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Compact Complement Topologies and k-Spaces

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Abstract. Let (X, τ) be a Hausdorff space, where X is an infinite set. The compact complement topology $\tau^\star$ on X is defined by: $\tau^\star = \{\emptyset\} \cup \{X \setminus M : M \text{ is compact in } (X, \tau)\}$. In this paper, properties of the space $(X, \tau^\star)$ are studied in **ZF** and applied to a characterization of k-spaces, to the Sorgenfrey line, to some statements independent of **ZF**, as well as to partial topologies that are among Delfs-Knebusch generalized topologies. Between other results, it is proved that the axiom of countable multiple choice (**CMC**) is equivalent with each of the following two sentences: (i) every Hausdorff first-countable space is a k-space, (ii) every metrizable space is a k-space. A **ZF**-example of a countable metrizable space whose compact complement topology is not first-countable is given.

1. Introduction

The compact complement topology of the real line was considered, for instance, in Example 22 of the celebrated book by Steen and Seebach "Counterexamples in Topology" ([19]). We investigate this notion in a much wider context of Hausdorff spaces and of partially topological spaces that belong to the class of generalized topological spaces in the sense of Delfs-Knebusch (cf. [2] and [14]). Our results are proved in **ZF** if this is not otherwise stated. All axioms of **ZF** can be found in [11].

In Section 2, we give elementary properties of the compact complement topology of a Hausdorff space. In particular, we show that if a Hausdorff space is locally compact and second-countable, then its compact complement topology is second-countable, while the compact complement topology of a non-locally compact metrizable space need not be first-countable. We give an example of a countable metrizable space whose compact complement topology is not first-countable. In Section 3, a necessary and sufficient condition for a set to be compact with respect to the compact complement topology of a given Hausdorff space leads us to a new characterization of k-spaces. A well-known theorem of **ZFC** states that all first-countable Hausdorff spaces are k-spaces (cf. Theorem 3.3.20 of [3]). We show that this theorem may fail in **ZF**. More precisely, we prove that, if $\mathcal{M}$ is a model of **ZF**, then all Hausdorff first-countable spaces of $\mathcal{M}$ are k-spaces if and only if all metrizable spaces in $\mathcal{M}$ are k-spaces which holds if and only if the axiom of

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countable multiple choice (Form 126 in [6]) is true in $\mathcal{M}$. In consequence, in some models of **ZF** there are metrizable spaces that are not k-spaces. We prove that if the Sorgenfrey line is a k-space, then the real line with its natural topology is sequential, so the Sorgenfrey line fails to be a k-space in some models of **ZF**. In section 4, we introduce a notion of a compact complement partial topology corresponding to a given partial topology. Partially topological Delfs-Knebusch generalized topological spaces were introduced in Definition 2.2.67 of [14]; however, a more convenient than in [14] definition of a partial topology was given in [13].

In this paper, definitions of compact, Lindelöf, regular, completely regular and normal spaces are not the same as in [3]. Namely, we call a topological space X *compact* (respectively, *Lindelöf*), if every open cover of X has a finite (respectively, countable) subcover. We omit separation axiom T_1 in the definitions of regular, completely regular and normal spaces from [3]. Our set-theoretic notation is standard. In particular, if X is a set, then $\mathcal{P}(X)$ denotes the power set of X. For weaker forms of the axiom of choice, we use mainly the notation from [5] and [9].

2. Basic Properties of Compact Complement Topologies

Throughout this article, we assume that τ is a topology on an infinite set X such that (X, τ) is a Hausdorff space.

Definition 2.1. We denote by $\mathcal{K}(\tau)$ the collection of all τ-compact sets, i.e. of all sets that are compact in the space (X, τ). The compact complement topology of (X, τ) is the collection

$$\tau^\star = \{\emptyset\} \cup \{X \setminus M : M \in \mathcal{K}(\tau)\}.$$

Since it is true in **ZF** that a compact subspace of a Hausdorff space is closed (see Theorem 3.1.8 of [3]), it is easy to check in **ZF** that $\tau^\star$ is a topology on X. Clearly, if (X, τ) were a finite Hausdorff space, then $\tau = \tau^\star = \mathcal{P}(X)$.

For a subset Y of X and a topology $\mathcal{T}$ on X, let

$$\mathcal{T}|Y = \{V \cap Y : V \in \mathcal{T}\}.$$

Then $(Y, \mathcal{T}|Y)$ is a topological subspace of $(X, \mathcal{T})$.

Theorem 2.2. *Let $Y \subseteq X$. The following conditions hold in* **ZF**:

(i) $\tau^\star|Y$ *is coarser than* $\tau|Y$, *i.e.,* $\tau^\star|Y \subseteq \tau|Y$;
(ii) *if Y is compact in (X, τ), then $\tau^\star|Y = \tau|Y$;*
(iii) $\tau^\star|Y = \tau|Y$ *if and only if there exists a τ-compact set C such that $Y \subseteq C$.*

Proof. To prove (i), it suffices to show that $\tau^\star \subseteq \tau$. Let $U \in \tau^\star$ and $U \neq \emptyset$. Then $U = X \setminus M$ for a compact subspace M of (X, τ). Since a compact subspace of a Hausdorff space is closed, M is closed in (X, τ). Hence, $U \in \tau$ and, in consequence, $\tau^\star \subseteq \tau$.

(ii) Suppose that Y is τ-compact and $V \in \tau$. Since (X, τ) is Hausdorff, the set Y is τ-closed, so $A = Y \cap (X \setminus V)$ is a τ-closed subset of the τ-compact set Y. Hence, A is τ-compact. Notice that $V \cap Y = Y \cap (X \setminus A)$. This implies that $V \cap Y \in \tau^\star|Y$ and $\tau|Y \subseteq \tau^\star|Y$.

(iii) If C is a τ-compact set such that $Y \subseteq C$, then since it follows from (ii) that $\tau|C = \tau^\star|C$, we immediately deduce that $\tau|Y = \tau^\star|Y$. Finally, suppose that Y is a subset of X such that $\tau^\star|Y = \tau|Y$. Let $V \in \tau$ and $\emptyset \neq V \cap Y \neq Y$. Since $V \cap Y \in \tau^\star|Y$, there exists a τ-compact set K_0 such that $V \cap Y = Y \setminus K_0$. Fix $x_0 \in V \cap Y$. Then $x_0 \notin K_0$. By the τ-compactness of K_0, there exists a pair U_1, U_2 of disjoint members of τ such that $x_0 \in U_1$ and $K_0 \subseteq U_2$. Of course, $U_2 \cap Y \neq \emptyset$ because $V \cap Y \neq Y$. Since $U_1 \cap Y$ and $U_2 \cap Y$ are both in $\tau^\star|Y$, there exist τ-compact sets K_1, K_2 such that $U_i \cap Y = Y \setminus K_i$ for $i \in \{1, 2\}$. Let $K = K_1 \cup K_2$. Then K is τ-compact and $Y = Y \setminus (U_1 \cap U_2) = (Y \setminus U_1) \cup (Y \setminus U_2) \subseteq K_1 \cup K_2 = K$. □

Corollary 2.3. (X, τ) *is compact if and only if* $\tau = \tau^\star$.

Remark 2.4. In general, $\tau^\star|Y$ is not equal to $(\tau|Y)^\star$. For instance, if Y is the open interval $(0,1)$ of $\mathbb{R}$, while τ_{nat} is the usual topology of $\mathbb{R}$, then $\tau^\star_{nat}|Y \neq (\tau_{nat}|Y)^\star$.

If this does not lead to misunderstanding, we shall denote the space $(\mathbb{R}, \tau_{nat})$ by $\mathbb{R}$ and call it the real line.

From the fact that $\tau^\star \subseteq \tau$, we have the following obvious results:

Proposition 2.5. (i) *If (X,τ) is separable, so is $(X,\tau^\star)$.*

(ii) *If (X,τ) is hereditarily separable, so is $(X,\tau^\star)$.*

(iii) *If (X,τ) is Lindelöf, so is $(X,\tau^\star)$.*

(iv) *If (X,τ) is hereditarily Lindelöf, so is $(X,\tau^\star)$.*

(v) *If (X,τ) is connected, so is $(X,\tau^\star)$.*

The statement that every infinite set is Dedekind infinite (Form 9 in [6]) is denoted by **Fin** in Definition 2.13 of [5].

Theorem 2.6. *The following sentences are equivalent in* **ZF**:

(i) ***Fin***.
(ii) *For every discrete space (X,τ), the space $(X,\tau^\star)$ is hereditarily separable.*
(iii) *For every uncountable discrete space (X,τ), the space $(X,\tau^\star)$ is separable.*

Proof. Let (X,τ) be a discrete space, i.e. $\tau = \mathcal{P}(X)$. If X is countable, then, of course, $(X,\tau^\star)$ is hereditarily separable. Consider the case when $Y \subseteq X$ and Y is uncountable. If **Fin** holds, then Y is Dedekind infinite, so Y contains an infinitely countable subset. It is clear that when D is an infinitely countable subset of Y, then D is dense in $(Y,\tau^\star|Y)$. Hence, (i) implies (ii). It is obvious that (ii) implies (iii) and that (iii) implies (i). □

Corollary 2.7. *It is consistent with* **ZF** *that there exists an uncountable discrete space (X,τ) such that $(X,\tau^\star)$ is separable and not hereditarily separable.*

Proof. Let $\mathcal{M}$ be any model of **ZF** in which $\mathbb{R}$ contains an uncountable Dedekind finite set. For instance, $\mathcal{M}$ can be Cohen's original model $\mathcal{M}1$ of [6]. Then if τ is the discrete topology on $\mathbb{R}$, the space $(\mathbb{R},\tau^\star)$ is not hereditarily separable because for each uncountable Dedekind finite subset Y of $\mathbb{R}$, the space $(Y,\tau^\star|Y)$ is not separable. Of course, $(\mathbb{R},\tau^\star)$ is separable because $\mathbb{R}$ is Dedekind infinite. □

Proposition 2.8. *For every Hausdorff space (X,τ), the space $(X,\tau^\star)$ is T_1.*

Proof. Let $x \in X$. Since finite sets are compact, we have that $X \setminus \{x\}$ is open in $(X,\tau^\star)$. Hence, $(X,\tau^\star)$ is a T_1-space. □

Proposition 2.9. *(X,τ) is not compact if and only if $(X,\tau^\star)$ is not Hausdorff. Moreover, if (X,τ) is not compact, then any two non-empty $\tau^\star$-open sets have a non-empty intersection.*

Proof. Assume that (X,τ) is not compact. Let U and V be any two non-empty open sets in $(X,\tau^\star)$. Then $X\setminus U$ and $X\setminus V$ are compact in (X,τ), so $(X\setminus U)\cup(X\setminus V)$ is compact in (X,τ). Hence, $X \neq (X\setminus U)\cup(X\setminus V) = X\setminus(U\cap V)$. This implies that $U\cap V \neq \emptyset$; thus, $(X,\tau^\star)$ is not Hausdorff. On the other hand, if we assume that $(X,\tau^\star)$ is not Hausdorff, then, since (X,τ) is Hausdorff, we have $\tau \neq \tau^\star$, so (X,τ) is not compact by Corollary 2.3. □

Corollary 2.10. *If (X,τ) is not compact, then the following conditions are satisfied:*

(i) *every set $V \in \tau^\star$ is connected in $(X,\tau^\star)$;*

(ii) $(X, \tau^\star)$ *is connected and locally connected.*

Proof. Suppose that (X, τ) is not compact and that $\emptyset \neq V \in \tau^\star$. If V were disconnected in $(X, \tau^\star)$, there would exist a pair U, W of non-empty disjoint members of $\tau^\star$ which would contradict Proposition 2.9. Hence, V is connected in $(X, \tau^\star)$. This is why (i) holds. Of course, (ii) follows from (i). □

Remark 2.11. Some authors call a topological space hyperconnected or irreducible if all open sets of this space are connected. In the light of Corollary 2.10, if (X, τ) is not compact, the space $(X, \tau^\star)$ is hyperconnected.

Theorem 2.12. *If (X, τ) is locally compact and second-countable, then $(X, \tau^\star)$ is second-countable.*

Proof. Assume that $\mathcal{B}$ is a countable open base of a locally compact Hausdorff space (X, τ). Let $\mathcal{A}$ be the collection of all sets $U \in \mathcal{B}$ which have compact closures $\mathrm{cl}_\tau U$ in (X, τ). By the local compactness of (X, τ), the collection $\mathcal{A}$ is an open base of (X, τ). Let $[\mathcal{A}]^{<\omega}$ be the collection of all finite subcollections of $\mathcal{A}$. We put

$$\mathcal{B}^\star = \{X \setminus \mathrm{cl}_\tau(\bigcup \mathcal{C}) : \mathcal{C} \in [\mathcal{A}]^{<\omega}\}.$$

Then $\mathcal{B}^\star$ is a countable subcollection of $\tau^\star$. To check that $\mathcal{B}^\star$ is an open base of $(X, \tau^\star)$, let us consider any non-empty set $V \in \tau^\star$ and a point $x \in V$. Let $K = X \setminus V$ and let $\mathcal{U}$ be the collection of all $U \in \mathcal{A}$ such that $x \notin \mathrm{cl}_\tau U$. Since (X, τ) is Hausdorff, we have $K \subseteq \bigcup \mathcal{U}$. By the compactness of K, there exists a finite $\mathcal{U}_K \subseteq \mathcal{U}$ such that $K \subseteq \bigcup \mathcal{U}_K$. Let $W = X \setminus \mathrm{cl}_\tau(\bigcup \mathcal{U}_K)$. Then $W \in \mathcal{B}^\star, x \in W$ and $W \subseteq V$. □

The axiom of countable choice, denoted by **CC** in [5], states that every non-empty countable collection of non-empty sets has a choice function (see Form 8 in [6]). The axiom of countable choice for $\mathbb{R}$, denoted by **CC**($\mathbb{R}$) in [5], states that every non-empty countable collection of non-empty subsets of $\mathbb{R}$ has a choice function (see Form 94 in [6]).

Remark 2.13. In view of Exercise E3 to Section 4.6 of [5], **CC**($\mathbb{R}$) is equivalent to the statement: for every second-countable topological space Z, every open base of Z contains a countable open base of Z. Let us notice that, if $\mathcal{M}$ is a model of **ZF** in which there exists a dense infinite Dedekind finite subset D of $\mathbb{R}$, then it holds true in $\mathcal{M}$ that the collection $\mathcal{B}^\star$ of all sets of the form $\mathbb{R} \setminus \bigcup_{i \in n+1}[a_i, b_i]$ with $n \in \omega, a_i, b_i \in D$ and $a_i < b_i$ for each $i \in n+1$ is an open base of $(\mathbb{R}, \tau^\star_{nat})$ which does not contain a countable open base of $(\mathbb{R}, \tau^\star_{nat})$.

We recall that a topological space $(Z, \mathcal{T})$ is *submetrizable* if there exists a metrizable topology $\mathcal{T}'$ on Z such that $\mathcal{T}' \subseteq \mathcal{T}$.

Theorem 2.14. *The following conditions are equivalent:*

(i) $(X, \tau^\star)$ *is metrizable;*
(ii) $(X, \tau^\star)$ *is submetrizable;*
(iii) (X, τ) *is a compact metrizable space.*

Proof. Of course, (i) implies (ii). Assume (ii). If (X, τ) is not compact, then $(X, \tau^\star)$ is not Hausdorff by Proposition 2.9. Hence, (ii) implies that (X, τ) is compact. In this case, $\tau = \tau^\star$ by Corollary 2.3. In consequence, (X, τ) is both compact and submetrizable. Since every compact submetrizable space is metrizable, (ii) implies (iii). That (iii) implies (i) follows from Corollary 2.3. □

Proposition 2.15. *Let $x_0 \in X$. Then $\{x_0\}$ is of type G_δ in $(X, \tau^\star)$ if and only if $X \setminus \{x_0\}$ is a σ-compact subspace of (X, τ).*

Proof. Necessity. Suppose that $\{U_n : n \in \omega\} \subseteq \tau^\star$ and $\{x_0\} = \bigcap_{n \in \omega} U_n$. Then the sets $K_n = X \setminus U_n$ are all compact in (X, τ) and $X \setminus \{x_0\} = \bigcup_{n \in \omega} K_n$, so $X \setminus \{x_0\}$ is σ-compact in (X, τ).

Sufficiency. Suppose that $X \setminus \{x_0\} = \bigcup_{n \in \omega} C_n$ where all the sets C_n are compact in (X, τ). Then the sets $V_n = X \setminus C_n$ are all open in $(X, \tau^\star)$ and $\{x_0\} = \bigcap_{n \in \omega} V_n$. □

Corollary 2.16. *If (X, τ) is not σ-compact, then the following conditions are satisfied:*

(i) *there does not exist a one-point set of type G_δ in $(X, \tau^\star)$;*
(ii) *$(X, \tau^\star)$ is not first-countable;*
(iii) *$(X, \tau^\star)$ is not second-countable;*
(iv) *$(X, \tau^\star)$ is not quasi-metrizable.*

Remark 2.17. We denote by $\mathbb{S}$ the Sorgenfrey line, i.e. the topological space $(\mathbb{R}, \tau_{\mathbb{S}})$ where $\tau_{\mathbb{S}}$ is the topology on $\mathbb{R}$ which has as an open base the collection of all half-open intervals $[a, b)$ where $a, b \in \mathbb{R}$ and $a < b$. The Sorgenfrey line is one of the most frequently used examples of a submetrizable, quasi-metrizable but not metrizable space, so we shall pay a special attention to it.

The countable union theorem (Form 31 in [6], abbreviated to **CUT** in [5]) states that countable unions of countable sets are countable sets. Let **CUT**($\mathbb{R}$) be the statement: for every family $\{A_n : n \in \omega\}$ of countable subsets of $\mathbb{R}$, the union $\bigcup_{n\in\omega} A_n$ is countable (see Form 6 in [6])). It is easy to prove in [**ZF** + **CUT**($\mathbb{R}$)] that the Sorgenfrey line is not σ-compact by using the following simple argument: since all compact subsets of $\mathbb{S}$ are countable, if $\mathbb{S}$ were σ-compact, $\mathbb{R}$ would be a countable union of countable sets; however, $\mathbb{R}$ cannot be a countable union of countable sets because $\mathbb{R}$ is uncountable. This is not a proof in **ZF** that the Sorgenfrey line is not σ-compact because **CUT**($\mathbb{R}$) fails in some models of **ZF** (see Theorem 10.6 of [7]).

Proposition 2.18. *In every model of* **ZF***, the Sorgenfrey line is not σ-compact.*

Proof. Consider any countable collection $\{K_n : n \in \omega\}$ of compact sets of the Sorgenfrey line. Then all the sets K_n are countable, closed in $\mathbb{R}$ and they do not have left accumulation points in $\mathbb{R}$. Therefore, each K_n is nowhere dense in $\mathbb{R}$. Since $\mathbb{R}$ is a separable completely metrizable space, by Theorem 4.102 of [5], the interior in $\mathbb{R}$ of the set $\bigcup_{n\in\omega} K_n$ is empty. Hence, $\mathbb{R} \neq \bigcup_{n\in\omega} K_n$. □

Corollary 2.19. *The compact complement topology of the Sorgenfrey line is not first-countable.*

Corollary 2.20. *The compact complement topology of the Sorgenfrey line is not quasi-metrizable.*

Proposition 2.21. *The compact complement topology of the real line $\mathbb{R}$ is quasi-metrizable.*

Proof. For $x \in \mathbb{R}$, let $m(x) = \min\{n \in \omega : |x| < n\}$. For each $x \in \mathbb{R}$ and $n \in \omega$, we define a set $G(n, x)$ by putting:

$$G(n, x) = (x - \frac{1}{2^{n+1}}, x + \frac{1}{2^{n+1}}) \cup (-\infty, -m(x) - n - 2) \cup (m(x) + n + 2, +\infty).$$

It is clear that, for each $x \in \mathbb{R}$, the collection $\mathcal{B}(x) = \{G(n, x) : n \in \omega\}$ is a base of neighbourhoods of x in $(\mathbb{R}, \tau^\star_{nat})$. One can check by a simple calculation that the following condition is satisfied: for all $x, y \in \mathbb{R}$ and $n \in \omega$, if $y \in G(n+1, x)$, then $G(n+1, y) \subseteq G(n, x)$. Let us notice that Theorem 10.2 of [4] (Chapter 10 of [12]) holds true in **ZF**, so we can infer from it that $(\mathbb{R}, \tau^\star_{nat})$ is quasi-metrizable in **ZF**. □

We are going to give a simple **ZF**-example of a countable metrizable space whose compact complement topology is not first-countable. We shall use the following lemma in this and in the third section:

Lemma 2.22. *Let us assume that $\{A_n : n \in \omega\}$ is a collection of non-empty pairwise disjoint sets, $A = \bigcup_{n\in\omega} A_n$ and $Z = A \cup \{\infty\}$ where $\infty \notin A$. For $x, y \in Z$ let $d(x, y) = d(y, x)$ and $d(x, x) = 0$; for each pair x, y of distinct points of Z, let $d(x, y) = \max\{\frac{1}{2^n}, \frac{1}{2^m}\}$ if $x \in A_n$ and $y \in A_m$; moreover, let $d(x, \infty) = \frac{1}{2^n}$ if $x \in A_n$. Then the function $d : Z \times Z \to \mathbb{R}$ is a metric on Z such that A is not closed in $(Z, \tau(d))$, while each A_n is a clopen discrete subspace of $(Z, \tau(d))$ where $\tau(d)$ is the topology on Z induced by d.*

Proof. Using the fact that $\max\{a, b\} \leq \max\{a, c\} + \max\{c, b\}$ for all non-negative real numbers a, b, c, one can easily check that d is a metric on Z. Since $\infty \in \mathrm{cl}_{\tau(d)} A$, the set A is not closed in $(Z, \tau(d))$. It is obvious that each A_n is a clopen discrete subspace of $(Z, \tau(d))$. □

Example 2.23. Let $A_n = \{n\} \times \omega$ for each $n \in \omega$ and let $A = \bigcup_{n\in\omega} A_n$ Take a point $\infty \notin A$ and put $Z = A \cup \{\infty\}$. Consider the metric d on Z defined in Lemma 2.22. Suppose that the point $(0, 0)$ has a countable base $\{V_n : n \in \omega\}$ of open neighbourhoods in $(Z, \tau(d)^\star)$ where $\tau(d)$ is as in Lemma 2.22. We may assume that $V_n \subseteq V_0$ for each $n \in \omega$. The sets $Z \setminus V_n$ are all $\tau(d)$-compact, while the sets A_n are not $\tau(d)$-compact because they are infinite discrete subspaces of $(Z, \tau(d))$. Hence, $A_n \cap V_n \neq \emptyset$ for each $n \in \omega$. For $n \in \omega$, let $a_n = \min\{m \in \omega : (n, m) \in A_n \cap V_n\}$. We define points $x_n \in A_n \cap V_n$ by putting $x_n = (n, a_n)$ for $n \in \omega$. Notice that the set $K = \{x_n : n \in \omega \setminus \{0\}\} \cup \{\infty\}$ is $\tau(d)$-compact, while $(0, 0) \notin K$. Then $V = Z \setminus K$ is an open neighbourhood of $(0, 0)$ in $(Z, \tau(d)^\star)$. There must exist $n \in \omega$ such that $V_n \subseteq V$. This is impossible because $V_n \subseteq V_0$ and $x_n \in V_n$ for each $n \in \omega \setminus \{0\}$. The contradiction obtained proves that $(Z, \tau(d)^\star)$ is not first-countable. Obviously, $(Z, \tau(d))$ is σ-compact because Z is countable. Of course, $(Z, \tau(d))$ is second-countable as a separable metrizable space. The point ∞ is not a point of local compactness of $(Z, \tau(d))$. This example shows that, in Theorem 2.12, the assumption of local compactness of (X, τ) cannot be replaced by the assumption that the set of points of non-local compactness of (X, τ) is finite.

An arbitrary example of a metrizable second-countable not σ-compact space also shows that the assumption of local compactness is essential in Theorem 2.12.

Example 2.24. Let $X = \mathbb{R} \setminus \mathbb{Q}$ and let $\tau = \tau_{nat}|X$. Then the space of irrationals (X, τ) is second-countable. That (X, τ) is not σ-compact in **ZF** can be shown by using the facts that the Baire category theorem holds in **ZF** in the class of separable completely metrizable spaces (see Theorem 4.102 of [5]) and that every compact subspace of (X, τ) is nowhere dense in (X, τ). This is why the compact complement topology $(\tau_{nat}|X)^\star$ is not first-countable, so it is not second-countable. Of course, the space of irrationals is not locally compact at each one of its points.

Remark 2.25. It was shown in Theorem 2.7 of [20] that if $\mathcal{T}$ is the co-finite topology on a set Z, then the space $(Z, \mathcal{T})$ is quasi-metrizable if and only if Z is a countable union of finite sets. Now, suppose that τ is the discrete topology on X, i.e. τ is the power set $\mathcal{P}(X)$ of X. Then $\tau^\star$ is the co-finite topology on X. Hence, for $\tau = \mathcal{P}(X)$, the space $(X, \tau^\star)$ is quasi-metrizable if and only if X is a countable union of finite sets. In some models of **ZF** in which a countable union of finite sets can fail to be countable, even when X is uncountable and $\tau = \mathcal{P}(X)$, then $(X, \tau^\star)$ can be quasi-metrizable (see [20]).

The following question does not seem to be trivial:

Question 2.26. *What are, expressed in terms of τ, simultaneously necessary and sufficient conditions for $(X, \tau^\star)$ to be quasi-metrizable when (X, τ) is a σ-compact quasi-metrizable space?*

Remark 2.27. Let us consider the case when (X, τ) is not compact. We notice that if p and $\hat{p}$ are properties such that a topological space Z has p if and only if Z is Hausdorff and has $\hat{p}$, then, in view of Proposition 2.9, the space $(X, \tau^\star)$ does not have p. In particular, $(X, \tau^\star)$ is not a T_i-space for $i \in \{2, 3, 3\frac{1}{2}, 4, 5, 6\}$. It is easily seen that $(X, \tau^\star)$ is neither regular, nor completely regular, nor normal. Every continuous mapping from $(X, \tau^\star)$ to a Hausdorff space is constant.

Theorem 2.28. *Let $A \subseteq X$. Then A is $\tau^\star$-compact if and only if $A \cap K$ is τ-closed for each τ-compact set K.*

Proof. Necessity. Suppose that A is $\tau^\star$-compact. Let K be a τ-compact set. Since (X, τ) is Hausdorff, to show that $A \cap K$ is τ-closed, it suffices to check that $A \cap K$ is τ-compact. Let $\mathcal{F}$ be a collection of τ-closed sets such that the collection $\mathcal{H} = \{F \cap A \cap K : F \in \mathcal{F}\}$ is centered. The sets $A \cap K \cap F$ for $F \in \mathcal{F}$ are all $\tau^\star|A$-closed. Since A is $\tau^\star$-compact and $\mathcal{H}$ is a centered collection of $\tau^\star|A$-closed sets, we have that $\bigcap \mathcal{H} \neq \emptyset$. This proves that $A \cap K$ is τ-compact.

Sufficiency. Now, suppose that $A \cap K$ is τ-compact for each τ-compact set K. We may assume that $A \neq \emptyset$. Let $\mathcal{U}$ be a non-empty collection of non-empty sets such that $\mathcal{U} \subseteq \tau^\star$, while $A \subseteq \bigcup \mathcal{U}$. Fix any set $U_0 \in \mathcal{U}$. The set $C_0 = X \setminus U_0$ is τ-compact, so $A \cap C_0$ is τ-compact as a τ-closed subset of a τ-compact set. Notice that $A \cap C_0 \subseteq \bigcup \mathcal{U}$ and, by Theorem 2.2, $\mathcal{U} \subseteq \tau$. By the τ-compactness of $A \cap C_0$, there exists a finite collection $\mathcal{V} \subseteq \mathcal{U}$ such that $A \cap C_0 \subseteq \bigcup \mathcal{V}$. Then $A \subseteq U_0 \cup \bigcup \mathcal{V}$. This proves that A is $\tau^\star$-compact. $\square$

Corollary 2.29. *For every Hausdorff space (X, τ), the space $(X, \tau^\star)$ is compact.*

A topological space $(Z, \mathcal{T})$ is called *jointly partially metrizable on compact subspaces*, if there is a metric d on Z such that, for every compact subspace A of $(Z, \mathcal{T})$, the restriction of d to $A \times A$ generates the subspace topology $\mathcal{T}|A$ on A (see [1]).

Example 2.30. The space $(\mathbb{R}, \tau_{nat})$ is metrizable, hence jointly partially metrizable on compact subspaces. But $(\mathbb{R}, \tau^\star_{nat})$ is not jointly partially metrizable on compact subspaces since it is compact and not metrizable for it is not Hausdorff.

A topological space Z is called C-*normal* if there exists a normal space Y and a bijective function $f : Z \to Y$ such that the restriction $f|A : A \to f(A)$ is a homeomorphism for each compact subspace A of Z (see [8]).

Example 2.31. The space $(\mathbb{R}, \tau_{nat})$ is C-normal. But $(\mathbb{R}, \tau^\star_{nat})$ is not C-normal since it is compact and not normal.

3. *k*-Spaces

Let us recall that a Hausdorff space Z is called a *k-space* if, for every set $A \subseteq Z$, it holds true that A is closed in Z if and only if $A \cap K$ is closed in Z for each compact set K in Z (see Section 3.3 of [3]).

We deduce directly from Theorem 2.28 the following characterization of *k*-spaces:

Theorem 3.1. *For every Hausdorff space (X, τ), it holds true in* **ZF** *that (X, τ) is a k-space if and only if every $\tau^\star$-compact subset of X is τ-closed.*

We recall definitions of sequential and Fréchet-Urysohn spaces for completeness.

Definition 3.2. Let Z be a topological space and $A \subseteq Z$. Then:

(i) A^s denotes the set of all points $z \in Z$ such that there exists a sequence $(z_n)_{n\in\omega}$ of points of $A \setminus \{z\}$ which converges in Z to the point z;
(ii) A is called sequentially closed if $A^s \subseteq A$;
(iii) the sequential closure of A in Z is the set $\mathrm{scl}_Z(A) = A^s \cup A$;
(iv) Z is called sequential (resp. Fréchet-Urysohn) if every sequentially closed subset of Z is closed in Z (resp. for every $F \in \mathcal{P}(Z)$ the equality $\mathrm{scl}_Z(F) = \mathrm{cl}_Z(F)$ holds).

In some texts, Fréchet-Urysohn spaces are called Fréchet spaces (see, for instance, [3] and [5]). It is well known that the following series of implications hold true in **ZFC** and, in general, none of them is reversible in **ZFC** (see, e.g. Sections 1.6 and Theorem 3.3.20 of [3]):

- Z is Hausdorff and first-countable $\to$ Z is Hausdorff and Fréchet-Urysohn $\to$ Z is Hausdorff and sequential $\to$ Z is a *k*-space.

Of course, the proof of Theorem 3.3.20 of [3] shows that it is true in **ZF** that every Hausdorff sequential space is a *k*-space. That even $\mathbb{R}$ can fail to be sequential in a model of **ZF** is shown in Theorem 4.55 of [5]. The second part of Theorem 3.3.20 of [3], which states that every first-countable Hausdorff space is a *k*-space, does not have a proof in **ZF**. Therefore, since we work in **ZF**, it is natural to ask about set-theoretical statuses of the following sentences:

(a) Every first-countable Hausdorff space is a *k*-space.
(b) $\mathbb{R}$ is a *k*-space.
(c) Every subspace of $\mathbb{R}$ is a *k*-space.

In this section, we are going to prove that (a) is equivalent with the axiom of countable multiple choice (i.e. Form 126 in [6]), while (b) holds in **ZF** and (c) is independent of **ZF**. We shall also show that even the Sorgenfrey line can fail to be a k-space in a model of **ZF**.

We recall that the axiom of countable multiple choice, denoted by **CMC** in [5], states that, for every collection $\{A_n : n \in \omega\}$ of non-empty sets there exists a collection $\{F_n : n \in \omega\}$ of non-empty finite sets such that $F_n \subseteq A_n$ for each $n \in \omega$. It was shown in [9] that **CMC** is equivalent with Form 126D of [6], i.e with the following sentence denoted by **WCMC**:

WCMC: For every denumerable family $\mathcal{A}$ of disjoint non-empty sets there is an infinite set $C \subseteq \bigcup \mathcal{A}$ such that, for each $A \in \mathcal{A}$ the intersection $A \cap C$ is finite.

More information about **WCMC** can be found in [9] and in Note 132 of [6].

If $\mathcal{A}$ is a denumerable collection of pairwise disjoint non-empty sets, then every infinite set $C \subseteq \bigcup \mathcal{A}$ such that $C \cap A$ is finite for each $A \in \mathcal{A}$ is called a *partial multiple choice* set of $\mathcal{A}$.

Theorem 3.3. *The following conditions are all equivalent in* **ZF**:

(i) **CMC**;
(ii) *every Hausdorff first-countable space is a k-space;*
(iii) *every metrizable space is a k-space.*

Proof. Let Y be a first-countable Hausdorff space and let D be a subset of Y which is not closed in Y. Fix in Y an accumulation point y of D such that $y \notin D$. Let $\mathcal{B}(y) = \{U_n : n \in \omega\}$ be a countable base of open neighbourhoods of y in Y such that $U_{n+1} \subset U_n$ for each $n \in \omega$. Since Y is Hausdorff, we can find a strictly increasing sequence $(k_n)_{n\in\omega}$ of positive integers such that the set $D_n = D \cap (U_{k_n} \setminus U_{k_{n+1}})$ is non-empty for each $n \in \omega$. Suppose that **CMC** holds. By **CMC**, there exists a sequence $(C_n)_{n\in\omega}$ of non-empty finite sets such that $C_n \subseteq D_n$ for each $n \in \omega$. Then the set $C = \{y\} \cup \bigcup_{n\in\omega} C_n$ is compact in Y, while y is an accumulation point of $D \cap C$ and $y \notin D \cap C$. Thus $D \cap C$ is not closed in Y. Therefore, Y is a k-space if **CMC** holds. Hence, (i) implies (ii). It is obvious that (ii) implies (iii). To complete the proof, it suffices to show that (iii) implies **WCMC**.

Now, let us assume that **WCMC** is false. Suppose that $\mathcal{A} = \{A_n : n \in \omega\}$ is a collection of pairwise disjoint non-empty sets without a partial multiple choice set. Put $A = \bigcup_{n\in\omega} A_n$. Take a point $\infty \notin A$ and put $Z = A \cup \{\infty\}$. Consider the metric d on Z defined in Lemma 2.22, as well as the topology $\tau(d)$ on Z induced by d. Let K be a compact subspace of $(Z, \tau(d))$. Since each A_n is a discrete clopen subspace of $(Z, \tau(d))$, the sets $K \cap A_n$ are all finite. If K were infinite, then K would be a partial multiple choice set of $\mathcal{A}$. Hence, K is finite, so $A \cap K$ is compact in $(Z, \tau(d))$. By Lemma 2.22, A is not closed in $(Z, \tau(d))$. This shows that $(Z, \tau(d))$ is not a k-space. Hence, (iii) implies (i). □

Corollary 3.4. *It is consistent with* **ZF** *that not every metrizable space is a k-space.*

Theorem 3.5. *If $\mathcal{M}$ is a model of* **ZF** *in which every metrizable space is sequential, then* **CMC** *holds in $\mathcal{M}$.*

Proof. Suppose $(Z, \tau(d))$ is the space from Lemma 2.22 and the proof to Theorem 3.3 where $\mathcal{A} = \{A_n : n \in \omega\}$ is a collection of pairwise disjoint non-empty sets without a partial multiple choice set. Then the set A is sequentially closed but not closed in $(Z, \tau(d))$. □

Remark 3.6. Let us notice that since **CMC** implies **CC**($\mathbb{R}$), it follows directly from Exercise E.3 to Section 4.6 of [5] that in every model of **ZF** in which **CMC** holds, every second-countable T_0-space (in particular, every second-countable metrizable space) is Fréchet-Urysohn, so sequential.

Theorem 3.7. $\mathbb{R}$ *is a k-space in every model of* **ZF**.

Proof. Let A be a subset of $\mathbb{R}$ such that $A \cap K$ is closed in $\mathbb{R}$ for each compact set K in $\mathbb{R}$. Suppose that $x \in (\mathrm{cl}_{\mathbb{R}} A) \setminus A$. Let $K_n = A \cap [x - \frac{1}{2^n}, x + \frac{1}{2^n}]$ for each $n \in \omega$. The sets K_n are all non-empty and compact in $\mathbb{R}$. We put $x_n = \inf(K_n)$ for each $n \in \omega$. It follows from the compactness of K_n that $x_n \in K_n$ for each $n \in \omega$. In this way, we define a sequence $(x_n)_{n\in\omega}$ of points of A which converges in $\mathbb{R}$ to x. The set $K = \{x\} \cup \{x_n : n \in \omega\}$ is compact in $\mathbb{R}$ but $A \cap K$ is not closed in $\mathbb{R}$ which is a contradiction. Hence, A must be closed in $\mathbb{R}$. This implies that $\mathbb{R}$ is a k-space in **ZF**. □

Proposition 3.8. (i) *It is consistent with* **ZF** *that a subspace of* $\mathbb{R}$ *can fail to be a k-space.*
(ii) *It is consistent with* **ZF** *that all subspaces of* $\mathbb{R}$ *are k-spaces.*

Proof. (i) Suppose that X is an infinite Dedekind finite subset of $\mathbb{R}$. Since X as a subspace of $\mathbb{R}$ is not discrete, there exists a set $A \subseteq X$ such that A is not closed in X. Let K be a compact subset of X. Then K is compact in $\mathbb{R}$, so, if K were infinite, then K would be Dedekind infinite. Since K is Dedekind finite, we deduce that K is finite. This implies $A \cap K$ is closed in X because $A \cap K$ is finite. To complete the proof to (i), it suffices to notice that in the model $\mathcal{M}1$ of [6] there is an infinite Dedekind finite subset of $\mathbb{R}$.

(ii) Let $\mathcal{M}$ be a model of **ZF** in which $\mathbf{CC}(\mathbb{R})$ holds. For instance, the model $\mathcal{M}2$ of [6] can be taken as $\mathcal{M}$. Since, by Theorem 4.54 of [5], it is true in $\mathcal{M}$ that every subspace of $\mathbb{R}$ is sequential, we infer that, in $\mathcal{M}$, every subspace of $\mathbb{R}$ is a k-space. □

Corollary 3.9. *It is independent of* **ZF** *that all subspaces of* $\mathbb{R}$ *are k-spaces.*

In what follows, as a metric space, $\mathbb{R}$ is considered with the metric ρ defined by $\rho(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$.

Using the notation from Theorem 4.55 of [5], we denote by $\mathbf{CC}(c\mathbb{R})$ the following sentence: Every non-empty countable collection of non-empty complete subspaces of $\mathbb{R}$ has a choice function.

Theorem 3.10. (i) *If the Sorgenfrey line is a k-space, then* $\mathbf{CC}(c\mathbb{R})$ *holds.*
(ii) *If* $\mathbf{CC}(\mathbb{R})$ *holds, then the Sorgenfrey line is a k-space.*

Proof. (i) Suppose that $\mathbf{CC}(c\mathbb{R})$ does not hold. Then, by Theorem 4.55 of [5], $\mathbb{R}$ is not sequential. Let A be a sequentially closed subset of $\mathbb{R}$ which is not closed in $\mathbb{R}$. Let $a \in (\mathrm{cl}_{\mathbb{R}} A) \setminus A$. The set $B = [(A - a) \cup (-A + a)] \cap (0, +\infty)$ is sequentially closed in $\mathbb{R}$ and not closed in $\mathbb{R}$. Since $0 \in (\mathrm{cl}_{\mathbb{S}} B) \setminus B$, the set B is not closed in $\mathbb{S}$. Let K be a compact set in $\mathbb{S}$. Then K is countable and compact in $\mathbb{R}$. The set $K \cap B$ is countable and sequentially closed in $\mathbb{R}$. Since, for every first-countable space X, it holds true in **ZF** that if C is a countable sequentially closed subset of X, then C is closed in X, we deduce that $K \cap B$ is closed in $\mathbb{R}$. This implies $K \cap B$ is closed in $\mathbb{S}$. Therefore, $\mathbb{S}$ is not a k-space.

(ii) Now, suppose that $\mathbf{CC}(\mathbb{R})$ holds. Let $F \subseteq \mathbb{R}$ be not closed in $\mathbb{S}$ and let $x \in \mathrm{cl}_{\mathbb{S}} F \setminus F$. Then $G = F \cap (x, +\infty)$ is not closed in $\mathbb{R}$ and $x \in (\mathrm{cl}_{\mathbb{R}} G) \setminus G$. In the light of Theorem 4.54 of [5], $\mathbb{R}$ is Fréchet. This implies that there exists a sequence $(x_n)_{n \in \omega}$ of points of G which converges in $\mathbb{R}$ to x. The set $K = \{x\} \cup \{x_n : n \in \omega\}$ is compact in $\mathbb{S}$ but $K \cap F$ is not closed in $\mathbb{S}$. This proves that $\mathbb{S}$ is a k-space. □

Corollary 3.11. *It is consistent with* **ZF** *that the Sorgenfrey line is not a k-space.*

From Theorems 2.28 and 3.10, we immediately obtain the following:

Corollary 3.12. *Let* τ *be the topology of the Sorgenfrey line. If every compact in* $(\mathbb{R}, \tau^{\star})$ *set is closed in* $(\mathbb{R}, \tau)$, *then* $\mathbf{CC}(c\mathbb{R})$ *holds.*

4. Compact Complement Partial Topology

Let us slightly reformulate Definition 2.1 of [13]:

Definition 4.1. A *partial topology* on a set X is a collection $Cov_X \subseteq \mathcal{P}(\mathcal{P}(X))$ which satisfies the following conditions:

(i) $\tau_X = \bigcup Cov_X$ is a topology on X;
(ii) if $\mathcal{U} \subseteq \tau_X$ and $\mathcal{U}$ is finite, then $\mathcal{U} \in Cov_X$;
(iii) if $\mathcal{U} \in Cov_X$ and $V \in \tau_X$, then $\{U \cap V : U \in \mathcal{U}\} \in Cov_X$;
(iv) if $\mathcal{U} \in Cov_X$ and, for each $U \in \mathcal{U}$, a collection $\mathcal{V}(U) \in Cov_X$ is given for which $U = \bigcup \mathcal{V}(U)$, then $\bigcup_{U \in \mathcal{U}} \mathcal{V}(U) \in Cov_X$;

(v) if $\mathcal{U} \subseteq \tau_X$ and $\mathcal{V} \in Cov_X$ are such that $\bigcup \mathcal{U} = \bigcup \mathcal{V}$ and, for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subseteq U$, then $\mathcal{U} \in Cov_X$.

Definition 4.2. If Cov_X is a partial topology on a set X, then the ordered pair (X, Cov_X) is called a partially topological space, while $\tau_X = \bigcup Cov_X$ is called the topology corresponding to Cov_X.

Remark 4.3. Let us notice that if (X, Cov_X) is a partially topological space, then the triple $(X, \bigcup Cov_X, Cov_X)$ is a Delfs-Knebusch generalized topological space (in abbreviation a D-K gts) in the sense of Definition 2.2.1 of [14] and, moreover, this D-K gts is partially topological in the sense of Definition 2.2.67 of [14]. Delfs-Knebusch gtses were studied, for instance, in [2, 10, 13–18]. We recall that, according to Remark 2.2.3 of [14], a D-K gts is an ordered pair (X, Cov_X) such that, for $Op_X = \bigcup Cov_X$, the triple (X, Op_X, Cov_X) satisfies the conditions of Definition 2.2.2 of [14]. In general, Op_X need not be a topology on X. If (X, Cov_X) is a D-K gts, then Cov_X is called a D-K (Delfs-Knebusch) generalized topology on X.

If ψ is a topological property, then we say that a partially topological space (X, Cov_X) has ψ if the topological space $(X, \bigcup Cov_X)$ has ψ. In particular:

Definition 4.4. We say that a partially topological space (X, Cov_X) is:

(i) Hausdorff if $(X, \bigcup Cov_X)$ is Hausdorff;
(ii) compact if $(X, \bigcup Cov_X)$ is compact.

Definition 4.5. Let (X, Cov_X) be a Hausdorff partially topological space, τ_X the topology corresponding to Cov_X and $\tau_X^\star$ the compact complement topology of (X, τ_X). Then the collection

$$Cov_X^\star = Cov_X \cap \mathcal{P}(\tau_X^\star)$$

will be called the compact complement partial topology of (X, Cov_X).

Remark 4.6. Let (X, Cov_X) be a Hausdorff partially topological space. That $Cov_X^\star$ is a D-K generalized topology follows from Fact 2.2.31 in [14] which says that the intersection of any non-empty family of D-K generalized topologies on X is a D-K generalized topology on X. Since $\bigcup(Cov_X \cap \mathcal{P}(\tau_X^\star)) = \tau_X^\star$, the D-K generalized topology $Cov_X^\star$ is a partial topology on X.

In what follows, we use the symbols $\cap_1, \setminus_1$ introduced on page 219 of [14]. We recall that, for collections $\mathcal{U}, \mathcal{V}$ of subsets of X, we have $\mathcal{U} \cap_1 \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ and, analogously, $\mathcal{U} \setminus_1 \mathcal{V} = \{U \setminus V : U \in \mathcal{U}, V \in \mathcal{V}\}$. Moreover, for a collection $\mathcal{A} \subseteq \mathcal{P}(\mathcal{P}(X))$, we denote by $\langle \mathcal{A} \rangle_X$ the intersection of all D-K generalized topologies on X that contain $\mathcal{A}$ (see page 242 of [17]).

Definition 4.7. For a subset Y of X and a partial topology Cov on X, let

$$Cov|Y = \langle \{\mathcal{V} \cap_1 \{Y\} : \mathcal{V} \in Cov\} \rangle_Y.$$

Then $Cov|Y$ is called the partial topology on Y induced by Cov and $(Y, Cov|Y)$ is called a partially topological subspace of (X, Cov).

The following theorem is an adaptation of Theorem 2.2 to partial topologies:

Theorem 4.8. *Let (X, Cov_X) be a Hausdorff partially topological space, $\tau_X = \bigcup Cov_X$ and $Y \subseteq X$. The following conditions are fulfilled:*

(i) $Cov_X^\star|Y \subseteq Cov_X|Y$;
(ii) *if Y is compact in (X, τ_X), then $Cov_X^\star|Y = Cov_X|Y$;*
(iii) *$Cov_X^\star|Y = Cov_X|Y$ if and only if there exists a τ_X-compact set K such that $Y \subseteq K$.*

Proof. Since $Cov_X^\star \subseteq Cov_X$, it is obvious that (i) is satisfied.

(ii) Suppose that Y is τ_X-compact and $\mathcal{V} \in Cov_X$. Since (X, τ_X) is Hausdorff, the set Y is τ_X-closed, so $\mathcal{A} = \{Y\} \cap_1 (\{X\} \setminus_1 \mathcal{V})$ is a collection of τ_X-compact subsets of Y. Notice that $\mathcal{V} \cap_1 \{Y\} = \{Y\} \cap_1 (\{X\} \setminus_1 \mathcal{A})$. This implies that $\mathcal{V} \cap_1 \{Y\} \in Cov_X^\star|Y$ and, in consequence, $Cov_X|Y \subseteq Cov_X^\star|Y$.

(iii) Now, assume that K is a τ_X-compact set such that $Y \subseteq K$. It follows from (ii) that $Cov_X|K = Cov_X^\star|K$. Hence, in view of Fact 10.3 of [17], we have $Cov_X|Y = (Cov_X|K)|Y = (Cov_X^\star|K)|Y = Cov_X^\star|Y$.

Finally, suppose that Y is a subset of X such that $Cov_X^\star|Y = Cov_X|Y$. Let $V \in \tau_X$ be such that $\emptyset \neq V \cap Y \neq Y$. Then $\{V\} \in Cov_X$. Since $\{V \cap Y\} \in Cov_X^\star|Y$, there exists a τ_X-compact set K_0 such that $V \cap Y = Y \setminus K_0$. Reasoning as in the proof to Theorem 2.2 (iii), we get that there exists a τ_X-compact set K such that $Y \subseteq K$. □

Corollary 4.9. *A Hausdorff partially topological space (X, Cov_X) is compact if and only if $Cov_X = Cov_X^\star$.*

In view of Proposition 2.8 and Corollary 2.29, the following proposition holds:

Proposition 4.10. *If (X, Cov_X) is a Hausdorff partially topological space, then the partially topological space $(X, Cov_X^\star)$ is compact and T_1.*

Remark 4.11. Similarly to the situation in Remark 2.4, we have that, in general, $Cov_X^\star|Y$ need not be equal to $(Cov_X|Y)^\star$.

Although it can be said more about compact complement partial topologies, let us finish with the following example:

Example 4.12. Consider the partially topological real lines considered in Definition 1.2 of [17]: $\mathbb{R}_{st} = (\mathbb{R}, Cov_{st})$, $\mathbb{R}_{lst} = (\mathbb{R}, Cov_{lst})$, $\mathbb{R}_{l^+st} = (\mathbb{R}, Cov_{l^+st})$. Let I be a bounded interval of $\mathbb{R}$. Then we get the following equalities of the induced partial topologies: $Cov_{st}|I = Cov_{lst}|I = Cov_{l^+st}|I = Cov_{st}^\star|I = Cov_{lst}^\star|I = Cov_{l^+st}^\star|I$.

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