Proof. (i) (Sufficiency) If $S$ is regular, every element in $S$ is regular, and so $V_S(a) \neq \emptyset$ for each $a \in S$. It follows from Theorem 3.1 that for any $a, b \in S$, $V_S(a) \cap V_S(b) \neq \emptyset$ implies that $V_S(a) = V_S(b)$. Thus, $S$ is an orthodox transversal of $S$ by Lemma 2.1.

(Necessity) If $S'$ is an orthogonal transversal, every element $x'$ in $S'$ is regular. For any $a \in S, a = a(x')$ with $e, f \in E, e\mathcal{L}a = E'$, $f\mathcal{R}a = E'$. Since $a$ is regular, $a\mathcal{L}a = E'$ implies that $a$ has a unique inverse $x \in R_\mathcal{L} \cap L_\mathcal{R}$, such that $ax = a', xa = a'$. Consequently, $axa = axa\cdot xa = a(xa)axa = e$ since $f\mathcal{R}a = \bar{x}a\mathcal{L}a = \bar{a}x$. That is, $a$ is a regular and therefore $S$ is a regular semigroup.

(ii) The necessary condition is obvious.

(Sufficiency) Let $a \in S, a = a(x')$ with $e, f \in E, e\mathcal{L}a = E'$, $f\mathcal{R}a = E'$. It follows from $e\mathcal{R}a\mathcal{R} \mathcal{L}b$ that $\bar{b}^{-} \cdot i \cdot \bar{b}^{-} = \bar{b}^{-} \cdot b^{-} = \bar{b}^{-} = \bar{e} = e$, so $\bar{b}^{-} = e$ is regular and by Theorem 3.1, $\bar{b}^{-} \in E$. Since $\bar{b}^{-} \mathcal{R}b^{-} \cdot e \mathcal{L}a$, if $S'$ is adequate, the idempotents in $S'$ commute and so $\bar{a}^{-} \mathcal{R} \mathcal{L} \bar{a}^{-} = \bar{a}^{-} \mathcal{R} \mathcal{L} \bar{a}^{-}$. By $S'$ is adequate, the idempotents in $S'$ commute and so $\bar{a}^{-} = \bar{b}^{-}$. Therefore $a = \bar{a}^{-} \bar{a} = \bar{b}^{-} \bar{a} = \bar{b}^{-} \bar{a} = \bar{b}$ and consequently, $S$ is in fact the adequate transversal of $S$. \qed

Therefore, by Theorem 3.3 we can say that quasi-Ehresmann transversals are the “real” common generalization of orthodox transversals and adequate transversals in the semi-abundant case.

4. The main theorem

In 1986, Saito [25] had proved that the product of any two quasi-ideal inverse transversals of a regular semigroup $S$ is a quasi-ideal inverse transversal of $S$. In 2011, we [19] had obtained that the product of any two quasi-ideal adequate transversals of an abundant semigroup $S$ which satisfy the regularity condition is a quasi-ideal adequate transversal of $S$. In this section, we acquire that the product of any two quasi-ideal quasi-Ehresmann transversals of a semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) and satisfies the regularity condition is a quasi-ideal quasi-Ehresmann transversal of $S$. Furthermore, all of the quasi-ideal quasi-Ehresmann transversals of the form a rectangular band.

Let $H$ and $J$ be subsets of a semigroup $S$ and write $HJ$ for $\{hj : h \in H, j \in J\}$. Clearly $(HJ, JK \subseteq S) (HJK) = H(JK)$ and we denote it by $HJK$.

Lemma 4.1 Let $S'$ be a quasi-ideal quasi-Ehresmann transversal of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) and $H$ a subset of $S$. Then

1. $HSS' = HS'$ and $SS'H = S'H$;
2. $HSS'$ and $SS'H$ are both subsemigroups and quasi-ideals of $S$;
3. for any $x \in RegS$, if $|V(x) \cap H| \geq 1$, then $|V(x) \cap HS'| \geq 1$ and $|V(x) \cap S'H| \geq 1$.

Proof. (1) Let $h \in H, x \in S$ and $s \in S'$. Then $h = e_hf_h$ with $f_h \mathcal{R}h \in E'$ and so $hxs = h\mathcal{R}f_hxs \in HS'SS' \subseteq HS'$. It is obvious that $h = e_hf_h$, $f_h \mathcal{R}h \in E'$ and so $hxs = h\mathcal{R}f_hxs \in HS'SS' \subseteq HS'$. Similarly, $S'HS' = HS'$.

(2) It is easy to see $HS' \subseteq H \cap SS'SS' \subseteq HS'$, thus $HS'$ is a subsemigroup of $S$. Similarly, $HS' \subseteq H \cap SS'SS' \subseteq HS'$, and so $HS'$ is a quasi-ideal of $S$. There is a dual result for $S'H$.

(3) For any regular element $x \in S$, take $x' \in V(x) \cap H$, then for any $x' \in V_S(x), x'xx' \in V(x) \cap HS'$, $V(x) \cap HSS' = HSS'$, that is $|V(x) \cap HSS'| \geq 1$. Similarly, $|V(x) \cap S'H| \geq 1$. \qed

Lemma 4.2 Let $S'$, $S''$ be quasi-ideal quasi-Ehresmann transversals of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL). For every $a \in RegS$, we have $V_{S'S'}(a) = V_{S''}(a) \cdot a \cdot V_S(a)$.

Proof. Let $a^e \in V_S(a), a^e \in V_S(a)$. Then $a^eaa^e = S'SS' = S'S'$ and $a^eaa^e \in V(a)$, and so $V_{S''}(a) \cdot a \cdot V_S(a) \subseteq
Lemma 4.3  Let $S^o$ be a quasi-ideal quasi-Ehresmann transversal of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL). For any $x, y \in S$, there exist $\xi, \eta \in S$ such that $x = e_\xi f_{\xi y} e_\eta L^{\eta \xi y}, f_{\xi y} R^{\eta \xi y}$ for some $\xi, \eta \in E$. Then

(1) $\xi S^o \xi \in E$,
(2) $e_\xi (\xi S^o \xi y)^* \in E$,
(3) $(f_{\xi y} y)^* f_{\xi y} \in \Lambda_{\xi y}$.

Proof.  Certainly $xy = e_\xi f_{\xi y} e_\eta f_{\eta y} = e_\xi (\xi S^o \xi y)^* (\xi S^o \xi y) (f_{\xi y} y)^* f_{\eta y}$,

where $e_\xi (\xi S^o \xi y)^* \in E; f_{\xi y} y)^* f_{\xi y} \in E \Lambda \subseteq E$ and $\xi S^o \xi \eta \xi y \in S^o$ since $S^o$ is a quasi-ideal. Since $R, R$ are left congruences and $L, L$ are right congruences, we have

$e_\xi (\xi S^o \xi y)^* L (\xi S^o \xi y)^* R (\xi S^o \xi y) = \xi S^o \xi \eta \xi y = \xi S^o \xi \eta \xi y = f_{\xi y} y)^* f_{\xi y} \xi S^o \xi \eta \xi y$.

Therefore the above properties valid.  \hfill \Box

In what follows $S^+$ and $S^o$ will denote a pair of quasi-Ehresmann transversal of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) and $E_S$ and $E_S$ will denote the idempotents of them respectively to avoid confusion. For the sake of simplicity, in $S^+$, we still denote the typical idempotent that $\tilde{L}$-related and $L$-related to $a \in S^+$ by $a^+$ and $a^*$ respectively. For any $x \in S$, we write $x = e_\xi f_{\xi x}$ and $x = i_x \lambda_x$, as the decompositions of $x$ in $S^+$ and $S^+$ respectively. Then $\lambda$ has the same meaning as in Definition 2.4. More precisely, $a, \lambda, \lambda_x \in E_S$ with $\tilde{L} L^{\eta \lambda}$ and $i_x \lambda_x \lambda, \lambda$, and so $i_x \lambda \lambda_x \lambda$.

Let $S^+$ and $S^+$ be quasi-Ehresmann transversals of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL). Denote

$I(S^+, S^+) = \{aa^+ : a \in RegS \cap S^+, a^+ \in V_S(a)\}$,

$\Lambda(S^+, S^+) = \{a^+ a : a \in RegS \cap S^+, a^+ \in V_S(a)\}$.

Theorem 4.4  Let $S^+$ and $S^+$ be a pair of quasi-ideal quasi-Ehresmann transversals of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) and satisfies the regularity condition. Then

$I(S^+, S^+) = \Lambda(S^+, S^+) = I_\circ \cap \Lambda_\circ$. 

V_{S^o}(a)$. For every $x^o y^o \in V_{S^o}(a)$, we have

\[ a = ax^o y^o a, \quad x^o y^o = x^o y^o : a \cdot x^o y^o. \]

Hence

\[ x^o y^o = x^o y^o : a^o a^o a \cdot x^o y^o = x^o y^o : a \cdot a^o a x^o y^o. \]

and

\[ x^o y^o a^o \in S^o S^o \subseteq S^+, \quad a^o a x^o y^o \in S^o S^o \subseteq S^+. \]

On the other hand,

\[ a \cdot x^o y^o a^o \cdot a = a \cdot x^o y^o a^o, \quad x^o y^o a^o \cdot a \cdot x^o y^o a^o = x^o y^o a^o. \]

Thus $x^o y^o a^o \in V_{S^+}(a)$ and dually, $a^o a x^o y^o \in V_{S^+}(a)$. Therefore $V_{S^o}(a) \subseteq V_S(a) \cap a \cdot V_{S^+}(a)$.  \hfill \Box
Proof. For any $aa^o \in I(S^0, S^0)$, where $a \in \text{Reg}S \cap S^0$, $a^o \in V_{S^0}(a)$, certainly, $a \in V_{S^0}(a^o)$ and so $aa^o = a^o a^o \in \Lambda(S^0, S^0)$. Thus $I(S^0, S^0) \subseteq \Lambda(S^0, S^0)$ and dually $\Lambda(S^0, S^0) \subseteq I(S^0, S^0)$. Consequently, $I(S^0, S^0) = \Lambda(S^0, S^0)$ and we denote it by $W$. From the above definitions, it is clear that $W \subseteq I_s \cap A_s$.

Conversely, suppose that $x \in I_s \cap A_s$. Since $x \in A_s$, we have $x = x^o x$ for some $x^o \in V_{S^0}(x)$ with $x^o \in E_{S^0}$ and so $x^o = x^o x^o$. Similarly, $x \in I_s$, implied that $x = xx^o$ for some $e^o \in V_{S^0}(x)$ with $e^o \in E_{S^0}$ and so $x^o = x x^o$. Let $x^o \in V_{S^0}(x)$. From $e^o \not\in \mathcal{X} x^o \mathcal{X}$, by Lemma 2.2, we deduce that $x^o \mathcal{X} x^o \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X}$ with $x^o x^o x^o \in E_{I_s} I_s \subseteq E_s$ since $S^0$ is a quasi-ideal and $S$ satisfies the regularity condition. Thus $x^o \not\in \mathcal{X} x^o \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X}$. Consequently, $x^o \mathcal{X} x^o \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X}$ and so by Lemma 2.2, $x^o \not\in \mathcal{X} x^o \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X}$ and $x^o x^o \mathcal{X} \mathcal{X} x^o \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X}$ in $I_s \cap A_s$. Therefore

\[ x^o \cdot x^o x^o \cdot x^o = xx^o = x^o \quad \text{and} \quad x^o x^o \cdot x^o \cdot x^o x^o = x^o x^o x^o x^o = x^o x^o x^o x^o \]

and so $x^o x^o \in V_{S^0}(x)$. Hence $x = x^o \cdot x^o x^o \in I(S^0, S^0) = W$.  

\[ \square \]

**Theorem 4.5** Let $S^0$ and $S^0$ be quasi-ideal quasi-Ehresmann transversals of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL) and satisfies the regularity condition. Then $S^0 S^0$ is a quasi-ideal quasi-Ehresmann transversal of $S$.

**Proof.** It is evident that $S^0 S^0$ is a subsemigroup and a quasi-ideal of $S$. For any $x \in S^0 S^0$, there exist $s^o \in S^0$, $t^o \in S^0$ such that $x = s^o t^o$. It follows from $S^0$ is a quasi-ideal of $S$ and Lemma 4.3 that $e_{S^0} (s^o f_{S^0} t^o)^o) \in I_{S^0 S^0}$, and we denote it by $e_{S^0}$. It is obvious that $I_{S^0 S^0}$. Since $s^o \in S^0$ and so from $e_{S^0} \mathcal{R}_{S^0} \in \mathcal{R}_{S^0}$ we deduce that $e_{S^0} \in I_{S^0 S^0}$. Thus by Theorem 4.4 there exists $a \in \text{Reg}(S^0)$ such that $e_{S^0} = a a^o$ and so

\[ e_{S^0} = e_{S^0} (s^o f_{S^0} t^o)^o = a a^o (s^o f_{S^0} t^o)^o \in S^0 S^0. \]

Similarly, $\lambda_s \in S^0 S^0$. Thus $e_{S^0}, \lambda_s \in \mathcal{R}_{S^0 S^0}$, and so from $e_{S^0} \not\in \mathcal{R}_{S^0 S^0}$ we deduce that $S^0 S^0$ is semi-abundant. It is a routine matter to show that $e_{S^0} \mathcal{R}_{S^0 S^0} \mathcal{L}_{S^0 S^0} \lambda_s$, thus $S^0 S^0$ is a quasi-ideal quasi-Ehresmann subsemigroup of $S$.

Let $e$ be an idempotent of $S^0 S^0$. Then $e = as$ for some $a \in S^0$, $s \in S^0$. Since $(asa)(asa) = asas = asasa = asasa^o$. Since $as \in S^0$, each idempotent of $S^0 S^0$ is of the form $b b^o$ for some regular element $b \in S^0$. Let $e$ and $f$ be idempotents of $S^0 S^0$. Then $e = be^o$ and $f = cc^o$ for some regular elements $b, c \in S^0$ with $b^o \in V_{S^0}(b)$ and $c^o \in V_{S^0}(c)$. For any $l \in E^o$, by the regularity condition, $l c c^o$ is regular and so $l c c^o \in E^o$ since $S^0$ is a quasi-Ehresmann transversal of the semi-abundant semigroup $S$ satisfies conditions (CR) and (CL). Thus $l c c^o \in E \cap S^0 = E^o$ since $S^0$ is also a quasi-ideal of $S$. Therefore $e f = b b^o c c^o = b^o(c c^o) = b^o c c^o \in I_{E^o E^o} \subseteq E$. Consequently, $S^0 S^0$ is a quasi-Ehresmann semigroup.

For any $x \in S$, there exist $a, b \in \text{Reg}S$ such that $e_a = a a^o$, $\lambda_s = b b^o$, where $a^o \in V_{S^0}(a)$, $b^o \in V_{S^0}(b)$. Thus

\[ x = e_{S^0} \mathcal{X} x \lambda_a = a a^o x b b^o = a a^o a^o a^o x b b^o b b^o b^o, \]

where $a^o \in V_{S^0}(a^o)$, $b^o \in V_{S^0}(b^o)$, and consequently

\[ e_a = a a^o \mathcal{X} x \lambda_a \in E_{S^0 S^0}, \quad \lambda_s = b b^o b b^o \in E_{S^0 S^0}. \]

Since $a^o a a^o x b b^o b^o \lambda_s = a^o a^o a^o x \lambda_a = a^o a^o x$, we have $a^o a^o x b b^o b b^o \mathcal{R}_{S^0 S^0} a^o x$. From $x \mathcal{R} e_a$ and $\mathcal{R}$ is a left congruence we deduce that

\[ a^o a^o x \mathcal{R} \mathcal{R} a^o a^o x e_a = a a^o a^o x \in E_{S^0 S^0}. \]

Similarly,

\[ a a^o a^o x b b^o \mathcal{L} x b b^o b b^o \mathcal{L} b b^o b b^o \in E_{S^0 S^0}. \]

Consequently, $x = e_{S^0 S^0} (a a^o a^o x b b^o b b^o) \lambda_s$ with $e_a, \lambda_s \in E \mathcal{L} (a a^o a^o x b b^o b b^o) \mathcal{R} a^o a^o x \mathcal{R} a^o a^o x \lambda_s \in E_{S^0 S^0}$ and $e_{S^0 S^0} (a a^o a^o x b b^o b b^o) \mathcal{R} b b^o b b^o \in E_{S^0 S^0}$. Therefore, $S^0 S^0$ is a generalized quasi-Ehresmann transversal of $S$.  

\[ \square \]
For regular elements \( c \in S, d \in S^o S^o \), take \( c' \in V_S S' (c), d' \in V_{S'} S' (d) \), then by Lemma 4.2, there exist \( c'' \in V_S (c), c'' \in V_S (c), d'' \in V_{S'} (d), d'' \in V_{S'} (d) \), such that \( c' = c'' c d, d' = d'' d d \). Since \( d \in S^o S^o, d' \in V_{S'} (d) \), we have \( d \in V_{S'} S' (d') \). By Lemma 4.2, there exist \( (d')^o \in V_{S'} (d'), (d')^o \in V_{S'} (d') \), such that \( d = (d')^o d (d')^o \). So \( c' c d d' = c'' c d d' c d = (d')^o d (d')^o c d = (d')^o d c \in A_o A_o \subseteq A_o \), and \( c' c d d' \) is idempotent. On the other hand, \( c' c d d' = c c e c d c d = (d')^o d (d')^o d c = (d')^o d c \in A_o A_o \subseteq A_o \), and \( c' c d d' \in E \). Thus \( c d c = c c' c d d' c d = (d')^o d (d')^o d c = (d')^o d c \in A_o A_o \subseteq A_o \), and so \( V_{S'} (d) V_{S'} (c) \subseteq V_{S'} (c (d) \). Similarly, \( V_{S'} (c) V_{S'} (d) \subseteq V_{S'} (c (d) \).

It follows from Theorem 3.2 that \( S^o S^o \) is a quasi-Ehresmann transversal. Since \( S^o S^o \) is a quasi-ideal, therefore \( S^o S^o \) is a quasi-ideal quasi-Ehresmann transversal of \( S \).

**Theorem 4.6** Let \( S \) be a semi-abundant semigroup satisfying conditions (CR) and (CL) and the the regularity condition. If \( S \) has a quasi-ideal quasi-Ehresmann transversal, then all quasi-ideal quasi-Ehresmann transversals of \( S \) form a rectangular band.

**Proof.** If \( S^o \) is a quasi-ideal quasi-Ehresmann transversal of \( S \), then \( S^o S^o = S^o \). To see this, for \( s^o \in S^o, s' = s' (s')^o \in S^o S^o \), hence \( S^o \subseteq S^o S^o \) and the reverse inclusion is obvious. By Theorem 4.5, all quasi-ideal quasi-Ehresmann transversals of \( S \) form a semigroup and so form a band.

Let \( S^o, S^o, S^o \) be arbitrary three quasi-ideal quasi-Ehresmann transversals of \( S \). For any \( x \in S^o, y \in S^o, \) we have \( x y (x y)^o \in S^o S^o S^o \subseteq S^o S^o \), \( x y (x y)^o \in S^o S^o S^o \subseteq S^o S^o \), \( x y (x y)^o \in S^o S^o S^o \subseteq S^o S^o \), where \( b^o \in E \) and \( c^o \in E \). Thus \( S^o S^o \subseteq S^o S^o \) and so \( S^o S^o \subseteq S^o S^o \) and \( S^o S^o \subseteq S^o S^o \). For every \( x \in S^o, y \in S^o, z \in S^o, \) then \( a^o c^o = a^o f_a (f_a)^o f_a c^o = f_a S^o S^o S^o = f_a S^o S^o \),

with \( f_a \) is an inverse in \( S^o \) of \( f_a \). Thus \( S^o S^o S^o = S^o S^o \) and therefore all quasi-ideal quasi-Ehresmann transversals of \( S \) form a rectangular band.

**Acknowledgements** The authors would like to express their sincere thanks to the referees for their valuable suggestions and corrections, which much improved this paper. Thanks also go to Professor Dijana Mosic for the timely communications. The first and corresponding author is a postdoctoral researcher of the Postdoctoral Station of Quufu Normal University and a Visiting Research Fellow of Curtin University.

**References**