



Application Measure of Noncompactness and Operator Type Contraction for Solvability of an Infinite System of Differential Equations in ℓ_p -Space

Bipan Hazarika^{a,b}, Reza Arab^c, M. Mursaleen^d

^aDepartment of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh-791 112, Arunachal Pradesh, India

^bDepartment of Mathematics, Gauhati University, Guwahati 781014, Assam, India.

^cDepartment of Mathematics, Sari Branch, Islamic Azad University, Sari-19318, Iran

^dDepartment of Mathematics, Aligarh Muslim University, Aligarh 202 002, India

Abstract. The aim of this paper is to obtain existence results for an infinite system of second order differential equations in the sequence space ℓ_p for $1 \leq p < \infty$ with the help of a technique associated with measures of noncompactness and contractive condition of operator type. We also provide some illustrative examples in support of our existence theorems.

1. Introduction

Kuratowski [12] was first introduced the measure of noncompactness which plays a very significant rôle in the study of infinite system of differential equations. There are many different types of measures of noncompactness in metric and topological spaces. We refer the interested reader to [9] for details on various measures of noncompactness. In the recent past, Banaś and Lecko [10] adopted the technique of measures of noncompactness to prove several existence results for infinite systems of differential equations in the classical Banach spaces c_0 , c and ℓ_1 , where c_0 and c denote the spaces of all null and convergent sequences, respectively, whereas ℓ_1 denotes the set of absolutely summable series. Mursaleen and Mohiuddine [17] presented a generalization of the existence theorem, which was proved by Banaś and Lecko [10] for ℓ_1 , by taking the space ℓ_p of absolutely p -summable series. Aghajani and Pourhadi [3] reported the Darbo type fixed point theorem and used it to obtain existence theorem in ℓ_1 for an infinite system of second-order differential equations through measures of noncompactness and the same result was proved by Mohiuddine *et al.* [16] in a more general setting by considering the space ℓ_p . On the other hand, Mursaleen and Rizvi [18] investigated this existence theorem for the Banach sequence spaces c_0 and ℓ_1 . The measure of noncompactness also has applications in several types of integral equations and differential equations (see, for example, [4, 6]).

2010 *Mathematics Subject Classification.* Primary 47H10, Secondary 34A34.

Keywords. System of differential equations; Measure of noncompactness; Meir-Keeler condensing operator; Fixed points.

Received: 04 May 2017; Accepted: 22 April 2018

Communicated by Dragan S. Djordjević

Email addresses: bh_rgu@yahoo.co.in (Bipan Hazarika), mathreza.arab@iausari.ac.ir (Reza Arab), mursaleenm@gmail.com (M. Mursaleen)

2. Preliminaries

In this section, we recall some basic facts concerning measures of noncompactness, which is defined axiomatically in terms of some natural conditions. Denote by \mathbb{R} the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. Let $\bar{B}(x, r)$ denote the closed ball centered at x with radius r . The symbol \bar{B}_r stands for the ball $\bar{B}(0, r)$. For X , a nonempty subset of E , we denote by \bar{X} and $ConvX$ the closure and the closed convex hull of X , respectively. Moreover, let us denote by \mathfrak{M}_E the family of nonempty bounded subsets of E and by \mathfrak{K}_E its subfamily consisting of all relatively compact subsets of E .

Definition 2.1. [8] A mapping $\mu : \mathfrak{M}_E \rightarrow \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

- (1°) The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{K}_E$;
- (2°) $X \subset Y \implies \mu(X) \leq \mu(Y)$;
- (3°) $\mu(\bar{X}) = \mu(X)$;
- (4°) $\mu(ConvX) = \mu(X)$;
- (5°) $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$;
- (6°) If $\{X_n\}$ is a sequence of closed sets from \mathfrak{M}_E such that $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$, and if $\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The subfamily $ker\mu$ defined (1°) represents the kernel of the measure μ of noncompactness and since

$$\mu(X_\infty) = \mu\left(\bigcap_{n=1}^{\infty} X_n\right) \leq \mu(X_n),$$

we see that

$$\mu\left(\bigcap_{n=1}^{\infty} X_n\right) = 0.$$

Therefore $X_\infty \in ker\mu$.

Theorem 2.2. (Schauder's fixed point theorem)[1] Let C be a closed, convex subset of a Banach space E . Then every compact, continuous map $T : C \rightarrow C$ has at least one fixed point.

In what follows, we recall the well known fixed point theorem of Darbo type [8].

Theorem 2.3. (Darbo's fixed point theorem) Let Ω be a nonempty, bounded, closed and convex subset of a space E and let $T : \Omega \rightarrow \Omega$ be a continuous mapping such that there exists a constant $k \in [0, 1)$ with the property

$$\mu(TX) \leq k\mu(X),$$

for any nonempty subset X of Ω . Then T has a fixed point in the set Ω .

In 1969 Meir and Keeler [13] established a fixed point theorem in a metric space (X, d) for mappings satisfying the condition that for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq d(x, y) < \epsilon + \delta(\epsilon) \implies d(Tx, Ty) < \epsilon, \tag{1}$$

for all $x, y \in X$. This condition is called the Meir-Keeler contractive type condition. This well-known result suggests the notion of a Meir-Keeler function.

Definition 2.4. A function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be a Meir-Keeler function if $\phi(0) = 0$ and for each $\epsilon > 0$ there exists $\delta > 0$ such that for any $t \in \mathbb{R}_+$,

$$\epsilon \leq t < \epsilon + \delta \implies \phi(t) < \epsilon.$$

Remark 2.5. It is obvious that if ϕ is a Meir-Keeler function then $\phi(t) < t$ for all $t > 0$.

We now introduce the notion of weaker Meir-Keeler function as follows:

Definition 2.6. [14, 15] We call $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a weaker Meir-Keeler function if for each $\epsilon > 0$, there exists $\delta > 0$ such that for any $t \geq 0$ with $\epsilon \leq t < \epsilon + \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(t) < \epsilon$.

Definition 2.7. [11] A function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be a Jachymski function if $\phi(0) = 0$ and for each $\epsilon > 0$ there exists $\delta > 0$ such that for any $t \in \mathbb{R}_+$,

$$\epsilon < t < \epsilon + \delta \implies \phi(t) \leq \epsilon.$$

Remark 2.8. [5] Obviously, each Meir-Keeler function is a Jachymski function. However, the converse does not follow even in the case that $\phi(t) < t$ for all $t > 0$.

The following concept of $O(f; \cdot)$ and its examples was given by Altun and Turkoglu [5]. Let $F([0, \infty))$ be class of all function $f : [0, \infty) \rightarrow [0, \infty)$ and let Θ be class of all operators

$$O(\bullet; \cdot) : F([0, \infty)) \rightarrow F([0, \infty)), \quad f \rightarrow O(f; \cdot)$$

satisfying the following conditions:

- (a) $O(f; t) > 0$ for $t > 0$ and $O(f; 0) = 0$,
- (b) $O(f; t) \leq O(f; s)$ for $t \leq s$,
- (c) $\lim_{n \rightarrow \infty} O(f; t_n) = O(f; \lim_{n \rightarrow \infty} t_n)$,
- (d) $O(f; \max\{t, s\}) = \max\{O(f; t), O(f; s)\}$ for some $f \in F([0, \infty))$.

Example 2.9. If $f : [0, \infty) \rightarrow [0, \infty)$ is a Lebesgue integrable mapping which is finite integral on each compact subset of $[0, \infty)$, non-negative and such that for each $t > 0$, $\int_0^t f(s)ds > 0$, then the operator defined by

$$O(f; t) = \int_0^t f(s)ds$$

satisfies the above conditions.

Example 2.10. If $f : [0, \infty) \rightarrow [0, \infty)$ non-decreasing, continuous function such that $f(0) = 0$ and $f(t) > 0$ for $t > 0$ then the operator defined by

$$O(f; t) = \frac{f(t)}{1 + f(t)}$$

satisfies the above conditions.

Let \mathbb{G} be class of all function $G : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ satisfying the following conditions:

- (i) $\max\{a, b\} \leq G(a, b)$ for $a, b \geq 0$;
- (ii) G is continuous.

3. Fixed point theorem

First, we define measure of noncompactness and Meir-Keeler condensing operator and to present some fixed point theorems.

Definition 3.1. Let Ω be a nonempty subset of a Banach space E and μ is a measure of noncompactness on E . We say that an operator $T : \Omega \rightarrow \Omega$ is a generalized Meir-Keeler type function if for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for any subset X of Ω

$$\epsilon \leq O(f; G(\mu(X), \varphi(\mu(X))) < \epsilon + \delta \implies O(f; G(\mu(T(X)), \varphi(\mu(TX)))) < \epsilon, \quad (2)$$

where $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous function, $O(\bullet; \cdot) \in \Theta$ and $G \in \mathbb{G}$.

We start this section with the first of our main theorems.

Theorem 3.2. Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and μ is an arbitrary measure of noncompactness on E . Let $T : \Omega \rightarrow \Omega$ be a continuous and generalized Meir-Keeler condensing operator then T has at least one fixed point and the set of all fixed points of T in Ω is compact.

Proof. By induction, we obtain a sequence $\{\Omega_n\}$ such that $\Omega_0 = \Omega$ and $\Omega_n = \text{Conv}(T\Omega_{n-1})$, $n \geq 1$. $T\Omega_0 = T\Omega \subseteq \Omega = \Omega_0$, $\Omega_1 = \text{Conv}(T\Omega_0) \subseteq \Omega = \Omega_0$, therefore by continuing this process we have

$$\Omega_0 \supseteq \Omega_1 \supseteq \dots \supseteq \Omega_n \supseteq \Omega_{n+1} \supseteq \dots$$

If there exists an integer $N \geq 0$ such that $\mu(\Omega_N) = 0$, then Ω_N is compact. Thus, Theorem 2.2 implies that T has a fixed point. Now assume that $\mu(\Omega_n) \neq 0$ for $n \geq 0$ and also, $G(\mu(\Omega_n), \varphi(\mu(\Omega_n))) > 0$ and $O(f; G(\mu(\Omega_n), \varphi(\mu(\Omega_n)))) > 0$. Define $\epsilon_n = O(f; G(\mu(\Omega_n), \varphi(\mu(\Omega_n))))$ and $\delta_n = \delta(\epsilon_n) > 0$. By the definition of Ω_n and $\epsilon_n < \epsilon_n + \delta_n$ we have

$$\begin{aligned} \epsilon_{n+1} &= O(f; G(\mu(\Omega_{n+1}), \varphi(\mu(\Omega_{n+1})))) = O(f; G(\mu(\text{Conv}(T\Omega_n)), \varphi(\mu(\text{Conv}(T\Omega_n)))) \\ &= O(f; G(\mu(T\Omega_n), \varphi(\mu(T\Omega_n)))) \\ &< O(f; G(\mu(\Omega_n), \varphi(\mu(\Omega_n)))) \\ &= \epsilon_n. \end{aligned}$$

Therefore $\{\epsilon_n\}$ is a positive decreasing sequence of real numbers and there exists $\gamma \geq 0$ such that $\epsilon_n \rightarrow \gamma$ as $n \rightarrow \infty$. We claim that $\gamma = 0$. Suppose, on the contrary, that $\gamma > 0$, then there exists n_0 such that $n > n_0$ implies $\gamma \leq \epsilon_n < \gamma + \delta(\gamma)$, therefore by the definition of generalized Meir-Keeler condensing operator, $\epsilon_{n+1} < \gamma$ which is a contradiction. Therefore $\gamma = 0$, that is, $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore

$$O(f; \lim_{n \rightarrow \infty} G(\mu(\Omega_n), \varphi(\mu(\Omega_n)))) = 0,$$

which from (a), implies that

$$\lim_{n \rightarrow \infty} G(\mu(\Omega_n), \varphi(\mu(\Omega_n))) = 0. \quad (3)$$

Using (3) and (i) of property \mathbb{G} , we get

$$\lim_{n \rightarrow \infty} \mu(\Omega_n) = \lim_{n \rightarrow \infty} \varphi(\mu(\Omega_n)) = 0.$$

Now since Ω_n is a nested sequence, in view of (6°) of Definition 2.1, we conclude that $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$ is nonempty, closed and convex subset of Ω . Moreover, we know that Ω_∞ belong to $\ker \mu$. So Ω_∞ is compact and invariant under the mapping T . Consequently, Theorem 2.2 implies that T has a fixed point

in Ω_∞ . Since $\Omega_\infty \subset \Omega$, the proof is completed. Now suppose that $\text{Fix}(T) = \{x \in \Omega : Tx = x\}$ and $\epsilon_0 = O(f; G(\mu(\text{Fix}(T)), \varphi(\mu(\text{Fix}(T))))$. If $\epsilon_0 \neq 0$ then by (2) and $T(\text{Fix}(T)) = \text{Fix}(T)$, we have

$$\begin{aligned} O(f; G(\mu(\text{Fix}(T)), \varphi(\mu(\text{Fix}(T)))) &= O(f; G(\mu(T(\text{Fix}(T))), \varphi(\mu(T(\text{Fix}(T)))))) \\ &< \epsilon_0 \\ &= O(f; G(\mu(\text{Fix}(T)), \varphi(\mu(\text{Fix}(T))))), \end{aligned}$$

which is a contradiction. So $\epsilon_0 = 0$ and $\text{Fix}(T)$ is relatively compact and since T is a continuous function so the set of fixed points of T in Ω is compact. \square

An immediate consequence of Theorem 3.2 is the following.

Theorem 3.3. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and μ is an arbitrary measure of noncompactness on E . Let $T : \Omega \rightarrow \Omega$ be a continuous and for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for any subset X of Ω*

$$\epsilon \leq G(\mu(X), \varphi(\mu(X))) < \epsilon + \delta \implies G(\mu(T(X)), \varphi(\mu(TX))) < \epsilon, \tag{4}$$

where $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous function and $G \in \mathbb{G}$. Then T has at least one fixed point and the set of all fixed points of T in Ω is compact.

Remark 3.4. *Take $G(a, b) = a + b$ and $\varphi \equiv 0$ in Theorem 3.3. Then Theorem 2.2 of [2] is obtained.*

Proposition 3.5. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and μ is an arbitrary measure of noncompactness on E . Let $T : \Omega \rightarrow \Omega$ and $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be two continuous functions. If for some $k \in (0, 1)$*

$$O(f; G(\mu(TX), \varphi(\mu(TX)))) \leq k O(f; G(\mu(X), \varphi(\mu(X)))), \tag{5}$$

is satisfied, where $O(\bullet; \cdot) \in \Theta$ and $G \in \mathbb{G}$. Then T is a generalized Meir-Keeler type function.

Proof. Suppose that (5) is satisfied. For all $\epsilon > 0$, one can easily check that (2) is satisfied with $\delta(\epsilon) = (\frac{1}{k} - 1)\epsilon$. \square

Remark 3.6. *Take $G(a, b) = a + b$, $\varphi \equiv 0$ and $O(f; t) = t$ in Proposition 3.5. Then Darbo’s fixed point theorem is obtained.*

The following corollary gives us a fixed point theorem with a contractive condition of integral type.

Corollary 3.7. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E , $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $T : \Omega \rightarrow \Omega$ are continuous functions, such that for any $X \subseteq \Omega$ one has*

$$\int_0^{G(\mu(T(X)), \varphi(\mu(TX)))} f(s) ds \leq k \int_0^{G(\mu(X), \varphi(\mu(X)))} f(s) ds,$$

where μ is an arbitrary measure of noncompactness, $G \in \mathbb{G}$ and $f : [0, \infty) \rightarrow [0, \infty)$ is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, \infty)$, non-negative and such that for each $\epsilon > 0$, $\int_0^\epsilon f(s) ds > 0$ and $k \in (0, 1)$. Then T has at least one fixed point in Ω .

Theorem 3.8. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and μ is an arbitrary measure of noncompactness on E . Let $T : \Omega \rightarrow \Omega$ be a continuous. Suppose that there exists a Jachymski function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\phi(t) < t$ for all $t > 0$ and*

$$O(f; G(\mu(T(X)), \varphi(\mu(TX)))) \leq \phi(O(f; G(\mu(X), \varphi(\mu(X))))), \tag{6}$$

for any subset X of Ω , where $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous function, $O(\bullet; \cdot) \in \Theta$ and $G \in \mathbb{G}$. Then T has at least one fixed point.

Proof. By induction, we obtain a sequence $\{\Omega_n\}$ such that $\Omega_0 = \Omega$ and $\Omega_n = \text{Conv}(F\Omega_{n-1}), n \geq 1$. If there exists an integer $N \geq 0$ such that $\mu(\Omega_N) = 0$, then Ω_N is compact. Thus, Theorem 2.2 implies that T has a fixed point. Now assume that $\mu(\Omega_n) \neq 0$ for $n \geq 0$ and also, $G(\mu(\Omega_n), \varphi(\mu(\Omega_n))) > 0$ and $O(f; G(\mu(\Omega_n), \varphi(\mu(\Omega_n)))) > 0$. By (6), we have

$$\begin{aligned} O(f; G(\mu(\Omega_{n+1}), \varphi(\mu(\Omega_{n+1})))) &= O(f; G(\mu(\text{Conv}(T\Omega_n)), \varphi(\mu(\text{Conv}(T\Omega_n)))) \\ &= O(f; G(\mu(T\Omega_n), \varphi(\mu(T\Omega_n)))) \\ &\leq \phi[O(f; G(\mu(\Omega_n), \varphi(\mu(\Omega_n))))] \\ &< O(f; G(\mu(\Omega_n), \varphi(\mu(\Omega_n)))). \end{aligned} \tag{7}$$

Then $\{O(f; G(\mu(\Omega_n), \varphi(\mu(\Omega_n))))\}_{n \in \mathbb{N}}$ is non-increasing and thus it converges to some point $\epsilon \geq 0$. Of course, $\epsilon < O(f; G(\mu(\Omega_n), \varphi(\mu(\Omega_n))))$ for all $n \in \mathbb{N}$. If $\epsilon > 0$, then there exists $\delta = \delta(\epsilon)$ such that

$$\epsilon < t < \epsilon + \delta \implies \phi(t) \leq \epsilon.$$

Take $n_\delta \in \mathbb{N}$ such that $O(f; G(\mu(\Omega_n), \varphi(\mu(\Omega_n)))) < \epsilon + \delta$ for all $n \geq n_\delta$. Therefore $\phi(O(f; G(\mu(\Omega_n), \varphi(\mu(\Omega_n)))) \leq \epsilon$, so by (7), $O(f; G(\mu(\Omega_{n+1}), \varphi(\mu(\Omega_{n+1})))) \leq \epsilon$ for all $n \in \mathbb{N}$, a contradiction. Consequently $\epsilon = 0$ and $\lim_{n \rightarrow \infty} \mu(\Omega_n) = 0$. Since the sequence (Ω_n) is nested, in view of axiom (6°) of Definition 2.1 we deduce that

the set $\Omega_\infty = \bigcap_{n=1}^\infty \Omega_n$ is nonempty, closed and convex subset of the set Ω . Moreover, the set Ω_∞ is invariant under the operator T and belongs to $\text{Ker}\mu$. Consequently, Theorem 2.2 implies that T has a fixed point in Ω_∞ . Since $\Omega_\infty \subset \Omega$, the proof is completed. \square

As a consequence of Theorems 3.8 we obtain the Corollary 3.9.

Corollary 3.9. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and μ is an arbitrary measure of noncompactness on E . Let $T : \Omega \rightarrow \Omega$ be a continuous function. Suppose that there exists a Meir-Keeler function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

$$O(f; G(\mu(T(X)), \varphi(\mu(TX)))) \leq \phi(O(f; G(\mu(X), \varphi(\mu(X))))),$$

for any subset X of Ω , where $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous function, $O(\bullet; \cdot) \in \Theta$ and $G \in \mathbb{G}$. Then T has at least one fixed point.

Proof. Apply Remarks 2.5 and 2.8, and Theorem 3.8. \square

4. Solvability of an infinite system of second order differential equations in ℓ_p ($1 \leq p < \infty$)

In our discussion, we study an infinite system of second order differential equation by transforming the system into an infinite system of integral equation with the help of Green’s function.

Mursaleen [19] defined the Hausdorff measure of noncompactness χ , for the Banach space $(\ell_p, \|\cdot\|_{\ell_p}), (1 \leq p < \infty)$ as follows.

$$\chi(B) = \lim_{n \rightarrow \infty} \left[\sup_{x \in B} \left(\sum_{k=n}^\infty |x_k|^p \right)^{1/p} \right], \tag{8}$$

where $x(t) = (x_i(t))_{i=1}^\infty \in \ell_p$ for each $t \in [0, 1]$ and $B \in \mathcal{M}_{\ell_p}$.

Consider the infinite system of second order differential equations

$$x_i''(t) + f_i(t, x_1, x_2, x_3, \dots) = 0, \tag{9}$$

where $x_i(0) = x_i(1) = 0, t \in [0, 1]$ and $i = 1, 2, 3, \dots$.

Let $C(I, \mathbb{R})$ denote the space of all continuous real functions on the interval $I = [0, 1]$ and let $C^2(I, \mathbb{R})$ be the

class of functions with second continuous derivatives on I . A function $u \in C^2(I, \mathbb{R})$ is a solution of (9) if and only if $u \in C(I, \mathbb{R})$ is a solution of the infinite system of integral equation

$$x_i(t) = \int_0^1 G(t, s) f_i(s, x(s)) ds, \tag{10}$$

where $f_i(t, x) \in C(I, \mathbb{R})$, $i = 1, 2, 3, \dots$ and $t \in I$ and the Green's function associated to (9) is given by (see [18])

$$G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{11}$$

The solution of the infinite system (9) in the sequence space ℓ_1 has been studied by Aghajani and Pourhadi [3]. In our study, we establish the existence of solution of the infinite system (9) for the sequence space ℓ_p ($1 \leq p < \infty$). Assume that

(A₁) The functions f_i are defined on the set $I \times \mathbb{R}^\infty$ and take real values. The operator f defined on the space $I \times \ell_p$ into ℓ_p as

$$(t, x) \rightarrow (fx)(t) = (f_1(t, x), f_2(t, x), f_3(t, x), \dots)$$

is such that the class of all functions $((fu)(t))_{t \in I}$ is equicontinuous at every point of the space ℓ_p ;

(A₂) The following inequality holds:

$$|f_n(t, x_1, x_2, x_3, \dots)|^p \leq h_n(t) |x_n(t)|^p$$

where $h_n(t)$ is real function defined on I , such that the sequence $(h_n(t))$ is equibounded on I .

Let us introduce

$$H_0 = \sup_{n \in \mathbb{N}, t \in I} \{h_n(t)\};$$

(A₃) The function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is nondecreasing and continuous such that $\varphi(\lambda t) \leq \lambda \varphi(t)$, for $\lambda \geq 0$, and $\varphi(0) = 0, \varphi(t) > 0$ for every $t > 0$.

Theorem 4.1. Under the hypothesis (A₁) – (A₂), infinite system (9) has at least one solution $x(t) = (x_i(t)) \in \ell_p$ for all $t \in [0, 1]$.

Proof. By using (10) and (A₂), we have for all $t \in I$,

$$\begin{aligned} \|x(t)\|_{\ell_p}^p &= \sum_{i=1}^\infty \left| \int_0^1 G(t, s) f_i(s, x(s)) ds \right|^p \\ &\leq \sum_{i=1}^\infty \int_0^1 |G(t, s)|^p \{h_i(s) |x_i|^p\} ds \\ &= \int_0^1 |G(t, s)|^p \left(\sum_{i=1}^\infty h_i(s) |x_i|^p \right) ds \end{aligned}$$

Since $x(t) \in \ell_p$ therefore $\sum_{i=1}^\infty |x_i(t)|^p < M < \infty$ (say) and $\int_0^1 |G(t, s)|^p ds \leq \frac{1}{4^p(p+1)}$.

Hence we get

$$\|x(t)\|_{\ell_p}^p \leq \frac{H_0 M}{4^p(p+1)} = r^p$$

i.e. $\|x(t)\|_{\ell_p} \leq r$.

Let $x^0(t) = (x_i^0(t))$ where $x_i^0(t) = 0, \forall t \in I$.

Consider $\bar{B} = \bar{B}(x^0, r_1)$, the closed ball centered at x_0 and radius $r_1 \leq r$, thus \bar{B} is an non-empty, bounded, closed and convex subset of ℓ_p . Consider the operator $\mathcal{F} = (\mathcal{F}_i)$ on $C(I, \bar{B})$ defined as follows. For $t \in I$,

$$(\mathcal{F}x)(t) = \{(\mathcal{F}_i x)(t)\} = \left\{ \int_0^1 G(t,s) f_i(s, x(s)) ds \right\},$$

where $x(t) = (x_i(t)) \in \bar{B}$ and $x_i(t) \in C(I, \mathbb{R})$. We have that $(\mathcal{F}x)(t) = \{(\mathcal{F}_i x)(t)\} \in \ell_p$ for each $t \in I$. Since $(f_i(t, x(t))) \in \ell_p$ for each $t \in I$, we have

$$\sum_{i=1}^{\infty} |(\mathcal{F}_i x)(t)|^p = \sum_{i=1}^{\infty} \left| \int_0^1 G(t,s) f_i(s, x(s)) ds \right|^p \leq r^p < \infty.$$

Also $(\mathcal{F}_i x)(t)$ satisfies boundary conditions i.e.

$$(\mathcal{F}_i x)(0) = \int_0^1 G(0,s) f_i(s, x(s)) ds = \int_0^1 0 \cdot f_i(s, x(s)) ds = 0,$$

and

$$(\mathcal{F}_i x)(1) = \int_0^1 G(1,s) f_i(s, x(s)) ds = \int_0^1 0 \cdot f_i(s, x(s)) ds = 0.$$

Since $\|(\mathcal{F}x)(t) - x^0(t)\|_{\ell_p} \leq r$ thus \mathcal{F} is self mapping on \bar{B} . The operator \mathcal{F} is continuous on $C(I, \bar{B})$ by the assumption (A_1) .

Now, we shall show that \mathcal{F} is a generalized Meir-Keeler type function.

For $\epsilon > 0$, we need to find $\delta > 0$ such that $\epsilon \leq \chi(\bar{B}) + \varphi(\chi(\bar{B})) < \epsilon + \delta \implies \chi(\mathcal{F}\bar{B}) + \varphi(\chi(\mathcal{F}\bar{B})) < \epsilon$.

We have

$$\begin{aligned} & \chi(\mathcal{F}\bar{B}) + \varphi(\mathcal{F}\bar{B}) \\ &= \lim_{n \rightarrow \infty} \left[\sup_{x(t) \in \bar{B}} \left\{ \sum_{k \geq n} \left| \int_0^1 G(t,s) f_k(s, x(s)) ds \right|^p \right\}^{\frac{1}{p}} \right] + \varphi \left[\lim_{n \rightarrow \infty} \left[\sup_{x(t) \in \bar{B}} \left\{ \sum_{k \geq n} \left| \int_0^1 G(t,s) f_k(s, x(s)) ds \right|^p \right\}^{\frac{1}{p}} \right] \right] \\ &\leq \lim_{n \rightarrow \infty} \left[\sup_{x(t) \in \bar{B}} \left\{ \sum_{k \geq n} \int_0^1 |G(t,s)|^p (h_{k(s)} |x_k(s)|^p) ds \right\}^{\frac{1}{p}} \right] + \varphi \left[\lim_{n \rightarrow \infty} \left[\sup_{x(t) \in \bar{B}} \left\{ \sum_{k \geq n} \int_0^1 |G(t,s)|^p (h_{k(s)} |x_k(s)|^p) ds \right\}^{\frac{1}{p}} \right] \right] \\ &\leq \lim_{n \rightarrow \infty} \left[\sup_{x(t) \in \bar{B}} \left\{ \int_0^1 |G(t,s)|^p \left(H_0 \sum_{k \geq n} |x_k(s)|^p \right) ds \right\}^{\frac{1}{p}} \right] + \varphi \left[\lim_{n \rightarrow \infty} \left[\sup_{x(t) \in \bar{B}} \left\{ \int_0^1 |G(t,s)|^p \left(H_0 \sum_{k \geq n} |x_k(s)|^p \right) ds \right\}^{\frac{1}{p}} \right] \right] \\ &\leq \frac{H_0^{1/p}}{4(p+1)^{1/p}} \chi(\bar{B}) + \varphi \left[\frac{H_0^{1/p}}{4(p+1)^{1/p}} \chi(\bar{B}) \right] \\ &\leq \frac{H_0^{1/p}}{4(p+1)^{1/p}} [\chi(\bar{B}) + \varphi(\chi(\bar{B}))]. \end{aligned}$$

Hence $\chi(\mathcal{F}\bar{B}) + \varphi(\mathcal{F}\bar{B}) \leq \frac{H_0^{1/p}}{4(p+1)^{1/p}} [\chi(\bar{B}) + \varphi(\chi(\bar{B}))] < \epsilon \implies \chi(\bar{B}) + \varphi(\chi(\bar{B})) < \frac{4\epsilon(p+1)^{1/p}}{H_0^{1/p}}$.

Taking $\delta = \frac{4(p+1)^{1/p} - H_0^{1/p}}{H_0^{1/p}} \epsilon$, we get $\epsilon \leq \chi(\bar{B}) + \varphi(\chi(\bar{B})) < \epsilon + \delta$. Therefore \mathcal{F} is a generalized Meir-Keeler

type function defined on the set $\bar{B} \subset \ell_p$. So \mathcal{F} satisfies all the conditions of Theorem 3.2 with $O(f; t) = t$ and $G(a, b) = a + b$ which implies \mathcal{F} has a fixed point in \bar{B} . This is a required solution of the system (9). \square

Example 4.2. Let us consider the following system of second order differential equations

$$\frac{d^2 x_n(t)}{dt^2} + f_n(t, x(t)) = 0, \quad (12)$$

where $f_n(t, x(t)) = e^t \cos\left(\frac{t}{n^2}\right)x_n(t)$, $\forall n \in \mathbb{N}$, $t \in I = [0, 1]$.

We have $\sum_{k=1}^{\infty} |f_n(t, x(t))|^p \leq e^p \sum_{k=1}^{\infty} |x_k(t)|^p < \infty$ if $x(t) = (x_i(t)) \in \ell_p$ where $1 \leq p < \infty$.

Let us consider a positive arbitrary real number $\epsilon > 0$ and $x(t) \in \ell_p$. Taking $y(t) \in \ell_p$ with

$$\|x(t) - y(t)\|_{\ell_p} < \delta = \frac{\epsilon}{e},$$

$$|f_n(t, x(t)) - f_n(t, y(t))| = |e^t \cos\left(\frac{t}{n^2}\right)x_n(t) - e^t \cos\left(\frac{t}{n^2}\right)y_n(t)| \leq e |u(t) - v(t)| < e\delta = \epsilon$$

which implies the equicontinuity of $(f_n(t))_{t \in I}$ on ℓ_p . Again, we have for all $n \in \mathbb{N}$ and $t \in I$

$$|f_n(t, x(t))|^p \leq e^{pt} |x_n(t)|^p = h_n(t) |x_n(t)|^p,$$

where $h_n(t) = e^{pt}$ is real function on I and the sequence $\{h_n(t)\}$ is equibounded on I . Thus, by Theorem 4.1, the system (12) has unique solution in ℓ_p .

References

- [1] R. P. Agarwal, D. O'Regan, Fixed point theory and applications, Cambridge University Press (2004).
- [2] A. Aghajani, M. Mursaleen, A. Shole Haghghi, Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness, Acta. Math. Sci. 35(3)(2015) 552-566.
- [3] A. Aghajani E. Pourhadi, Application of measure of noncompactness to ℓ_1 -solvability of infinite systems of second order differential equations, Bull. Belg. Math. Soc. Simon Stevin 22(2015) 105-118.
- [4] A. Aghajani, J. Banaś, N. Sabzali, Some generalizations of Darbo fixed point theorem and applications, Bull. Belg. Math. Soc. Simon Stevin 20 (2013) 345-358.
- [5] C. Alegre, J. Marn, S. Romaguera; A fixed point theorem for generalized contractions involving w-distances on complete quasi-metric spaces, Fixed Point Theory Appl. 2014(2014):40.
- [6] A. Alotaibi, M. Mursaleen, S. A. Mohiuddine, Application of measure of noncompactness to infinite system of linear equations in sequence spaces, Bull. Iranian Math. Soc. 41 (2015) 519-527.
- [7] I. Altun, D. Turkoglu, A fixed point theorem for mappings satisfying a general contractive condition of operator type, J. Comput. Anal. Appl. 9 (1) (2007) 9-14.
- [8] J. Banaś, D. O'Regan, K. Sadarangani, On solutions of a quadratic hamperstein integral equation on an unbonded interval, Dynam. Systems Appl. 18 (2009) 251-264.
- [9] J. Banaś, K. Goebel, Measure of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics, Vol. 60, Marcel Dekker, New York, 1980.
- [10] J. Banaś, M. Lecko, Solvability of infinite systems of differential equations in Banach sequence spaces, J. Comput. Appl. Math. 137 (2001) 363-375.
- [11] J. Jachymski; Equivalent conditions and the Meir-Keeler type theorems. J. Math. Anal. Appl. 194(1995) 293-303.
- [12] K. Kuratowski, Sur les espaces complets, Fund. Math. 15 (1930) 301-309.
- [13] A. Meir, E. Keeler; A theoremon contraction mappings. J. Math. Anal. Appl. 28(1969) 326-329.
- [14] C. Ming Chen, Fixed point theory for the cyclic weaker Meir-Keeler function in complete metric spaces, Fixed Point Theory Appl. 2012:17 (2012), doi:10.1186/1687-1812-2012-17.
- [15] C. Ming Chen, Fixed point theorems for cyclic Meir-Keeler type mappings in complete metric spaces, Fixed Point Theory Appl. 2012:41 (2012), doi:10.1186/1687-1812-2012-41.
- [16] S. A. Mohiuddine, H. M. Srivastava, A. Alotaibi, Application of measures of noncompactness to the infinite system of second-order differential equations in ℓ_p spaces, Adv. Difference Equ. 2016 (2016), Article ID 317, 1-13.
- [17] M. Mursaleen, S. A. Mohiuddine, Applications of measures of noncompactness to the infinite system of differential equations in ℓ_p spaces, Nonlinear Anal. 75 (2012) 2111-2115.
- [18] M. Mursaleen, Syed M. H. Rizvi, Solvability of infinite systems of second order differential equations in c_0 and ℓ_1 by Meir-Keeler condensing operators, Proc. Amer. Math. Soc. 144(10)(2016) 4279-4289.
- [19] M. Mursaleen, Application of measure of noncompactness to infinite system of differential equations, Canad. Math. Bull. 56 (2013) 388-394.