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Low-Level Separation Axioms From the Viewpoint of Computational Topology

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Abstract. The present paper studies certain low-level separation axioms of a topological space, denoted by $A(X)$, induced by a geometric AC-complex X. After proving that whereas $A(X)$ is an Alexandroff space satisfying the semi-$T_{\frac{1}{2}}$-separation axiom, we observe that it does neither satisfy the pre $T_{\frac{1}{2}}$-separation axiom nor is a Hausdorff space. These are main motivations of the present work. Although not every $A(X)$ is a semi-T_1 space, after proceeding with an edge to edge tiling (or a face to face crystallization) of $\mathbf{R}^n, n \in \mathbf{N}$, denoted by $T(\mathbf{R}^n)$ as an AC complex, we prove that $A(T(\mathbf{R}^n))$ is a semi-T_1 space. Furthermore, we prove that $A(E^n)$, induced by an nD Cartesian AC complex $C^n = (E^n, N, dim)$, is also a semi-T_1 space, $n \in \mathbf{N}$. The paper deals with AC-complexes with the locally finite (LF-, for brevity) property, which can be used in the fields of pure and applied mathematics as well as digital topology, computational topology, and digital geometry.

1. Introduction

The well-known Alexandroff space X [1], *i.e.* every point $x \in X$ has the smallest open neighborhood of x, plays an important role in the fields of pure and applied topology as well as computational topology. Thus the works [14–16] have intensively studied about locally finite spaces with a special kind of axioms, the so called *axiomatic locally finite* (ALF, for short) spaces based on an *abstract cell* (AC for brevity) complexes [24, 29]. Indeed, AC complexes, cell complexes and simplicial complexes have been often used for studying digital (or discrete) spaces in computer imaginary and discrete (or digital) geometry [16, 30, 31]. Unlike simplicial and cell complexes in geometric topology, the present paper deals with AC complexes and an edge to edge tiled space or a face to face crystallized spaces of the Euclidean n-dimensional space $\mathbf{R}^n$ from the viewpoint of computational topology.

The papers [3, 5, 7, 18, 19, 21] introduced the notions of $T_{\frac{1}{2}}$-separation axiom, preopen set, preclosed set, semi-open set, semi-closed set, and nowhere dense set. Motivated by these notions, the present paper proves that a topological space, denoted by $A(X)$, derived from a geometric AC-complex X is an Alexandroff space satisfying the semi-$T_{\frac{1}{2}}$-separation axiom instead of the pre-$T_{\frac{1}{2}}$-separation axiom. Furthermore, let us

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denote $T(\mathbf{R}^n)$ by an edge to edge tiled (or face to face crystallized) space of $\mathbf{R}^n, n \in \mathbf{N}$, as an *AC*-complex. Then $A(T(\mathbf{R}^n))$ is proved to be a semi-T_1 space. Besides, the paper studies some properties of an $A(X)$.

The rest of this paper is organized as follows: Section 2 investigates some properties of smallest open neighborhoods of elements of *AC* complex. Besides, it also refers to several examples for *AC* complexes over the *n*-dimensional real space via their edge to edge tilings and face to face crystallizations. Section 3 investigates some topological properties of $A(X)$. Section 4 studies some properties of a certain singleton of $A(X)$ related to a semi-open and a semi-closed set. Furthermore, it studies low-level separation axioms of $A(X)$. In addition, we prove that $A(X)$ is a good example showing that the semi-$T_{\frac{1}{2}}$-separation axiom need not imply the pre-$T_{\frac{1}{2}}$-separation axiom. Section 5 concludes the paper with a summary.

Based on *AC* complexes, Kovalevsky [15] developed an *ALF* space [29]. The author [6] also studied some properties of *ALF* spaces. Let **R**, **Z**, and **N** be the sets of real numbers, integers, and natural numbers, respectively. For $a, b \in \mathbf{Z}$ we use the notation $[a, b]_{\mathbf{Z}} := \{x \in \mathbf{Z} \mid a \leq x \leq b\}$. For a set X we follow the notation $|X|$ as the cardinality of the set. In order to establish a special kind of neighborhood space, we shall use various properties of the tiling (or tessellation) of the plane [2], a face to face quasi-crystallization of the 3 dimensional real space [26] and an $(n-1)$ dimensional face to face quasi-crystallization of $\mathbf{R}^n, n \geq 4$. The recent paper [4] proved that an Alexandroff space satisfying the T_0-separation axiom is a semi-$T_{\frac{1}{2}}$ space. But the converse does not hold. Thus we have the following queries.

[Question] (1) What is an Alexandroff space satisfying the semi-$T_{\frac{1}{2}}$-separation axiom ?

(2) What separation axiom characterizes $A(X)$?

(3) For $A(X)$, what about the *preopen* property ?

(4) For a given $A(X)$, under what condition does a semi-$T_{\frac{1}{2}}$ space imply a semi-T_1 space ?

(5) What is a locally finite space satisfying the semi-T_1-separation axiom ?

The neighborhood was developed in a way characteristic for a so-called neighborhood space described in [25]. Indeed, the original neighborhood space was based on the notion of abstract cell complex established by J. Listing [20], E. Steinitz [28], A.W Tucker [29] and K. Reidemeister [24]. Unlike an abstract cell complex $C := (E, B, dim)$ [15], where E is an abstract set, B is a bounding relation [15, 16] among the elements of E and *dim* is a function assigning a non-negative integer to each element of E.

The present paper prefers to the neighborhood relation for establishing an abstract cell complex (see Definition 2.1, Proposition 2.11, and Definition 2.10).

Before studying an abstract cell complex, we need to establish a certain criteria for both elements of an abstract cell complex and the neighborhoods of $e \in E$, as follows:

(2-1) Any element (or cell) of an abstract cell complex is not a subset of another element (or cell).

(2-2) We follow the notion of neighborhood space in the book [25] (see the Axioms A and B below). Indeed, the book [25] takes only two axioms for a neighborhood space such as **Axiom A** and **Axiom B** below. A finite or an infinite nonempty set of mathematical objects, which will be called points (or elements), is called a neighborhood space if, to each point, certain subsets are assigned as neighborhoods of that point; these neighborhoods must satisfy both of the following axioms:

Axiom A: Each element p of a neighborhood space has at least one neighborhood; each neighborhood of p contains p.

If E is a neighborhood space, a neighborhood of p is denoted by $N(p \mid E)$ (or $N(p)$ if there is no danger of ambiguity).

Axiom B: Given a neighborhood $N(p \mid E)$, each subset of E containing this neighborhood is also a neighborhood of p.

At the moment the terminology "neighborhood" need not be a topological neighborhood.

2. Smallest Neighborhoods of Elements of an *AC* Complex

By using this neighborhood relation, we can represent an *AC* complex, as follows:

Definition 2.1. ([6, 15]) An abstract cell (for short, *AC*) complex $C := (X, N, dim)$ is a nonempty set X of elements provided with

(1) a reflexive, antisymmetric and transitive binary relation $N \subset X \times X$ called the neighborhood relation, where a is an element of a neighborhood of b if $(a, b) \in N$, and

(2) a dimension function dim: $X \to I$ from X into the set I of non-negative integers such that $(a, b) \in N$ implies $dim(a) \geq dim(b)$.

In Definition 2.1 if for $x \in X$, $dim(x) = i$, then:

(2.1): x is called an i-dimensional cell (or i-cell, for brevity) denoted by $c^i_j \in X = \{c^i_j \,|\, i \in M, j \in M'_i\}$ (see Fig.1(f)) and the subscript j of the cell means the only index for discriminating the i-dimensional cells, where the index set M is finite and M'_i need not be finite. For instance, as shown in Fig.1(f), as an example we can suggest a 0-, 1-, 2-, 3-cell of an AC complex [14–16]. Besides, we can see the dimension property of Definition 2.1 from the objects in Fig.1 (a) and (b) (see Example 3.2 of the present paper).

(2.2): In case there is no confusion, we may denote c^i instead of c^i_j by the i-cell for short. Besides, depending on the situation, we may use a more convenient notation if there is no confusion (see Fig.1).

(2.3): In view of Definition 2.1(2), the neighborhood relation N of an AC complex $C := (X, N, dim)$ is not a simple partial order. We need to remind that it is associated with the function "dim" from Definition 2.1(2).

(2.4): Although an element of an AC-complex can be objects or set or ordinary element of set theory and so forth, the present paper focuses only on geometric objects.

(2.5): Hereafter, we denote $a \leq b$ by the relation $(a, b) \in N$ satisfying the properties (1) and (2) of Definition 2.1. Then the AC-complex of Definition 2.1 can be considered to be a special kind of *partially ordered set* (for short, poset) denoted by $(X, \leq)$.

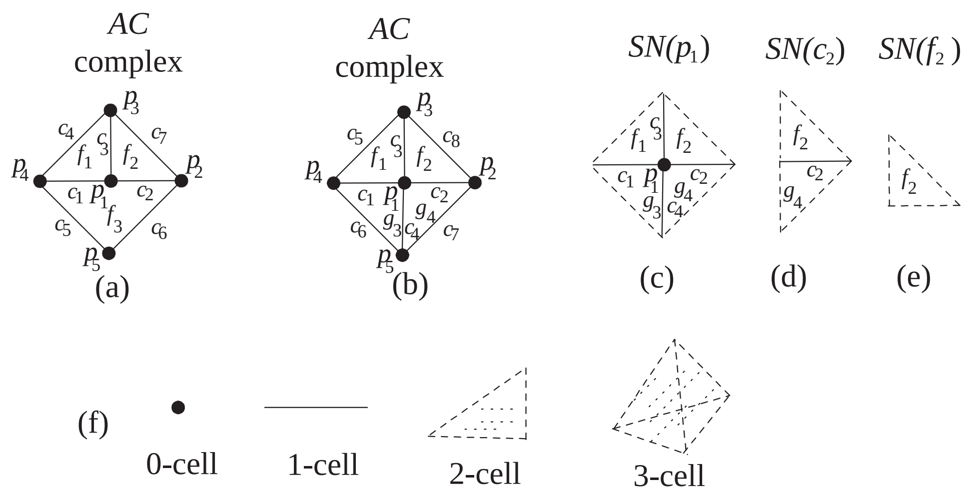

Figure 1: (a) Example of an AC-complex with fifteen elements such as five 0-cells, seven 1-cells and three 2-cells; (b) Another AC-complex with seventeen elements; (c)-(e) Based on the AC-complex in (b), examples of several types of smallest open neighborhoods $SN(p_1)$, $SN(c_2)$ and $SN(f_2)$ of the given elements p_1, c_2 and f_2, respectively; (f) Examples of n-cells, $n \in [0, 3]_{\mathbf{Z}}$.

As mentioned in (2.1) above, the complex is called "abstract" since its elements called "cells" are not subsets of a Hausdorff space as it is the case in a Euclidean and CW complex. Besides, they are neither simplices in simplicial complexes nor cells in cell complexes. Namely, any element of an AC-complex does not consist of a part of the other element. Unlike the number i of the i-cell in Fig.(f), it was demonstrated that the dimension of a cell c of an abstract cell complex is equal to the length (number of cells minus 1) of the maximum neighborhood path leading from any cell of the complex to the cell c (see Definition 3.4 of the present paper). Namely, the dimension of an element of an AC-complex depends on the situation. Besides, it is quite distinctive from that in ordinary geometry as well as topology. The neighborhood path is a sequence of cells in which each cell is a neighbor of the next one in terms of the relation "$\leq$" from Definition 2.1 [15] (see the property (2.5)).

Definition 2.2. ([15, 16]) A nonempty neighborhood space E is called a locally finite (*LF*-, for brevity) space if to each element $e(\in E)$ certain subsets of E are assigned as neighborhoods of e and some of them are finite. Hereafter, we call this space E a neighborhood space with the *LF*-property, *i.e.*, an *LF*-space.

Since the paper stresses on studying geometric *AC*-complexes from the viewpoint of computational topology, in what follows we shall consider only neighborhood spaces satisfying the *LF*-property. Namely, for a given element of an *LF*-space, there is the smallest neighborhood of e that is the intersection of all neighborhoods of e. Thus each neighborhood of e contains its smallest neighborhood with a finite cardinality. Hereafter, we shall denote $SN(e)$ by the smallest neighborhood of e. By using this neighborhood SN, we define the notion of neighborhood relation on an *LF*-space.

Definition 2.3. ([15, 16]) The neighborhood relation N on a neighborhood space E with the *LF*-property is a binary relation in the set of the elements of the space E. The ordered pair (a, b) is in N if and only if $a \in SN(b)$. And it is denoted by $a \leq b$ as a partial relation on E.

Let us now introduce the smallest neighborhood of an element of an *AC* complex.

Definition 2.4. ([8]) Let $C := (X, N, dim)$ be an *AC* complex. For an element $a \in X$ let $SN(a, X) = \{b \in X \,|\, b \leq a\}$ (for short, $SN(a)$ if there is no confusion). Then we call $SN(a)$ the smallest neighborhood of a in X.

In this state we need to point out that the smallest neighborhood of Definition 2.4 is used without topology (see Fig.1(c)-(e)). Besides, we need to speak out the following: The notion of 'bounding relation' in an *AC* complex can be defined as follows: In case $a \neq b$, $a \in SN(b)$, we say that ' b bounds a' or ' a is bounded by b' [15].

Let us now define the terminology "adjacent (or joins)" between two cells of an *AC* complex, as follows:

Definition 2.5. ([6, 15]) Let $C := (X, N, dim)$ be an *AC* complex. For two distinct elements a and b in X we say that a is adjacent to (or joins) b if $a \in SN(b)$ (or $a \leq b$) or $b \in SN(a)$ (or $b \leq a$).

For instance, consider the object in Fig.1(b). Then we say that each of the 2-cells f_1 and f_2 is adjacent to the 1-cell c_3. Besides, each of the 2-cells $f_i, i \in \{1, 2\}$, $g_j, j \in \{3, 4\}$ and the 1-cells $c_k, k \in [1, 4]_{\mathbb{Z}}$ is adjacent to 0-cell p_1. Consider an *AC* complex $C := (X, N, dim)$, where $X := \{c^i_j \,|\, i \in M, j \in M'_i\}$. According to Definitions 2.4 and 2.5, we can represent the smallest neighborhood of an element $c^i_j \in X$ via the notion of adjacency of Definition 2.5, as follows:

$$SN(c^i_j) = \{c^i_j\} \cup \{c^{i_1}_{j_1} \mid c^{i_1}_{j_1} \text{ is adjacent to } c^i_j, i_1 > i\}. \tag{2.6}$$

By using the property (2.6), we obtain the following: Let $C := (X, N, dim)$ be an *AC* complex, where $X := \{c^i_j \,|\, i \in M, j \in M'_i\}$. Consider an element $c^t_j \in X$ with the maximal dimension in X. Then we have $SN(c^t_j) = \{c^t_j\}$. For instance, consider the *AC*-complex in Fig.1(b). Since each of the 2-cells $f_i, i \in \{1, 2\}$ and $g_j, j \in \{3, 4\}$ has the maximal dimension 2 in the *AC*-complex in Fig.1(b). Hence we have $SN(f_i) = \{f_i\}, i \in \{1, 2\}$ and $SN(g_j) = \{g_j\}, j \in \{3, 4\}$.

Example 2.6. In Fig.1(c) we can observe that $SN(p_1) = \{p_1, c_i, f_j, g_k \,|\, i \in [1, 4]_{\mathbb{Z}}, j \in [1, 2]_{\mathbb{Z}}, k \in [3, 4]_{\mathbb{Z}}\}$, $SN(c_2) = \{c_2, f_2, g_4\}$ and $SN(f_2) = \{f_2\}$.

Remark 2.7. ([6]) The binary relation SN of the property (2.6) is reflexive, antisymmetric, and transitive.

The following boundary (see Definition 2.8 is a slight generalization of the earlier version of [15, 16]. More precisely, the original boundary of an element c^i_j of an *AC* complex in [6, 15] was defined only for the case $i \gneq 1$. However, the present paper considers it for the case $i \geq 0$.

Definition 2.8. ([8]) Let $C := (X, N, dim)$ be an *AC* complex, where $X := \{c^i_j \,|\, i \in M, j \in M'_i\}$. For each m-cell c^m in X, its boundary, denoted by $\partial(\{c^m\})$ (or ∂c^m), is defined as follows:
$\partial c^m := \{c^i_j \mid c^i_j$ is adjacent to (or joins) $c^m, i \lneq m\}$. In addition, for a 0-cell (or point), denoted by c^0, we say that $\partial c^0 = \emptyset$.

In this state we also need to point out that the notion of boundary is used without topology.

After proceeding with several types of edge to edge tilings (or face to face crystallizations) of $\mathbf{R}^n$ [2, 26], we have the following example.

Example 2.9. Let $T(\mathbf{R}^n)$ be an edge to edge tiled (or a face to face crystallized) space of $\mathbf{R}^n, n \in \mathbf{N}$. Then $(T(\mathbf{R}^n), N, dim)$ is an AC complex. For instance, in case $n = 1$, consider $T(\mathbf{R})$ in Fig.3(h). Then we have $T(\mathbf{R})$ as an AC complex.
Next, in case $n = 2$, consider $T(\mathbf{R}^2)$ in Fig.3(a). Then we can have $T(\mathbf{R}^2)$ as an AC complex [9].
Finally, by using the method similar to the above approach, we can obtain an AC complex structure over $T(\mathbf{R}^n), n \geq 3$.

It is well known that in general a poset $(X, \leq)$ induces an Alexandroff topological space with T_0 separation axiom [1, 22] induced by the set of all $U_x := \{x' \in X \,|\, x' \leq x, x \in X\}$ as a base.

Definition 2.10. ([6]) Let $C := (X, N, dim)$ be an AC-complex. Let (X, B) be a binary set, where $B = \{SN(x) \,|\, x \in X\}$. Then we obtain the topology on X generated by the set B as a base. Then this topology is denoted by $A(X)$ in the present paper.

Since an AC-complex is a special kind of poset, according to Definition 2.4 and the property (2.5), we have the following.

Proposition 2.11. *Owing to the poset structure $(X, \leq)$ derived from an AC-complex $C := (X, N, dim)$, we have an Alexandroff topological space $A(X)$ with T_0-separation axiom generated by the set $\{SN(x) \,|\, x \in X\}$ as a base.*

For an $A(X)$, we need to point out that the smallest neighborhood of (2.6) is exactly that of Definition 2.10 so that it is the smallest open set of an element of $A(X)$. Furthermore, if $|X| \geq 2$, then a connected $A(X)$ cannot be a discrete topological space [8].

3. Dimension and Topological Properties of $A(X)$

Let us now introduce a neighborhood path to characterize a dimension of an AC complex $C := (X, N, dim)$.

Definition 3.1. ([6, 8, 15]) In an AC complex $C := (X, N, dim)$, consider the following sequence

$$\text{“}aNbNc \cdots Nk\text{”} \tag{3.1}$$

of pairwise distinct cells of X in which each cell belongs to the smallest neighborhood of the next one: $a \in SN(b)$ and $a \neq b$, *i.e.* $(a, b) \in N$; $b \in SN(c)$ and $b \neq c$, *i.e.* $(b, c) \in N$ etc. We shall call it the *neighborhood path* from a to k. The number of cells in the sequence minus one is called the *length* of the *neighborhood path*. For an element $x \in X$ consider the longest neighborhood path containing the element x. Then the first element (resp. the last element) of this path is called the maximal (resp. minimal) element of the given longest neighborhood path containing the element x. Besides, for each element $x \in X$ the first element (resp. the last element) of the given longest neighborhood path containing the element x is called the maximal (resp. minimal) element of X(or C) related to the element x.

Example 3.2. (1) For the AC complex X of Fig.2, each of 0-cells $p_i, i \in [1, 7]_{\mathbf{Z}}$ is a minimal element of X.

(2) Consider the element c_4 in Fig.2. Then we have the chain $f_1 \leq c_4 \leq p_2$ such that $f_1 \in SN(c_4)$ and $c_4 \in SN(p_2)$. Hence the element f_1 is the maximal element related to the element c_4 and further, the element p_2 is the minimal element related to the element c_4.

(3) The object X in Fig.2 has three types of maximal elements such as the 1-cell c_1, the 2-cell f_1 and the 3-cell, *i.e.* the open tetrahedron based on the elements p_4, p_5, p_6 and p_7.

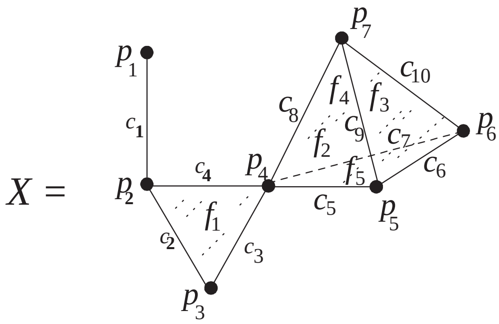

Figure 2: An explanation of a maximal element and a minimal element of an *AC*-complex *X* related to the dimension of a given element in the *AC* complex $C := (X, N, dim)$.

Remark 3.3. Owing to the property (1) of Definition 2.1, we need to point out that in the neighborhood path of (3.1) the relation $a \in SN(k)$ holds because of the transitivity of the neighborhood relation.

The notion of dimension of a cell $c(\in X)$ in an *AC* complex $C := (X, N, dim)$ can be defined by using the notion of neighborhood path, as follows:

Definition 3.4. ([8, 15]) Let $C := (X, N, dim)$ be an *AC* complex. For $c \in X$ the dimension of the cell c of C, denoted by $dim(c, C)$, is the length of a longest neighborhood path from c to any element of C. For a cell that does not belong to any smallest neighborhood of another cell, its length and dimension are zero. The dimension of an *AC* complex is the greatest dimension of its cells.

For instance, as mentioned in Example 3.2(2), the maximal dimension related to the elements p_2, c_4 and f_1 is 2. Besides, in view of Definition 3.4, the dimension of the given *AC* complex *X* of Fig.2 is three because we have the following neighborhood path

$$t \leq f_2 \leq c_7 \leq p_4 \quad \text{or} \quad t \leq f_3 \leq c_9 \leq p_7,$$

where t is the open tetrahedron, which is an element of an *AC*-complex, generated by the elements p_4, p_5, p_6, and p_7.

Motivated by Remark 3.3, we introduce the following:

Definition 3.5. ([10]) Consider $A(X)$ derived from an *AC* complex $C := (X, N, dim)$. Then we say that the dimension of $A(X)$, denoted by $dim(A(X))$, is that of the *AC*-complex inducing the given $A(X)$.

Remark 3.6. Consider $A(X)$ induced by an *AC* complex $C := (X, N, dim)$. Then the existence of a maximal n-cell of $C := (X, N, dim)$ does not imply $dim(A(X)) = n$. For instance, consider the set $\{c_j^1 \mid j \in [1, 4]_{\mathbb{Z}}\} := V$ in Fig.3(g). Namely, the set V consists of only four 1-cells. Hence, whereas V has the maximal 1-cell, we have $dim(A(V)) = 0$.

Motivated by Definitions 2.10 and 3.5, we introduce the following: For an $A(X)$ induced by an *AC* complex $C := (X, N, dim)$ and for each element $x_i \in X$, we obtain a finite and the longest neighborhood path such as

$$\cdots \leq x_k \leq x_j \leq x_i \leq \cdots \tag{3.2}$$

satisfying both $x_k \in SN(x_j), x_j \neq x_k$ and $x_j \in SN(x_i), x_j \neq x_i$ and so forth. Then we say that the first element (resp. the last element) of the path is the maximal (resp. minimal) element of $A(X)$ (see also (3.1)).

The following statement related to the property (2.6), Definition 3.4, and the property (3.2) will be used in Section 4.

Hereafter, we assume that $A(X)$ has both a maximal and a minimal element.

Proposition 3.7. *Consider $A(X)$ derived from an AC complex $C := (X, N, dim)$, where $X := \{c^i_j \mid i \in M, j \in M'_i\}$. Let $c^{i_1}_j \in X$ (resp. $c^{i_2}_j \in X$) be a minimal element (resp. a maximal element) of $A(X)$. Then each i-cell c^i_j has the following property in $A(X)$.*

(1) The singleton $\{c^{i_1}_j\}$ is closed.

(2) Let the i-cell c^i_j be neither a maximal element nor a minimal element. Then the singleton $\{c^i_j\}$ is neither open nor closed.

(3) The singleton $\{c^{i_2}_j\}$ is open.

Proof. Owing to the topological structure of $A(X)$ established in Proposition 2.11 and Definition 2.10, the proofs of (1) and (3) are completed. Thus we suffice to prove the assertion (2) for $A(X)$ with $dim(A(X)) \geq 2$. Let us now prove that for each i with $i_1 \lneq i \lneq i_2$ the singleton $\{c^i_j\}$ is not open in $A(X)$ because there is no open set included in the singleton $\{c^i_j\}$ (see the property (2.6)), where i_1 (resp. i_2) is a minimal dimension (resp. a maximal dimension) related to the given element c^i_j.

It is clear that for each i with $i_1 \lneq i \lneq i_2$ the singleton $\{c^i_j\}$ is not closed in $A(X)$ because $X \setminus \{c^i_j\}$ is not an open set in $A(X)$. □

Based on the above properties of an $A(X)$, let us now investigate some properties of $A(X)$ related to the $T_{\frac{1}{2}}$-separation axiom. We say that a topological space (X, T) satisfies the *$T_{\frac{1}{2}}$-separation axiom* if each singleton as a subset of X is either open or closed in (X, T) [18]. For instance, the Khalimsky line is an example for a topological space satisfying the $T_{\frac{1}{2}}$-separation axiom, where the Khalimsky line is the topological space on $\mathbf{Z}$ generated by the set $\{[2n-1, 2n+1]_{\mathbf{Z}} \mid n \in \mathbf{Z}\}$ as a subbase [12]. Then the Khalimsky line topology is denoted by $(\mathbf{Z}, \kappa)$ [12].

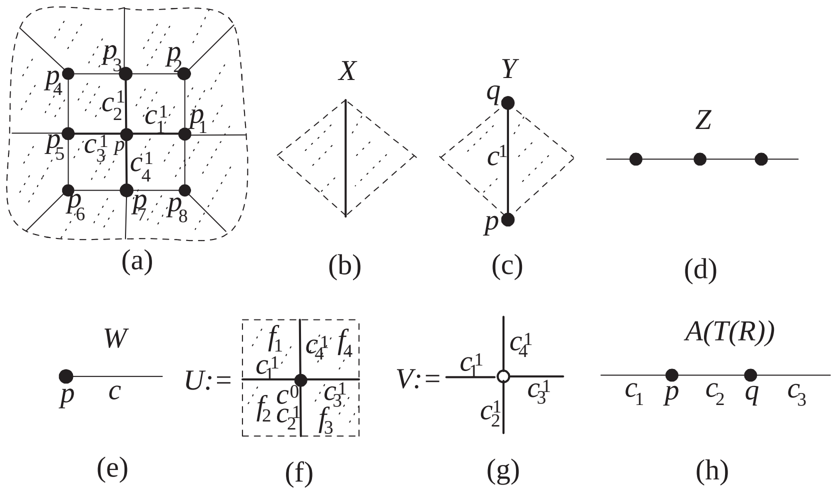

Figure 3: (a) A portion of an edge to edge tiling of $\mathbf{R}^2$, *i.e.* $T(\mathbf{R}^2)$; (b),(d) Configuration of the $T_{\frac{1}{2}}$-separation axiom of $A(X)$(resp. $A(Z)$) over the given AC complex X(resp. Z); (c) Configuration of the non-$T_{\frac{1}{2}}$-separation axiom of $A(Y)$ over the given AC complex Y (see the element c^1).

In view of Proposition 3.7, we obtain the following:

Corollary 3.8. *Consider $A(X)$ derived from an AC-complex $C := (X, N, dim)$, where $X := \{c^i_j \mid i \in M, j \in M'_i\}$. Then we obtain the following:*

(1) If $dim(A(X)) \leq 1$, *then* $A(X)$ *satisfies the* $T_{\frac{1}{2}}$*-separation axiom.*

(2) If $dim(A(X)) \geq 2$, *then* $A(X)$ *does not satisfy the* $T_{\frac{1}{2}}$*-separation axiom.*

Example 3.9. (1) Consider the AC complex X of Fig.3(b) with $dim(A(X)) = 1$ and $|X| = 3$. Then it is clear that $A(X)$ satisfies the $T_{\frac{1}{2}}$-separation axiom.

(2) Consider the AC complex Z of Fig.3(d) with $dim(A(Z)) = 1$ and $|Z| = 7$. Then it is clear that $A(Z)$ satisfies the $T_{\frac{1}{2}}$-separation axiom.

(3) Assume the AC complex Y of Fig.3(c) with $dim(A(Y)) = 2$. Then it is obvious that $A(Y)$ does not satisfy the $T_{\frac{1}{2}}$-separation axiom. To be specific, the singleton $\{c^1\}(\subset Y)$ is neither open nor closed in $A(Y)$.

4. Low-Level Separation Axioms of $A(X)$

This section examines if $A(X)$ satisfies some low-level separation axioms such as *pre-*$T_{\frac{1}{2}}$, *semi-*$T_{\frac{1}{2}}$ *and semi-*T_1*-separation axiom*. In particular, we prove that $A(X)$ satisfies the semi-$T_{\frac{1}{2}}$-separation axiom which is weaker than the separation axiom $T_{\frac{1}{2}}$. Furthermore, we investigate some conditions which make $A(X)$ a semi-T_1 space. To do this work, let us now recall basic notions associated with low level separation axioms.

The papers [5, 18, 19, 21] introduced the notions of *preopen set* and *semi-open* or *semi-closed*, as follows.

Definition 4.1. ([21]) A subset B of a topological space (X, T) is called preopen (resp. preclosed) if $B \subset Int(Cl(B))$ (resp. $X \setminus B$ is preopen), where the notation "Int" means the operator 'interior' of a topological space.

Definition 4.2. ([18]) A subset B is called a semi-open set of a topological space (X, T) if there is an open set $O \in T$ such that $O \subset B \subset Cl(O)$ (or $B \subset Cl(Int(B))$). A subset $F \subset X$ is called semi-closed of a topological space (X, T) if $X \setminus F$ is semi-open.

Lemma 4.3. *Consider* $A(X)$ *induced by an AC-complex* $C := (X, N, dim)$, *where* $X := \{c^i_j \mid i \in M, j \in M'_i\}$. *Let* $c^{i_1}_j$ *(resp.* $c^{i_2}_j$*) be a minimal (resp. maximal) element of* $A(X)$. *Let the i-cell* c^i_j *be neither a maximal element nor a minimal element (i.e.* $i_1 \lneq i \lneq i_2$*). Then the singleton* $\{c^i_j\}$ *is semi-closed in* $A(X)$.

Proof. Consider the singleton $\{c^i_j\}$, where c^i_j is neither a maximal element nor a minimal element. According to Proposition 3.7(2), the singleton $\{c^i_j\}$ is neither open nor closed, Then we obtain an open set $X \setminus Cl(\{c^i_j\}) := O$ in $A(X)$ such that

$$O \subset X \setminus \{c^i_j\} \subset Cl(O) = X,$$

which implies that the singleton $\{c^i_j\}$ is semi-closed in $A(X)$. □

To support Lemma 4.3, as an example, consider the element c_8 in the AC-complex X in Fig.2, Then the singleton $\{c_8\}$ is obviously neither a maximal element nor a minimal element. Since

$$Cl(\{c_8\}) = \{c_8, p_4, p_7\},$$

we have an open set $O := X \setminus Cl(\{c_8\})$ such that

$$O \subset X \setminus \{c_8\} \subset Cl(O) = X,$$

which implies that the singleton $\{c_8\}$ is semi-closed in $A(X)$.

Thus we may pose the following query.

[Question 2] What separation axiom characterizes $A(X)$?

We will address this topic with Theorems 4.7 and 4.12, and Example 4.11 in the paper.

Definition 4.4. ([5]) We say that a topological space (X, T) has the semi-$T_{\frac{1}{2}}$-separation axiom if every singleton of X is either semi-open or semi-closed.

By Proposition 3.7 and Lemma 4.3, we obtain the following:

Proposition 4.5. *The topological space $A(X)$ satisfies the semi-$T_{\frac{1}{2}}$-separation axiom.*

At this moment, we may pose the following query.

[Question 3] Under what condition does a semi-$T_{\frac{1}{2}}$ space imply a semi-T_1 space ?

Definition 4.6. ([5]) We say that a topological space (X, T) has the semi-T_1-separation axiom if every singleton of X is semi-closed. The space (X, T) is called a semi-T_1 space.

Unlike Proposition 3.7(3), $A(T(\mathbf{R}^n)), n \in \mathbf{N}$, has the following property.

Theorem 4.7. *$A(T(\mathbf{R}^n))$ is a semi-T_1 space, $n \in \mathbf{N}$.*

Proof. By using Proposition 3.7, it is obvious that first of all the singleton $\{c_j^0\}$ of $A(T(\mathbf{R}^n))$ is not semi-open but semi-closed. Second, for each i-cell c_j^i of $A(T(\mathbf{R}^n)), 0 \lneq i \lneq n, n \in \mathbf{N}$ which is neither a maximal element nor a minimal element, we see that the singleton $\{c_j^i\}$ is semi-closed (see Proposition 3.7). Third, although the singleton $\{c_j^n\}$ is semi-open (see Proposition 3.7(3)), in particular, we need to further prove that it is also semi-closed in $(A(T(\mathbf{R}^n)), n \in \mathbf{N}$.

To be precise, indeed, $A(T(\mathbf{R}^n)) \setminus \{c_j^n\}$ is semi-open according to the following property with $A(T(\mathbf{R}^n)) \setminus Cl(\{c_j^n\}) := O$

$$O \subset A(T(\mathbf{R}^n)) \setminus \{c_j^n\} \subset Cl(O) = A(T(\mathbf{R}^n)) \setminus \{c_j^n\}, \tag{4.1}$$

because $Cl(\{c_j^n\}) = \{\{c_j^n\}\} \cup \partial(\{c_j^n\})$, which implies that $\{c_j^n\}$ is a semi-closed subset of $A(T(\mathbf{R}^n))$. □

To support Theorem 4.7, let us examine the proof of the property (4.1) in $A(T(\mathbf{R}))$, $A(T(\mathbf{R}^2))$ and $A(T(\mathbf{R}^n)), n \geq 3$, as follows.

(Case 1) Let us take any singleton $\{c_j^1\} \subset A(T(\mathbf{R}))$ such as the singleton $\{c_2\}$ consisting of the element $c_2 \in T(\mathbf{R})$ in Fig.3(h). Then, to guarantee the property (4.1), putting $O := A(T(\mathbf{R})) \setminus Cl(\{c_2\}) = \{c_1, c_3\}$ as an open set, we have the following property

$$O \subset A(T(\mathbf{R})) \setminus \{c_2\} \subset Cl(O) = A(T(\mathbf{R})) \setminus \{c_2\}, \tag{4.2}$$

which implies that the singleton $\{c_2\}$ is semi-closed in $A(T(\mathbf{R}))$.

(Case 2) Let us take any singleton $\{c_j^2\} \subset A(T(\mathbf{R}^2))$ such as the singleton $\{f_1\}$ consisting of the element $f_1 \in T(\mathbf{R}^2)$ in Fig.3(f). Then, to support the property (4.1), putting $O := A(T(\mathbf{R}^2)) \setminus Cl(\{f_1\})$ as an open set because $Cl(\{f_1\}) = \{f_1, c_1^1, c_4^1, c^0\}$, we have the following property

$$O \subset A(T(\mathbf{R}^2)) \setminus \{f_1\} \subset Cl(O) = A(T(\mathbf{R}^2)) \setminus \{f_1\}, \tag{4.3}$$

which implies that the singleton $\{f_1\}$ is semi-closed in $A(T(\mathbf{R}^2))$.

(Case 3) Let us take any singleton $\{c_j^n\} \subset A(T(\mathbf{R}^n)), n \geq 3$. Then, by using the similar method as that of Cases 1 and 2, we conclude that the singleton $\{f_1\}$ is semi-closed in $A(T(\mathbf{R}^n))$.

Owing to Proposition 3.7 and Lemma 4.3, we obtain the following:

Corollary 4.8. *(1) An $A(X)$ with $dim(A(X)) = 0$ is a semi-T_1 space.*

(2) A connected $A(X)$ with $dim(A(X)) = 1$ with $|X| = 2$ is not a semi-T_1 space.

Proof. (1) By Theorem 4.7, the assertion is trivial.

(2) Under the hypothesis, not every each singleton of $A(X)$ is semi-closed. For instance, consider the singleton $\{c\}$ in Fig.3(e). Then it is not semi-closed in $A(X)$ because $X \setminus \{c\}$ is not semi-open. To be specific, since the biggest open set contained in $X \setminus \{c\}$ is the empty set, $X \setminus \{c\}$ is not semi-open. □

Owing to Corollary 4.8(2), we conclude that $A(X)$ with $dim(A(X)) \geq 1$ need not be a semi-T_1 space. Namely, it turns out that not every $A(X)$ is a semi-T_1 space, which makes us further study some properties of $A(X)$ related to the semi-T_1-separation axiom. To perform topological and geometrical calculations with abstract cell complexes, it is necessary to assign "names" to the cells. One of the possibilities of doing this consists in introducing coordinates. Consider an AC complex $C = (E, N, dim)$ whose elements compose a sequence $E = (e_0, e_1, e_2, \cdots, e_{2m}), m \geq 1$ (see Fig.4(a) with $e_i := i$). The smallest neighborhoods $SN(e_i)$ are the following [8, 15]:

$$\left\{\begin{matrix} SN(e_i) = \{e_{i-1}, e_i, e_{i+1}\} \cap E, \text{ if } i \text{ is even and} \\ SN(e_i) = \{e_i\}, \text{ if } i \text{ is odd.} \end{matrix}\right\} \tag{4.4}$$

The dimension of e_i is defined according to Definition 3.5:

$dim(e_i) = 0$ if i is even and $dim(e_i) = 1$ if i is odd. Thus C becomes a one-dimensional AC complex consisting of a single adjacent path (see Definition 2.1). We shall call such a complex a *path complex* [15].

We now provide a path complex C with a coordinate function $X : E \to I(\subset \mathbf{Z})$ assigning subsequent integers (not necessarily positive) to subsequent cells of E in such a way that a cell of dimension 0 (a 0-cell for short) obtains an even number while a cell of dimension 1 (1-cell) obtains an odd one. We call the numbers *combinatorial coordinates* in C [15].

Definition 4.9. ([15]) A combinatorial coordinate axis $A = (E, X)$ is a one-dimensional path complex provided with a coordinate function $X : E \to I(\subset \mathbf{Z})$ assigning subsequent integers to subsequent cells of E. It is also called a coordinate axis for short.

In view of Definition 4.9, it is obvious that $dim(A(E)) = 1$.

To address the above Question 2, we need to recall the Cartesian product of n combinatorial coordinate axes in [8, 16]. It possesses the product topology derived from the topology over a coordinate axis in terms of the smallest neighborhood suggested in (2.6), as follows:

Definition 4.10. ([8, 15]) An nD Cartesian AC complex $C^n = (E^n, N, dim)$ is the Cartesian product of n combinatorial coordinate axes $A_i = (E_i, X_i); i = 1, 2, \cdots, n;$ provided with a coordinate function X^n on the set $E^n = E_1 \times E_2 \times \cdots \times E_n$. A cell c of C^n is an n-tuple, *i.e.* an ordered sequence of cells a_i of the axes: $a_i \in A_i$. For an n-tuple $c = (a_1, a_2, \cdots, a_n) \in E^n$ we define its smallest neighborhood as $SN(c) = SN_1(a_1) \times SN_2(a_2) \times \cdots \times SN_n(a_n)$ where $SN_i(a_i)$ is the smallest neighborhood of a_i in A_i. The coordinate function $X^n : E^n \to I^n(\subset \mathbf{Z}^n)$ assigns the n-tuple $X^n(c) = (X_1(a_1), X_2(a_2), \cdots, X_1(a_n))$ to a cell $c = (a_1, a_2, \cdots, a_n)$ of E^n. The dimension function dim assigns to each cell $c \in E^n$ its dimension according to Definition 3.4. It is equal to the number of odd coordinates $X_i(a_i)$ of c.

Example 4.11. In the subspace of the 2D combinatorial coordinate, for example see Fig.4(c), we name the 2-cells $f_1 := (1, 7)$, $f_5 := (5, 1)$, etc. Besides, we name the 1-cells $c_1 := (1, 6)$, $c_5 := (5, 2)$ and so forth. In addition, we also name the 0-cells $p_1 := (2, 4)$ and $p_2 := (4, 2)$.

As we can see several examples for subsets of the combinatorial coordinate and the Cartesian product of n combinatorial coordinate axes, we may have the notation such as $e_i := i$ and $(e_i, e_j) := (i, j)$ for convenience.

Unlike Corollary 4.8, in case $dim(A(X)) = 1$ with $|X| = \aleph_0$, we have the following:

Theorem 4.12. *Consider $A(E^n)$ induced by an nD Cartesian AC complex $C^n = (E^n, N, dim)$. Then $A(E^n)$ is a semi-T_1 space, $n \in \mathbf{N}$.*

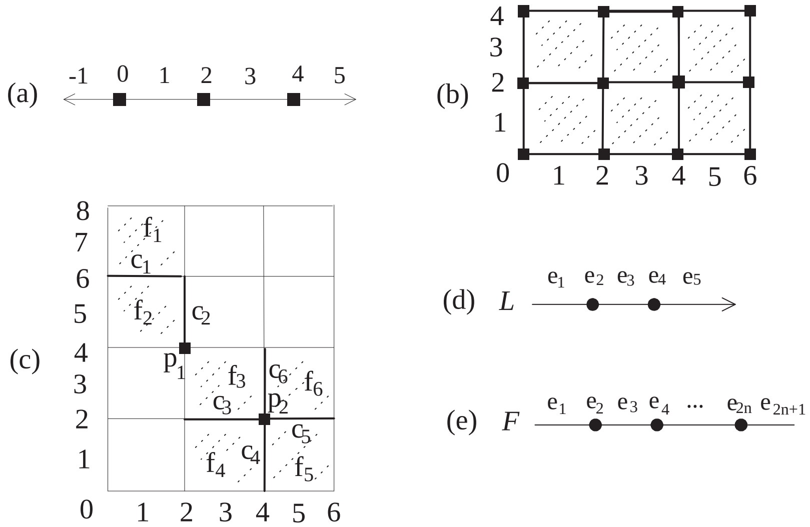

Figure 4: (a) A portion of the one dimensional combinatorial coordinate with $e_i := i \in \mathbf{Z}$ [15]; (b) A portion of the 2D combinatorial coordinate with $e_i := i$ and $(e_i, e_j) := (i, j)$; (c) A finite portion of an $A(E^2)$; (d) An infinite subset of the one dimensional combinatorial coordinate, *i.e.* $L := \{e_i \,|\, i \in \mathbf{N}\}$; (e) A finite subset of the one dimensional combinatorial coordinate, *i.e.* $F := \{e_i \,|\, i \in [1, 2n+1]_{\mathbf{Z}}\}$.

Proof. First, since the singleton $\{(e_{2i_1}, e_{2i_2}, \cdots, e_{2i_n})\}$ is a closed set in $A(E^n)$, it is semi-closed because it is minimal element of $A(X)$ (Proposition 3.7(1)). Second, since $dim(A(E^n)) = n$, we have the following neighborhood path

$$\begin{Bmatrix} (e_{2i_1+1}, e_{2i_2+1}, \cdots, e_{2i_n+1}) \lneq \cdots \lneq (e_{2i_1+1}, e_{2i_2+1}, \cdots, e_{2i_{n-1}+1}, e_{2i_n}) \\ \lneq (e_{2i_1}, e_{2i_2}, \cdots, e_{2i_n}) \end{Bmatrix} \tag{4.5}$$

By Proposition 3.7(2), any singleton consisting of an element which is neither a minimal element nor a maximal element of $A(X)$ is semi-closed. Third, we need to only prove that every singleton $\{(e_{2i_1+1}, e_{2i_2+1}, \cdots, e_{2i_n+1})\}$ is a semi-closed set in $A(E^n)$. Since we have the following property with $A(E^n) \setminus Cl(\{(e_{2i_1+1}, e_{2i_2+1}, \cdots, e_{2i_n+1})\}) := O$ in $A(E^n)$

$$\begin{Bmatrix} O \subset A(E^n) \setminus \{(e_{2i_1+1}, e_{2i_2+1}, \cdots, e_{2i_n+1})\} \\ \subset Cl(O) = A(E^n) \setminus \{(e_{2i_1+1}, e_{2i_2+1}, \cdots, e_{2i_n+1})\} \end{Bmatrix} \tag{4.6}$$

which completes the proof. □

At this moment, we may pose the following remark.

Remark 4.13. Unlike the subspace $(V, A(E)_V)$ in the proof of Theorem 4.12, to have a semi-T_1 subspace of $A(E)$, we can take the following two ways:

(1) In studying an infinite semi-T_1 subspace of $A(E)$, we need to take $(L, A(E)_L)$ induced by $A(E)$ on $L \subset E$, where L is an infinite subset of the combinatorial coordinate axis $E := \{e_i \,|\, i \in \mathbf{N}\}$ (see Fig.4(d)). Then any singleton consisting of each 1-cell c^1 such as $c^1 \in \{e_{2n-1} \,|\, n \in \mathbf{N}\}$ is semi-closed in $(L, A(E)_L)$ because

$$L \setminus Cl(\{c^1\}) = L \setminus \{c^1\} \subset Cl(L \setminus Cl(\{c^1\})) = L,$$

which implies that the subspace $(L, A(E)_L)$ is a semi-T_1 space.

(2) When dealing with a finite semi-T_1 subspace of $A(E)$, we need to take $(F, A(E)_F)$ induced by $A(E)$ on $F \subset E$, where F is a finite subset of the combinatorial coordinate axis E represented by the set $\{e_i \,|\, i \in$

$[1, 2n+1]_Z\} \subset E$ (see Fig.4(e)). Then any singleton consisting of each 1-cell $c^1 \in \{e_{2m-1} \mid m \in [1, n+1]_Z\} \subset F$ is semi-closed in $(F, A(E)_F)$ because

$$F \setminus Cl(\{e_{2m-1}\}) = F \setminus \{e_{2m-1}\} \subset Cl(F \setminus Cl(\{e_{2m-1}\})) = F,$$

which implies that the subspace $(F, A(E)_F)$ is a semi-T_1 space. □

In order to take a semi-T_1 subspace of $A(E^n)$, in view of Theorem 4.12 and Remark 4.13, we obtain the following:

Corollary 4.14. *(1) In taking an infinite and connected subspace of an $A(E^n)$ over an nD Cartesian AC complex $C^n = (E^n, N, dim)$, we need to point out that we should take the n-dimensional product space, for example $(L^n, (A(E)_L)^n$, induced by $(L, A(E)_L)$ of Remark 4.13(1) because $(L^n, (A(E)_L)^n$ is a semi-T_1 space.*

(2) In taking a finite and connected subspace of an $A(E^n)$ over an nD Cartesian AC complex $C^n = (E^n, N, dim)$, we need to point out that we should take the n-dimensional product space, denoted by $(F^n, (A(E)_F)^n)$, induced by $(F, A(E)_F)$ of Remark 4.13(2) because $(F^n, (A(E)_F)^n)$ is a semi-T_1 space.

As a weaker form of the $T_{\frac{1}{2}}$-separation axiom we can propose the following:

Definition 4.15. We say that a topological space (X, T) has the pre-$T_{\frac{1}{2}}$-separation axiom if every singleton of X is either preopen or preclosed.

According to the definition, we obtain the following:

Remark 4.16. An $A(X)$ does not satisfy the pre-$T_{\frac{1}{2}}$-separation axiom.

5. Summary and Further Work

We have studied some properties of an $A(X)$ related to an Alexandroff space, the notions of $T_{\frac{1}{2}}$-separation axiom, a preopen set, a preclosed set, a semi-open set, a semi-closed set and so forth. Furthermore, an $A(X)$ is proved to be a locally finite topological spaces satisfying the semi-$T_{\frac{1}{2}}$-separation axiom instead of both the pre-$T_{\frac{1}{2}}$-separation axiom. Besides, $A(X)$ with some conditions can satisfy the semi-T_1-separation axiom. Finally, we have addressed the above four questions.

As a further work we have the following query.

[Question 4] Let (X, T) be an Alexandroff space satisfying the semi-$T_{\frac{1}{2}}$-separation axiom. What semi-T_i-separation axiom, $\frac{1}{2} \lneq i \lneq 1$ such as $i \in \{\frac{3}{4}, \cdots\}$, characterizes the given space (X, T) ?

[Question 5] As a further work related to Corollary 4.8, we need to find a condition of which $A(X)$ with $dim(A(X)) \geq 1$ is a semi-T_1 space.

The topological space $A(X)$ can be used in the field of computer geometry and digital imaginary. Furthermore, based on the properties of $A(X)$ we can study a function space of AC complexes in terms of fuzzy topology. Using the results in the paper, we can study Jordan curve theorem under a locally finite topology instead of Hausdorff topology, boundary tracking, finding an interior and an exterior which can be used in computational topology.

References

[1] P. Alexandroff, Diskrete Räume, Mat. Sb. 2 (1937) 501–518.
[2] B. Grunbaum, G.C.Shepard, Tilings and Patterns, New York, W. H. Freeman & Co., 1987.
[3] J. Cao, M. Ganster, I. Reilly, On generalized closed sets, Topol. Appl. 123 (2002) 37–46.
[4] V.A. Chatyrko, S.-E. Han, Y. Hattori, Some remarks concerning semi-$T_{\frac{1}{2}}$ spaces, Filomat 28 (2014) 21–25.
[5] N. Hamlett, A correction to the paper "Semi-open sets and semi-continuity in topological spaces" by Norman Levine, Proc. Amer. Math. Soc. 49 (1975) 458–460.

[6] S.-E. Han, Continuity of maps between axiomatic locally finite spaces and its applications, Internat. J. Comput. Math. 88 (2011) 2889–2900.
[7] S.-E. Han, Study on topological spaces with the semi-$T_{\frac{1}{2}}$, Honam Math. J. 35 (2013) 707–716.
[8] S.-E. Han, Properties of space set topological spaces, Filomat 30 (2016) 2474–2487.
[9] S.-E. Han, Jordan curve theorem for a locally finite space, (2018), submitted.
[10] S.-E. Han, S. Lee, An equivalence between the dimension and the height for a locally finite topological space, Far East J, Math. Sci. 102 (2017) 2861–2876.
[11] D.S. Janković, I.L Reilly, On semiseparation properties, Indian J. Pure Appl. Math. 16 (1985) 957–964.
[12] E. Khalimsky, Motion, deformation, and homotopy in finite spaces, Proceedings IEEE International Conferences on Systems, Man, and Cybernetics (1987) 227–234.
[13] E. Khalimsky, R. Kopperman, P.R. Meyer, Computer graphics and connected topologies on finite ordered sets, Topol. Appl. 36 (1990) 1–17.
[14] V. Kovalevsky, Finite topology as applied to image analysis, Computer Vision, Graphics, Image Processing 46 (1989) 141–161.
[15] V. Kovalevsky, Axiomatic Digital Topology, J. Math. Imaging Vis. 26 (2006) 41–58.
[16] V. Kovalevsky, Geometry of Locally Finite Spaces, Monograph, Berlin, 2008.
[17] E.H. Kronheimer, The topology of digital images, Topol. Appl. 46 (1992) 279–303.
[18] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly 70 (1963) 36–41.
[19] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo 19 (1970) 89–96.
[20] J. Listing, Der Census raumlicher Complexe, Abhandlungen der Koniglichen Gesellschaft der Wissenschaften zu Gottingen 10 (1862) 97–182.
[21] A.S. Mashhour, M.E. Abd El-Monsef, S.N. El-Deeb, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt 53 (1982) 47–57.
[22] J.P. May, Finite topological spaces, Lecture note, University of Chigago, 2010, 1–15.
[23] J.R. Munkres, Topology, Prentice-Hall Inc., Englewood Cliffs, N.J., 2000 (1st version 1975).
[24] K. Reidemeister, Topologie der Polyeder und kombinatorische Topologie der Komplexe, Akademische Verlagsgesellschaft Geest & Portig, Leipzig, 1938 (second edition 1953).
[25] H. Seifert, W. Threlfall, A Textbook of Topology, Academic Press, 1980.
[26] M. Senechal, Quasicrystals and Geometry, Cambridge Univ. Press, 1995.
[27] E. Steinitz, Sitzungsbericht Berliner Mathematischen Gesellschaft, Beitrage zur Analysis 7 (1908) 29–49.
[28] E.H. Spanier, Algebraic Topology, McGraw-Hill Inc., New York, 1966.
[29] A.W. Tucker, An abstract approach to manifolds, Ann. Math. 34 (1933) 191–243.
[30] J. Webster, Cell complexes and digital convexity, LNCS 2243, Springer Verlag, Berlin, 2001, pp 272–282.
[31] P. Wiederhold, S. Morales, Thinning on cell complexes from polygonal tilings, Discrete Appl. Math. 157 (2009) 3424–3434.