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Approximation by Generalized Integral Favard-Szász Type Operators Involving Sheffer Polynomials

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Abstract: This article deals with the approximation properties of a generalization of an integral type operator in the sense of Favard-Szász type operators including Sheffer polynomials with graphics plotted using Maple.We investigate the order of convergence, in terms of the first and the second order modulus of continuity, Peetre's K-functional and give theorems on convergence in weighted spaces of functions by means of weighted Korovkin type theorem. At the end of the work, we give some numerical examples.

1. Introduction

Approximation theory plays an essential role in mathematics literature, providing the convergence for whole space of functions by using a finite number of functions. This theory is also closely related to the other branches of mathematics. The existence of such a relationship can be explained by the fact that there are many important problems of the approximation theory and that these problems have been solved in the development process of other mathematical topics. A considerable amount of research about well-known Korovkin type approximation theory has been done by numerous mathematicians since 1953. S. M. Mazhar and V. Totik [1] modified the Szász operator [2] and have given another class of positive linear operators

$$S_n^*(f;x) = ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_0^{\infty} e^{-nt} \frac{(nt)^k}{k!} f(t)\,dt \tag{1}$$

for functions $f \in L_1[0,\infty)$. Similarly, we will revise an operator introduced by A. Jakimovski and D. Leviatan [3]. Now we need to remind these operators which are obtained with the help of Appell polynomials. It is known that Appell polynomials can be defined as follows

$$g(u)\,e^{ux} = \sum_{k=0}^{\infty} p_k(x)\,u^k \tag{2}$$

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where $g(z)=\sum_{n=0}^{\infty}a_nz^n$ is an analytic function in the disk $|z|<R, (R>1)$ and $g(z)\neq 0$. By using the generatig function (2)

$$P_n(f;x):=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right) \tag{3}$$

is defined by A. Jakimovski and D. Leviatan. Then A. Ciupa [4] modified the operator P_n as below

$$P_n^*(f;x):=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}f(t)\,dt \tag{4}$$

where $\lambda\geq 0$ and Γ is gamma function. For the special case $g(z)=1$ and $\lambda=0$, the operators defined by (4) turn into operators S_n^*. Let p_k be Sheffer polynomials defined by

$$A(t)e^{xH(t)}=\sum_{k=0}^{\infty}p_k(x)t^k,\ \ |t|<R \tag{5}$$

where

$$A(z)=\sum_{n=0}^{\infty}a_nz^n,\ a_0\neq 0$$

$$H(z)=\sum_{n=1}^{\infty}h_nz^n,\ h_1\neq 0 \tag{6}$$

and the following properties are used:

(i) $p_k(x)\geq 0$, for $x\in[0,\infty)$,

$$(ii)\ A(1)\neq 0 \text{ and } H'(1)=1 \tag{7}$$

By taking into account the condition (7), Ismail [5] investigated some approximation properties of the positive linear operators:

$$T_n(f;x):=\frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right),\ \text{for } n\in\mathbb{N}. \tag{8}$$

whenever function f is an exponential type. S.Sucu and İ.Büyükyazıcı [6] modified the operators which are given in (8) and gave some approximation properties of the operators.

Now we will revise the operators T_n as follows

$$\begin{aligned} L_n^*(f;x) &= \frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\left(\frac{n}{b_n}\right)^{\lambda+k+1} \\ &\quad\times\frac{1}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-\frac{n}{b_n}t}t^{\lambda+k}f(t)\,dt \end{aligned} \tag{9}$$

where the parameter $\lambda \geq 0$, Γ is gamma function with b_n a positive increasing sequence such that

$$\lim_{n\to\infty} b_n = \infty, \ \lim_{n\to\infty} \frac{b_n}{n} = 0 \tag{10}$$

and p_k are Sheffer polynomials defined by (5). For an other generalization of operators (8) one can refer to [7].

In Section 2, we present the some approximation properties of the operators (9) with the help of classical Korovkin type theorem. Moreover, the order of convergence is obtained by means of the first and the second order of modulus of continuity and Peetre's K-functional. In addition to this, we examine convergence of these operators in a weighted space of functions on a positive semi-axis.

2. Approximation properties of $L_n^*(f;x)$

In this section, we will give some important results for the operator L_n^*. We denote by $E[0,\infty)$ the set of all functions f on $[0,\infty)$ with the property $\left|f(x)\right| \leq \beta e^{\alpha x}$ for all $x \geq 0$ and some positive finite α, β and denote by $C_E[0,\infty)$ the set of all continuous functions $f \in E[0,\infty)$. Also, for a fixed $r \in \mathbb{N}$, we denote by $C_E^r = \left\{f \in C_E[0,\infty) : f', f'', f''', ..., f^{(r)} \in C_E[0,\infty)\right\}$. For L_n^*, we can easily get the following auxiliary result. For the proofs of the next theorems the following simple results are needed:

Lemma 2.1. *Let $e_i(x) = x^i$, $i \in \{0,1,2\}$. For the operator L_n^* defined by (9) and for all $x \in [0,\infty)$ and $\lambda \geq 0$, the following statements are hold.*

$$L_n^*(e_0;x) = 1, \tag{11}$$

$$L_n^*(e_1;x) = x + \frac{b_n}{n}\left(\frac{A'(1)}{A(1)} + \lambda + 1\right) \tag{12}$$

$$\begin{aligned} L_n^*(e_2;x) &= x^2 + \frac{b_n}{n}\left((2\lambda+4) + 2\frac{A'(1)}{A(1)} + H''(1)\right)x \\ &+ \frac{b_n^2}{n^2}\left((\lambda+1)(\lambda+2) + (2\lambda+3)\frac{A'(1)}{A(1)} + \frac{A'(1)+A''(1)}{A(1)}\right) \end{aligned} \tag{13}$$

It follows from Lemma 2.1 that,

$$L_n^*((e_1-x);x) = \frac{b_n}{n}\left(\frac{A'(1)}{A(1)} + \lambda + 1\right) \tag{14}$$

$$\begin{aligned} L_n^*\left((e_1-x)^2;x\right) &= \frac{b_n^2}{n^2}\left((\lambda+1)(\lambda+2) + (2\lambda+3)\frac{A'(1)}{A(1)} + \frac{A'(1)+A''(1)}{A(1)}\right) \\ &+ \frac{b_n}{n}\left(2 + H''(1)\right)x \end{aligned} \tag{15}$$

Theorem 2.2. *For $f \in C_E[0,\infty)$, $L_n^*(f) \to f$ uniformly on $[0,a]$ as $n \to \infty$.*

Proof. According to (11)-(13), we have

$$\lim_{n\to\infty} L_n^*(e_i;x) = e_i(x), \ i \in \{0,1,2\}$$

Applying the Korovkin theorem [9], we easily obtain the desired result. □

Example 2.3. *For $A(t) = 1$, $H(t) = t$ and $\lambda = 0, \frac{1}{2}, 1$; the convergence of the operators $L_n^*(f;x)$ to $f(x) = xe^{-\frac{1}{2}x}$(dash) are displayed in Figs. 1, 2, and 3 respectively, where $n = 10, 50, 100, 300, 500$ and $b_n = \sqrt{n}$.*

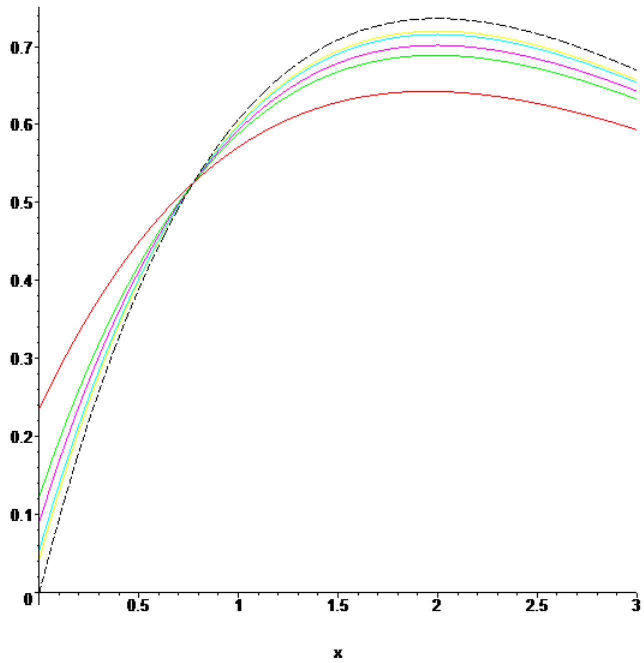

Figure 1: The convergence of $L_n^*(f;x)$ to $f(x) = xe^{-\frac{1}{2}x}$(dash). ($\lambda = 0$, $L_{10}^*(f;x)$(red), $L_{50}^*(f;x)$(green), $L_{100}^*(f;x)$(magenta), $L_{300}^*(f;x)$(cyan), $L_{500}^*(f;x)$(yellow)).

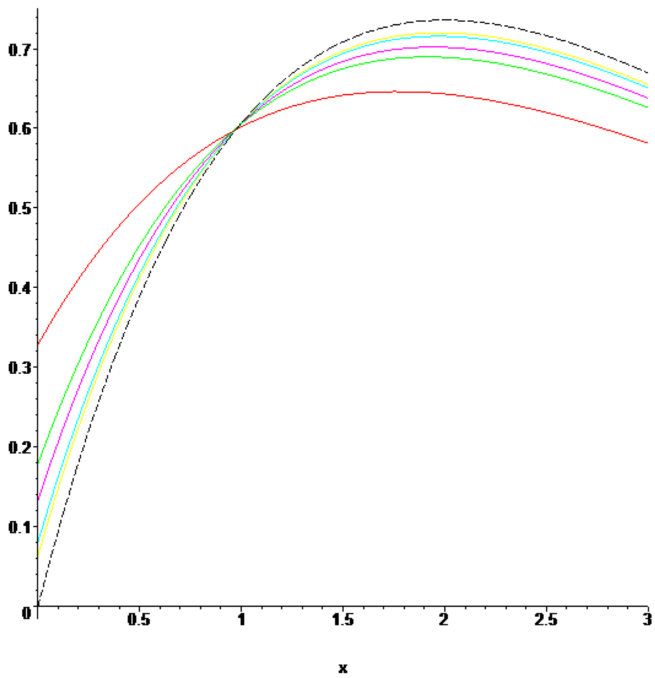

Figure 2: The convergence of $L_n^*(f;x)$ to $f(x) = xe^{-\frac{1}{2}x}$(dash). ($\lambda = 1/2$, $L_{10}^*(f;x)$(red), $L_{50}^*(f;x)$(green), $L_{100}^*(f;x)$(magenta), $L_{300}^*(f;x)$(cyan), $L_{500}^*(f;x)$(yellow))

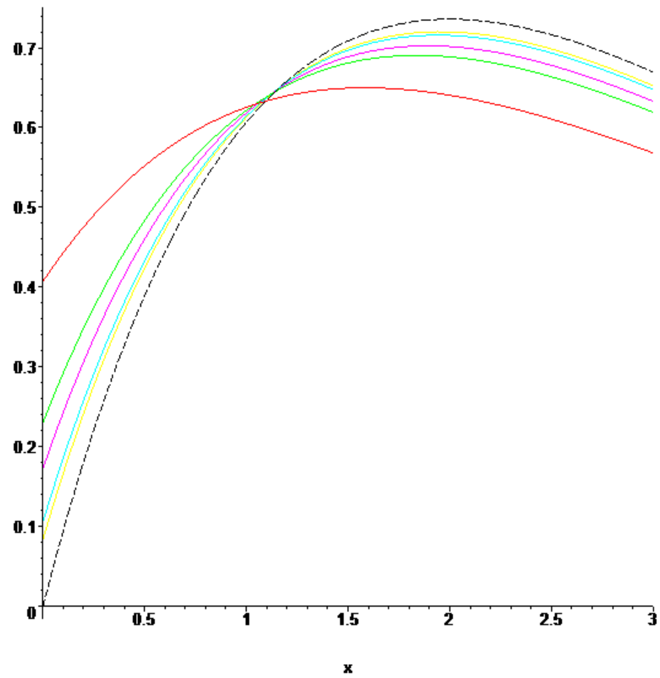

Figure 3: The convergence of $L_n^*(f;x)$ to $f(x) = xe^{-\frac{1}{2}x}$(dash). ($\lambda = 1$, $L_{10}^*(f;x)$(red), $L_{50}^*(f;x)$(green), $L_{100}^*(f;x)$(magenta), $L_{300}^*(f;x)$(cyan), $L_{500}^*(f;x)$(yellow)).

Example 2.4. *For $A(t) = 1$, $H(t) = t$ and $\lambda = 0, \frac{1}{2}, 1$; the convergence of the operators $L_n^*(f;x)$ to $f(x) = (1+x)e^{-x}$(circle) are displayed in Figs. 4, 5 and 6 respectively, where $n = 10, 50, 100, 300, 500$ and $b_n = \sqrt{n}$.*

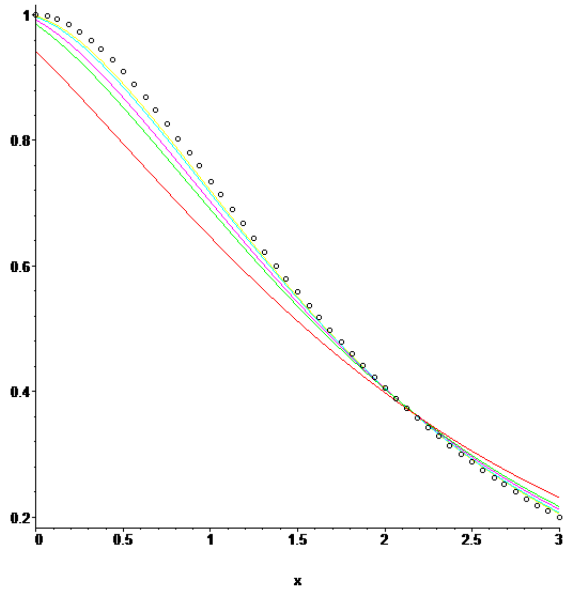

Figure 4: The convergence of $L_n^*(f;x)$ to $f(x) = (1+x)e^{-x}$(circle). ($\lambda = 0$, $L_{10}^*(f;x)$(red), $L_{50}^*(f;x)$(green), $L_{100}^*(f;x)$(magenta), $L_{300}^*(f;x)$(cyan), $L_{500}^*(f;x)$(yellow)).

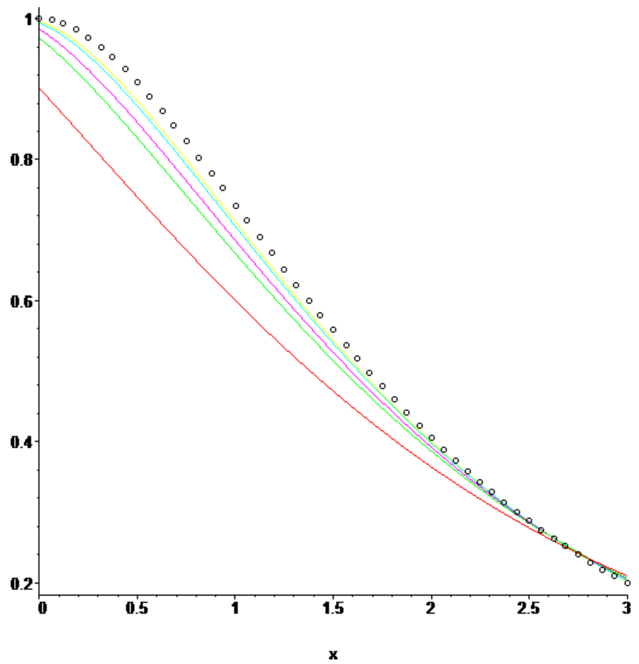

Figure 5: The convergence of $L_n^*(f;x)$ to $f(x) = (1+x)e^{-x}$(circle). ($\lambda = 1/2$, $L_{10}^*(f;x)$(red), $L_{50}^*(f;x)$(green), $L_{100}^*(f;x)$(magenta), $L_{300}^*(f;x)$(cyan), $L_{500}^*(f;x)$(yellow)).

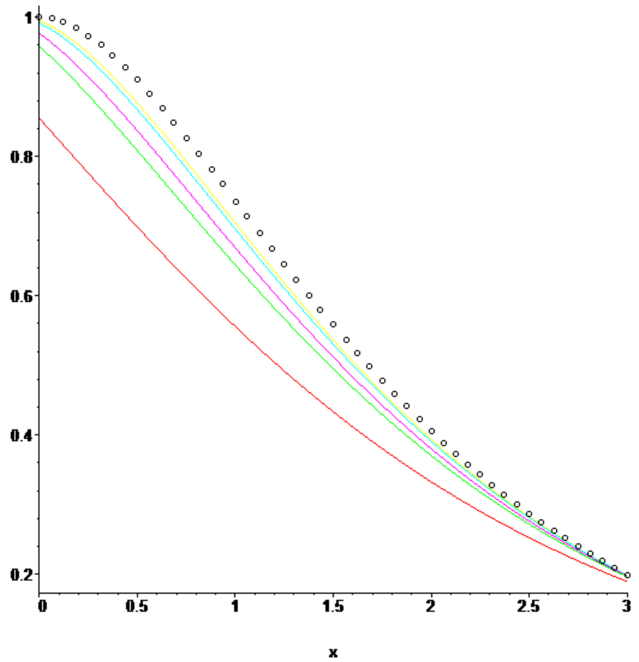

Figure 6: The convergence of $L_n^*(f;x)$ to $f(x) = (1+x)e^{-x}$(circle). ($\lambda = 1$, $L_{10}^*(f;x)$(red), $L_{50}^*(f;x)$(green), $L_{100}^*(f;x)$(magenta), $L_{300}^*(f;x)$(cyan), $L_{500}^*(f;x)$(yellow)).

• If $\delta > 0$, the modulus of continuity $\omega(f, \delta)$ of $f \in C[a, b]$ is defined by

$$\omega(f, \delta) = \sup_{x,y\in[a,b],\, |x-y|\leq\delta} |f(x) - f(y)|.$$

• The second order modulus of continuity of $f \in C_B[0, \infty)$ is defined by

$$\omega_2(f, \delta) = \sup_{0<t\leq\delta} \|f(\cdot + 2t) - 2f(\cdot + t) + f(\cdot)\|_{C_B}$$

where $C_B[0, \infty)$ is the class of real valued functions on $[0, \infty)$ which are bounded and uniformly continuous with the norm $\|f\|_{C_B} = \sup_{x\in[0,\infty)} |f(x)|$.

• The Peetre's K-functional [8] of the function $f \in C_B[0, \infty)$ is defined by

$$K(f, \delta) := \inf_{g\in C_B^2[0,\infty)} \left\{\|f - g\|_{C_B} + \delta \|g\|_{C_B^2}\right\}$$

where

$$C_B^2[0, \infty) := \left\{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\right\}$$

and the norm

$$\|g\|_{C_B^2} := \|g\|_{C_B} + \|g'\|_{C_B} + \|g''\|_{C_B}$$

It is known that the following relationship between the second order modulus of smoothness and Peetre's K-functional as below [4] :

$$K(f, \delta) \leq M\left\{\omega_2\left(f, \sqrt{\delta}\right) + \min(1, \delta)\|f\|_{C_B}\right\}$$

is valid for all $\delta > 0$.The constant M is independent of f and δ.

Lemma 2.5. *([10]) Let $g \in C^2[0, \infty)$ and $(P_n)_{n\geq0}$ be a sequence of positive linear operators with the property $P_n(1; x) = 1$.Then*

$$\left|P_n(g; x) - g(x)\right| \leq \|g'\| \sqrt{P_n\left((s - x)^2; x\right)} + \frac{1}{2}\|g''\| P_n\left((s - x)^2; x\right).$$

Lemma 2.6. *([11]) Let $f \in C[a, b]$ and $h \in \left(0, \frac{b-a}{2}\right)$. Let f_h be the second-order Steklov function attached to the function f. Then the following inequalities are satisfied:*

(i) $\|f_h - f\| \leq \frac{3}{4}\omega_2(f, h)$,
(ii) $\|f_h''\| \leq \frac{3}{2h^2}\omega_2(f, h)$.

Theorem 2.7. *If $f \in C_E[0, \infty)$, then for any $x \in [0, a]$, we have*

$$\left|L_n^*(f; x) - f(x)\right| \leq (1 + \xi_n)\ \omega\left(f, \sqrt{\frac{b_n}{n}}\right)$$

where

$$\xi_n = \sqrt{\left((2 + H''(1))\,a + \frac{b_n}{n}\left((\lambda + 1)(\lambda + 2) + (2\lambda + 3)\frac{A'(1)}{A(1)} + \frac{A'(1) + A''(1)}{A(1)}\right)\right)}.$$

Proof. We will use the relation (15) and the well-known properties of the modulus of continuity. We have

$$\left|L_n^*(f;x)-f(x)\right|\le \frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\left(1+\frac{1}{\delta}\left(\frac{n}{b_n}\right)^{\lambda+k+1}\frac{1}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-\frac{n}{b_n}t}t^{\lambda+k}\left|t-x\right|dt\right)\omega(f,\delta)$$

By using the Cauchy-Schwartz inequality for the integral term on the right hand side of the above inequality, we get

$$\left|L_n^*(f;x)-f(x)\right|\le\left\{1+\frac{1}{\delta}\sqrt{\left(\frac{b_n}{n}\right)^2(\lambda+1)(\lambda+2)+(2\lambda+3)\frac{A'(1)}{A(1)}+\frac{A'(1)+A''(1)}{A(1)}+\frac{b_n}{n}(2+H''(1))x}\right\}\omega(f,\delta)$$

$$\le\left\{1+\frac{1}{\delta}\sqrt{\frac{b_n}{n}}\sqrt{(2+H''(1))x+\frac{b_n}{n}\left((\lambda+1)(\lambda+2)+(2\lambda+3)\frac{A'(1)}{A(1)}+\frac{A'(1)+A''(1)}{A(1)}\right)}\right\}\omega(f,\delta)$$

In the previous inequality, for $0\le x\le a$, choosing $\delta=\sqrt{\frac{b_n}{n}}$ one obtains the desired result. □

Now, let us compute the rate of convergence of the operators L_n^* with the help of the second order modulus of smoothness.

Theorem 2.8. *For $f\in C[0,a]$, we have, the following inequality:*

$$\left|L_n^*(f;x)-f(x)\right|\le\frac{2}{a}\left\|f\right\|h^2+\frac{3}{4}\left(a+2+h^2\right)\omega_2(f,h)$$

is provided where

$$h:=h_n(x)=\sqrt[4]{L_n^*((e_1-x)^2;x)}$$

and the second order modulus of continuity is given by $\omega_2(f,h)$ with the norm $\left\|f\right\|=\max_{x\in[a,b]}\left|f(x)\right|$.

Proof. Let f_h be the second-order Steklov function associated to the function f. By means of the identity (11), one can write

$$\begin{aligned}\left|L_n^*(f;x)-f(x)\right| &\le \left|L_n^*(f-f_h;x)\right|+\left|L_n^*(f_h;x)-f_h(x)\right|+\left|f_h(x)-f(x)\right|\\ &\le 2\left\|f-f_h\right\|+\left|L_n^*(f_h;x)-f_h(x)\right| \end{aligned}\tag{16}$$

Taking into account the fact that $f_h\in C^2[0,a]$ and if we use Lemma 2.5

$$\left|L_n^*(f_h;x)-f_h(x)\right|\le\left\|f_h'\right\|\sqrt{L_n^*((e_1-x)^2;x)}+\frac{1}{2}\left\|f_h''\right\|L_n^*\left((e_1-x)^2;x\right)$$

Combining the Landau inequality and Lemma 2.6, we can write

$$\begin{aligned}\left\|f_h'\right\| &\le \frac{2}{a}\left\|f_h\right\|+\frac{a}{2}\left\|f_h''\right\|\\ &\le \frac{2}{a}\left\|f\right\|+\frac{3a}{4}\frac{1}{h^2}\omega_2(f,h)\end{aligned}\tag{17}$$

Owing to the fact $f_h\in C^2[0,a]$, using Lemma 2.6 and (17) one can have the estimate

$$\begin{aligned}\left|L_n^*(f_h;x)-f_h(x)\right| &\le \left(\frac{2}{a}\left\|f\right\|+\frac{3a}{4}\frac{1}{h^2}\omega_2(f,h)\right)\sqrt{L_n^*((e_1-x)^2;x)}\\ &\quad+\frac{3}{4}\frac{1}{h^2}L_n^*\left((e_1-x)^2;x\right)\omega_2(f,h)\end{aligned}$$

Choosing $h=\sqrt[4]{L_n^*((e_1-x)^2;x)}$ in above inequality

$$\left|L_n^*(f_h;x)-f_h(x)\right|\le\frac{2}{a}\left\|f\right\|h^2+\frac{3a}{4}\omega_2(f,h)+\frac{3}{4}h^2\omega_2(f,h)$$

Substituting the last inequality in (16), Lemma 2.6 hence gives the proof of the theorem. □

If the function f is a smooth function then the following theorem gives the estimation of approximation to function f.

Theorem 2.9. *Let $f \in C_B^2[0,\infty)$. Then*

$$\left|L_n^*(f;x) - f(x)\right| \le \varphi_n(x)\left\|f\right\|_{C_B^2} \tag{18}$$

where

$$\varphi(x) := \varphi_n(x) = \frac{1}{2}L_n^*\left((e_1 - x)^2; x\right)$$

Proof. Applying Taylor expansion of f

$$f(\varsigma) = f(x) + f'(x)(\varsigma - x) + \frac{f''(\eta)}{2}(\varsigma - x)^2$$

where $\eta \in (x,\varsigma)$. By virtue of linearity property of operators L_n^*, one can write

$$L_n^*(f;x) - f(x) = f'(x)\; L_n^*(\varsigma - x; x) + \frac{f''(\eta)}{2}L_n^*\left((\varsigma - x)^2; x\right).$$

From this truth and using Lemma 2.1, we have

$$\begin{aligned}\left|L_n^*(f;x) - f(x)\right| &\le \frac{b_n}{n}\left(\lambda + 1 + \frac{A'(1)}{A(1)}\right)\left\|f'\right\|_{C_B} + \frac{1}{2}\left\{\begin{array}{c}\frac{b_n}{n}\left(2 + H''(1)\right)x \\ +\frac{b_n^2}{n^2}\left[\begin{array}{c} +(\lambda+1)(\lambda+2) + (2\lambda+3)\frac{A'(1)}{A(1)} \\ +\frac{A'(1)+A''(1)}{A(1)}\end{array}\right]\end{array}\right\}\left\|f''\right\|_{C_B} \\ &\le \frac{1}{2}\left\{\begin{array}{c}\frac{b_n}{n}\left(2 + H''(1)\right)x \\ +\frac{b_n^2}{n^2}\left[\begin{array}{c} +(\lambda+1)(\lambda+2) + (2\lambda+3)\frac{A'(1)}{A(1)} \\ +\frac{A'(1)+A''(1)}{A(1)}\end{array}\right]\end{array}\right\}\left\|f\right\|_{C_B^2}\end{aligned}$$

By making a simple calculation in above inequality, we derive (18). □

The following theorem is about the quantitative estimate via Peetre's K-functional.

Theorem 2.10. *Let $f \in C_B[0,\infty)$. Then we have*

$$\left|L_n^*(f;x) - f(x)\right| \le 2M\left\{\omega_2\left(f, \sqrt{\delta}\right) + \min(1,\delta)\left\|f\right\|_{C_B}\right\}$$

where $\delta := \delta_n(x) = \frac{1}{2}\varphi_n(x)$ and $M > 0$ is a constant independent of the function f and δ. Note that $\varphi_n(x)$ is defined as in Theorem 2.9.

Proof. Let $g \in C_B^2[0,\infty)$. From the previous theorem and property of the K-functional we can write

$$\left|L_n^*(f;x) - f(x)\right| \le \left|L_n^*(f-g;x)\right| + \left|L_n^*(g;x) - g(x)\right| + \left|g(x) - f(x)\right|$$

$$\le 2\left\|f - g\right\|_{C_B} + \frac{1}{2}\left\{L_n^*\left((t-x)^2; x\right)\right\}\left\|g\right\|_{C_B^2}$$

$$= 2\left\{\left\|f - g\right\|_{C_B} + \frac{1}{4}\left[L_n^*\left((t-x)^2; x\right)\right]\left\|g\right\|_{C_B^2}\right\}$$

$$= 2\left\{\left\|f-g\right\|_{C_B}+\delta\left\|g\right\|_{C_B^2}\right\} \tag{19}$$

Since the left hand side of inequality (19) does not depend on the function $g \in C_B^2[0,\infty)$, it provides that

$$\left|L_n^*(f;x)-f(x)\right| \le 2K(f,\delta) \tag{20}$$

By the connection between Peetre's K-functional and second modulus of smoothness

$$\left|L_n^*(f;x)-f(x)\right| \le 2M\left\{\omega_2\left(f,\sqrt{\delta}\right)+\min(1,\delta)\left\|f\right\|_{C_B}\right\}$$

is obtained. □

3. Approximation properties in weighted spaces

This section is devoted to the approximation properties of L_n^* in the weighted spaces of continuous with exponential growth on $\mathbb{R}_0^+=[0,\infty)$ and the study is motivated by weighted Korovkin type theorem introduced by Gadjiev in [12], [13]. Firstly, the concepts of weighted spaces are introduced. Let the weighted function $\rho(x)=1+x^2$ and M_f a positive constant. We denote the set of functions that satisfy inequality $\left|f(x)\right| \le M_f\rho(x)$ by $B_\rho(\mathbb{R}_0^+)$ to obtain:

$$B_\rho(\mathbb{R}_0^+)=\left\{f\in E(\mathbb{R}_0^+):\left|f(x)\right|\le M_f\rho(x)\right\}.$$

Then the followings are defined:

$$\begin{aligned} C_\rho(\mathbb{R}_0^+) &= \left\{f\in B_\rho(\mathbb{R}_0^+): f \text{ is continuous}\right\}, \\ C_\rho^k(\mathbb{R}_0^+) &= \left\{f\in C_\rho(\mathbb{R}_0^+): \lim_{x\to\infty}\frac{f(x)}{\rho(x)}=K_f<\infty, \text{ a constant}\right\} \end{aligned}$$

It is clear that $B_\rho(\mathbb{R}_0^+)\supset C_\rho(\mathbb{R}_0^+)\supset C_\rho^k(\mathbb{R}_0^+)$. The associated norm of the space $B_\rho(\mathbb{R}_0^+)$ is as follows

$$\left\|f\right\|_\rho=\sup_{x\in\mathbb{R}_0^+}\frac{\left|f(x)\right|}{\rho(x)}$$

Below results make use of us.

Lemma 3.1. *([12], [13]) The sequence of positive linear operators $(L_n)_{n\ge1}$ which act from $C_\rho(\mathbb{R}_0^+)$ to $B_\rho(\mathbb{R}_0^+)$ if and only if there exists a positive constant k such that*

$$L_n(\rho;x)\le k\rho(x), \text{ i.e.}$$

$$\left\|L_n(\rho;x)\right\|_\rho\le k.$$

Theorem 3.2. *([12], [13]) Let $(L_n)_{n\ge1}$ be the sequence of positive linear operators which act from $C_\rho(\mathbb{R}_0^+)$ to $B_\rho(\mathbb{R}_0^+)$ satisfying the conditions*

$$\lim_{n\to\infty}\left\|L_n\left(t^i;x\right)-x^i\right\|_\rho=0,\ i\in\{0,1,2\},$$

then for any function $f\in C_\rho^k(\mathbb{R}_0^+)$

$$\lim_{n\to\infty}\left\|L_nf-f\right\|_\rho=0$$

Lemma 3.3. *Let $\rho(x) = 1 + x^2$ be a weight function. If $f \in C_\rho(\mathbb{R}_0^+)$, then*

$$\left\|L_n^*(\rho;x)\right\|_\rho \leq 1 + M$$

Proof. With the help of (11) and (13), one has

$$\begin{aligned} L_n^*(\rho;x) &= 1 + x^2 + \frac{b_n}{n}\left((2\lambda+4) + 2\frac{A'(1)}{A(1)} + H''(1)\right)x \\ &\quad + \frac{b_n^2}{n^2}\left((\lambda+1)(\lambda+2) + (2\lambda+3)\frac{A'(1)}{A(1)} + \frac{A'(1)+A''(1)}{A(1)}\right) \end{aligned}$$

here

$$\begin{aligned} \left\|L_n^*(\rho;x)\right\|_\rho &= \sup_{x\in\mathbb{R}_0^+} \frac{\left|L_n^*(\rho;x)\right|}{\rho(x)} \\ &= \sup_{x\geq 0}\left\{\frac{1}{1+x^2}\left[\begin{array}{c} 1 + x^2 + \frac{b_n}{n}\left((2\lambda+4) + 2\frac{A'(1)}{A(1)} + H''(1)\right)x \\ +\frac{b_n^2}{n^2}\left((\lambda+1)(\lambda+2) + (2\lambda+3)\frac{A'(1)}{A(1)} + \frac{A'(1)+A''(1)}{A(1)}\right) \end{array}\right]\right\} \end{aligned}$$

$$\begin{aligned} \left\|L_n^*(\rho;x)\right\|_\rho &\leq 1 + \frac{b_n}{n}\left((2\lambda+4) + 2\frac{A'(1)}{A(1)} + H''(1)\right) \\ &\quad + \frac{b_n^2}{n^2}\left((\lambda+1)(\lambda+2) + (2\lambda+3)\frac{A'(1)}{A(1)} + \frac{A'(1)+A''(1)}{A(1)}\right) \end{aligned}$$

$$\lim_{n\to\infty}\frac{b_n}{n} = 0,$$

Since there exists a positive M such that

$$\left\|L_n^*(\rho;x)\right\|_\rho \leq 1 + M$$

thus the proof is completed. □

Using Lemma 3.1, it can be seen that the operators L_n^* defined by (9) act from $C_\rho(\mathbb{R}_0^+)$ to $B_\rho(\mathbb{R}_0^+)$.

Theorem 3.4. *Let the operators L_n^* defined by (9) and $\rho(x) = 1 + x^2$.Then for every $f \in C_\rho^k(\mathbb{R}_0^+)$*

$$\lim_{n\to\infty}\left\|L_n^*(f;x) - f(x)\right\|_\rho = 0$$

Proof. We verify that the conditions of the weighted Korovkin type theorem given by Theorem 3.2 are valid. From (11)

$$\lim_{n\to\infty}\left\|L_n^*(e_0;x) - e_0(x)\right\|_\rho = 0 \tag{21}$$

By using (12)

$$\left\|L_n^*(e_1;x) - e_1(x)\right\|_\rho = \frac{b_n}{n}\left(\frac{A'(1)}{A(1)} + \lambda + 1\right) \tag{22}$$

this implies that

$$\lim_{n\to\infty}\left\|L_n^*(e_1;x) - e_1(x)\right\|_\rho = 0. \tag{23}$$

By means of (13) we have

$$\begin{aligned}\left\|L_n^*(e_2;x)-e_2(x)\right\|_\rho &= \sup_{x\in\mathbb{R}_0^+}\left|\begin{array}{c}\frac{b_n^2}{n^2}\left((\lambda+1)(\lambda+2)+(2\lambda+3)\frac{A'(1)}{A(1)}+\frac{A'(1)+A''(1)}{A(1)}\right)\frac{1}{1+x^2}\\ +\frac{b_n}{n}\left((2\lambda+4)+2\frac{A'(1)}{A(1)}+H''(1)\right)\frac{x}{1+x^2}\end{array}\right| \\ &\le \frac{b_n^2}{n^2}\left((\lambda+1)(\lambda+2)+(2\lambda+3)\frac{A'(1)}{A(1)}+\frac{A'(1)+A''(1)}{A(1)}\right) \\ &\quad +\frac{b_n}{n}\left((2\lambda+4)+2\frac{A'(1)}{A(1)}+H''(1)\right)\end{aligned} \tag{24}$$

and using the conditions (10), it follows that

$$\lim_{n\to\infty}\left\|L_n^*(e_2;x)-e_2(x)\right\|_\rho=0. \tag{25}$$

From (21), (23) and (25), for $i\in\{0,1,2\}$ we have

$$\lim_{n\to\infty}\left\|L_n^*(e_i;x)-e_i(x)\right\|_\rho=0.$$

Finally, if we apply Theorem 3.2, the desired result is obtained. □

Next, we find the approximation and rate of approximation of the functions $f\in C_\rho^k(\mathbb{R}_0^+)$ by using the operators L_n^* on $\mathbb{R}_0^+=[0,\infty)$. We use the following new type of weighted modulus of continuity introduced by Gadjiev and Aral in [14], since the usual first modulus of continuity does not tend to zero as $\delta\to 0$ on $\mathbb{R}_0^+$,

$$\Omega_\rho(f,\delta)=\Omega(f,\delta)_{\mathbb{R}_0^+}=\sup_{\substack{x,t\in\mathbb{R}_0^+\\ |\rho(t)-\rho(x)|\le\delta}}\frac{\left|f(t)-f(x)\right|}{\left[\left|\rho(t)-\rho(x)\right|+1\right]\rho(x)} \tag{26}$$

where ρ is such that:
(*i*) ρ is a continuously differentiable function on $\mathbb{R}_0^+$ and $\rho(0)=1$,
(*ii*) $\inf_{x\ge 0}\rho'(x)\ge 1$.
The weighted modulus of continuity $\Omega_\rho(f,\delta)$ has the following properties.

Lemma 3.5. *([14]) For any $f\in C_\rho^k(\mathbb{R}_0^+)$ then*

$$\lim_{\delta\to 0}\Omega_\rho(f,\delta)=0,$$

and for each $x,t\in\mathbb{R}_0^+$ the inequality

$$\left|f(t)-f(x)\right|\le 2\rho(x)\left(1+\delta^2\right)\left(1+\frac{(\rho(t)-\rho(x))^2}{\delta^2}\right)\Omega_\rho(f,\delta)$$

holds, where δ is any fixed positive number.

The estimates of the approximation of functions by positive linear operators by means of the new type of modulus of continuity are given in the following theorem [14]:

Theorem 3.6. *([14]) Let $\rho(x)\le\psi_k(x)$, $k=1,2,3$ and the sequences of the positive linear operators $(L_n)_{n\ge1}$ satisfying the conditions*

$$\|L_n(1;x)-1\|_{\psi_1}=\alpha_n, \tag{27}$$

$$\left\| L_n(\rho;x) - \rho \right\|_{\psi_2} = \beta_n, \tag{28}$$

$$\left\| L_n\left(\rho^2;x\right) - \rho^2 \right\|_{\psi_3} = \gamma_n, \tag{29}$$

where α_n, β_n and γ_n tend to zero as $n \to \infty$ and $\psi(x) = \max\{\psi_1(x),\ \psi_2(x),\ \psi_3(x)\}$. Then for all $f \in C_\rho^k(\mathbb{R}_0^+)$, the inequality

$$\left\| L_n(f;x) - f(x) \right\|_{\psi\rho^2} \le 16\Omega_\rho\left(f,\ \sqrt{\alpha_n + 2\beta_n + \gamma_n}\right) + \alpha_n \left\| f \right\|_\rho$$

holds for sufficiently large n.

Now, we define the positive linear operators P_n^ρ by

$$\begin{aligned} P_n^\rho(f;x) \quad : \quad &= \frac{\rho^2(x)\, e^{-\frac{n}{b_n}xH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)^{\lambda+k+1} \frac{1}{\Gamma(\lambda+k+1)} \\ &\times \int_0^\infty e^{-\frac{n}{b_n}t} t^{\lambda+k} \frac{f(t)}{\rho^2(t)} dt \end{aligned} \tag{30}$$

Theorem 3.7. *Let P_n^ρ be the sequence of the positive linear operators defined by (30) and $\psi(x) = 1 + x^2$. If $f \in C_\rho^k(\mathbb{R}_0^+)$, then*

$$\left\| P_n^\rho(f;x) - f(x) \right\|_{\rho^4\psi} \le 16\Omega_\rho\left(f,\ \sqrt{\alpha_n + 2\beta_n}\right) + \alpha_n \left\| f \right\|_\rho \tag{31}$$

Proof. By simple calculations we get

$$P_n^\rho(1;x) - 1 = \rho^2(x) \left[\begin{array}{c} \frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)} \sum\limits_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)\left(\frac{n}{b_n}\right)^{\lambda+k+1} \frac{1}{\Gamma(\lambda+k+1)} \\ \times \int_0^\infty e^{-\frac{n}{b_n}t} t^{\lambda+k} \frac{1}{\rho^2(t)} dt - \frac{1}{\rho^2(x)} \end{array} \right] \tag{32}$$

$$P_n^\rho(\rho;x) - \rho(x) = \rho^2(x) \left[\begin{array}{c} \frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)} \sum\limits_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)\left(\frac{n}{b_n}\right)^{\lambda+k+1} \frac{1}{\Gamma(\lambda+k+1)} \\ \times \int_0^\infty e^{-\frac{n}{b_n}t} t^{\lambda+k} \frac{1}{\rho(t)} dt - \frac{1}{\rho(x)} \end{array} \right] \tag{33}$$

$$P_n^\rho\left(\rho^2;x\right) - \rho^2(x) = 0 \tag{34}$$

From (23) and (25) we have

$$\lim_{n\to\infty} \left\| \frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)\left(\frac{n}{b_n}\right)^{\lambda+k+1} \frac{1}{\Gamma(\lambda+k+1)} \int_0^\infty e^{-\frac{n}{b_n}t} t^{\lambda+k} \frac{1}{\rho^2(t)} dt - \frac{1}{\rho^2(x)} \right\|_\psi = 0,$$

$$\lim_{n\to\infty} \left\| \frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)\left(\frac{n}{b_n}\right)^{\lambda+k+1} \frac{1}{\Gamma(\lambda+k+1)} \int_0^\infty e^{-\frac{n}{b_n}t} t^{\lambda+k} \frac{1}{\rho(t)} dt - \frac{1}{\rho(x)} \right\|_\psi = 0,$$

Using (24) and (32) we obtain

$$\begin{aligned} \left\| P_n^\rho(1;x) - 1 \right\|_{\rho^2\psi} \quad &= \quad \left\| \frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)\left(\frac{n}{b_n}\right)^{\lambda+k+1} \frac{1}{\Gamma(\lambda+k+1)} \int_0^\infty e^{-\frac{n}{b_n}t} t^{\lambda+k} \frac{1}{\rho^2(t)} dt - \frac{1}{\rho^2(x)} \right\|_\psi \\ &\le \quad \frac{b_n}{n}\left((2\lambda+4) + \frac{2A'(1)}{A(1)} + H''(1)\right) \\ &\quad + \frac{b_n^2}{n^2}\left((\lambda+1)(\lambda+2) + (2\lambda+3)\frac{A'(1)}{A(1)} + \frac{A'(1) + A''(1)}{A(1)}\right) \\ &= \quad \alpha_n \end{aligned}$$

By means of (22) and (32), one gets

$$\begin{aligned}\left\|P_n^{\rho}(\rho;x)-\rho\right\|_{\rho^2\psi} &= \left\|\frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\left(\frac{n}{b_n}\right)^{\lambda+k+1}\frac{1}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-\frac{n}{b_n}t}t^{\lambda+k}\frac{1}{\rho(t)}dt-\frac{1}{\rho(x)}\right\|_{\psi}\\ &\le \frac{b_n}{n}\left(\frac{A'(1)}{A(1)}+\lambda+1\right)\\ &= \beta_n,\end{aligned}$$

Finally from (34), it is clear that $\gamma_n = 0$. Thus the (27)-(29) assumptions of Theorem 3.6 are satisfied for the operators (30). From Theorem 3.6, we have

$$\left\|P_n^{\rho}(f;x)-f(x)\right\|_{\rho^4\psi} \le 16\Omega_{\rho}\left(f,\sqrt{\alpha_n+2\beta_n+\gamma_n}\right)+\alpha_n\left\|f\right\|_{\rho}$$

for each $f \in C_{\rho}^{k}(\mathbb{R}_0^+)$. This fulfills the proof. □

4. Numerical Examples for Approximation

Example 4.1. *The sequence $\left\{x^k\right\}_{k=1}^{\infty}$ that is Sheffer sequence for $A(t) = 1$, $H(t) = t$ has the generating function as follows*

$$e^{xt} = \sum_{k=0}^{\infty}\frac{x^k}{k!}t^k$$

Let us pick $p_k(x) = \frac{x^k}{k!}$. Since for $x \in [0,\infty)$, $p_k(x) \ge 0$ and $A(1) \ne 0$, $H(t) = t$ are verified. Taking these polynomials in (9), we get operators as follows

$$L_n^*(f;x) = e^{-\frac{n}{b_n}x}\sum_{k=0}^{\infty}\frac{\left(\frac{n}{b_n}x\right)^k}{k!}\left(\frac{n}{b_n}\right)^{\lambda+k+1}\frac{1}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-\frac{n}{b_n}t}\,t^{\lambda+k}\,f(t)\,dt$$

The error bounds for the functions $f(x) = xe^{-\frac{1}{2}x}$, $f(x) = (1+x)e^{-x}$ under the choice $A(t) = 1$, $H(t) = t$ and $\lambda = 0, \frac{1}{2}$, 1 are calculated in the following Table 1 and Table 2, respectively.

Error estimate by L_n^* operators including $\left\{x^k\right\}_{k=1}^{\infty}$ sequence			
n	$\lambda = 0$	$\lambda = 1/2$	$\lambda = 1$
10	0.517778349	1.562474678	1.617483267
10^3	0.563323810	0.565147928	0.567481145
10^5	0.188671471	0.188733186	0.188812492
10^7	0.060800846	0.060802837	0.060805396
10^9	0.019343514	0.019343577	0.019343658
10^{11}	0.006128707	0.006128709	0.006128711
10^{13}	0.001939245	0.001939245	0.001939245
10^{15}	0.000613361	0.000613361	0.000613361
10^{17}	0.000193973	0.000193973	0.000193973
10^{19}	0.000061341	0.000061341	0.000061341
10^{21}	0.000019397	0.000019397	0.000019397

Table 1: The error bound of function $f(x) = xe^{-\frac{1}{2}x}$ by using modulus of continuity.

Error estimate by L_n^* operators including $\left\{x^k\right\}_{k=1}^{\infty}$ sequence			
n	$\lambda = 0$	$\lambda = 1/2$	$\lambda = 1$
10	0.730041031	0.751539660	0.777998417
10^3	0.226207339	0.226939829	0.227876751
10^5	0.071378205	0.071401553	0.071431556
10^7	0.022566848	0.022567587	0.022568537
10^9	0.007136108	0.007136131	0.007136162
10^{11}	0.002256631	0.002256631	0.002256632
10^{13}	0.000713610	0.000713610	0.000713610
10^{15}	0.000225662	0.000225662	0.000225662
10^{17}	0.000071360	0.000071360	0.000017360
10^{19}	0.000022566	0.000022566	0.000022566
10^{21}	$0.713 * 10^{-5}$	$0.713 * 10^{-5}$	$0.713 * 10^{-5}$

Table 2: The error bound of function $f(x) = (1 + x)\, e^{-x}$ by using modulus of continuity.

Conclusion 4.2. *A generalization of integral Favard-Szász type operators by the help of Sheffer polynomials is introduced and some approximation results are obtained. Important convergence theorems in weighted spaces of functions are given. Approximations to some convenient functions are examined by visualizing with the help of graphics. Some numerical examples are also established and the error bounds of given functions are calculated by means of modulus of smoothness.*

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