Filomat 33:8 (2019), 2295–2305
https://doi.org/10.2298/FIL1908295H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Involutes of Order k of a Null Cartan Curve in Minkowski Spaces

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Abstract. In this paper, we define an involute and an evolving involute of order k of a null Cartan curve in Minkowski space $\mathbb{E}_1^n$ for $n \geq 3$ and $1 \leq k \leq n-1$. In relation to that, we prove that if a null Cartan helix has a null Cartan involute of order 1 or 2, then it is Bertrand null Cartan curve and its involute is its Bertrand mate curve. In particular, we show that Bertrand mate curve of Bertrand null Cartan curve can also be a non-null curve and find the relationship between the Cartan frame of a null Cartan curve and the Frenet or the Cartan frame of its non-null or null Cartan involute of order $1 \leq k \leq 2$. We show that among all null Cartan curves in $\mathbb{E}_1^3$, only the null Cartan cubic has two families of involutes of order 1, one of which lies on B-scroll. We also give some relations between involutes of orders 1 and 2 of a null Cartan curve in Minkowski 3-space. As an application, we show that involutes of order 1 of a null Cartan curve in $\mathbb{E}_1^3$, evolving according to null Betchov-Da Rios vortex filament equation, generate timelike Hasimoto surfaces.

1. Introduction

The notions of *evolute* and *involute* are introduced by C. Huygens in 1673 in order to describe the geometric properties of isochronous pendulum clock ([19]). He discovered that isochronous curve is an arc of a cycloid and that involute of a cycloid is a similar cycloid. Later, he applied this discovery to find the radius of curvature of a given plane curve. In particular, he obtained an interesting relationship between two plane curves γ_i and γ_e - the locus of centers of curvature for points P on a given curve γ_i lies on the second curve γ_e. The curves γ_e and γ_i are called *evolute* of γ_i and *involute (evolvent)* of γ_e, respectively. Hence the evolute γ_e is the envelope of normal lines of γ_i and the involute γ_i can be seen as the trajectory described by the end of stretched string unwinding from a point of the curve.

In classical differential geometry, the tangent vector field T of a regular plane curve α can be regarded as its 1-dimensional osculating space. Accordingly, a regular curve $\alpha^\star$ in $\mathbb{E}^2$ is called an *involute* of α, if $\alpha^\star$ is orthogonal to the 1-dimensional osculating space of α. Clearly, the tangent vector field $T^\star$ of $\alpha^\star$ is parallel with the principal normal vector field N of α. Involutes of plane curves represent the special case of *tanvolutes*, introduced in [1] as the curves which intersect every tangent line of a given plane curve at an arbitrary fixed angle. Involutes of order $k > 1$ and generalized evolutes in Euclidean n-space $\mathbb{E}^n$ and

2010 *Mathematics Subject Classification*. Primary 53A04; Secondary 53C40
Keywords. involute of order k, null Cartan curve, null vortex filament equation, Minkowski space
Received: 07 August 2018; Accepted: 05 November 2018
Communicated by Mića Stanković
The first and the second author were supported by National Natural Science Foundation of China (No.61473059). The third author was partially supported by the Serbian Ministry of Education, Science and Technological Development (grant number 174012)
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in Minkowski space-time $\mathbb{E}^4_1$, are defined and studied in [13, 21, 22]. In a simply isotropic space $I^{(1)}_n$ and Galilean space $\mathbb{G}_4$, involutes of order k are introduced in [7, 17]. For further properties of evolutes and involutes in Euclidean and Minkowski spaces, we refer to [2, 12, 14, 24, 25].

In Euclidean 3-space, a regular smooth curve α is called *Bertrand curve*, if there exist another regular smooth curve $\bar{\alpha}$ and a bijection $\varphi : \alpha \mapsto \bar{\alpha}$ such that at the corresponding points of the curves, the principal normal lines of α coincide with the principal normal lines of $\bar{\alpha}$ ([9]). A pair of curves $(\alpha, \bar{\alpha})$ is called *Bertrand pair* and $\bar{\alpha}$ is called *Bertrand mate (partner) curve* of α. In Minkowski space $\mathbb{E}^3_1$, null Bertrand curves are studied in [3], where Bertrand mate curve of a null Bertrand curve is defined as a null (Cartan) curve.

It is known that the velocity $v(s,t)$ of a vortex filament $\alpha(s,t)$ regarded as a curve in Euclidean space $\mathbb{E}^3$ and parameterized by the arc-length parameter s for all time t, is given by *Betchov-Da Rios vortex filament equation* (localized induction equation)([4, 6])

$$v = \alpha_t = \alpha_s \times \alpha_{ss}. \tag{1}$$

In particular, a space curve $\alpha(s,t)$ evolving according to the vortex filament equation (1), generates *Hasimoto surface* ([16, 23]). It is shown in [10] that a non-null curve $\alpha(s,t)$ with a non-null principal normal in Minkowski 3-space, evolving according to equation (1) generates non-degenerate Hasimoto surface.

On the other hand, the null Cartan curve $\beta(s,t)$ in Minkowski space $\mathbb{E}^3_1$ evolves according to *null Betchov-Da Rios vortex filament equation* (null localized induction equation) ([11])

$$\beta_t = \beta_{ss} \times \beta_{sss}. \tag{2}$$

The example of a null Cartan curve $\beta(s,t)$ in Minkowski space-time $\mathbb{E}^4_1$, which evolves according to null Betchov-Da Rios vortex filament equation $\beta_t = \beta_{ss} \times \beta_{sss} \times \beta_{ssss}$, is obtained in [15].

In this paper, we define an involute and an evolving involute of order k of a null Cartan curve in Minkowski space $\mathbb{E}^n_1$ for $n \geq 3$ and $1 \leq k \leq n-1$. In relation to that, we obtain a new characterizations of Bertrand null Cartan curves in $\mathbb{E}^3_1$ in terms of involutes of orders 1 and 2. Namely, we prove that if a null Cartan helix has a null Cartan involute of order 1 or 2, then it is Bertrand null Cartan curve and its involute is its Bertrand mate curve. We show that Bertrand mate curve of Bertrand null Cartan curve can also be a non-null curve. This implies that the definition of Bertrand null curves given in [3] is not completely correct. Hence we have redefined Bertrand null Cartan curves (Definition 2.1 in Section 2) with respect to possible causal characters of their Bertrand mate curves. In particular, we find the relationship between the Cartan frame of a null Cartan curve and the Frenet or the Cartan frame of its non-null or null Cartan involute of order 1 or 2, respectively. We also show that among all null Cartan curves in $\mathbb{E}^3_1$, only null Cartan cubic has two families of involutes of order 1, one of which lies on B-scroll. More precisely, we prove that every non-null curve lying on B-scroll whose base curve is null Cartan cubic α is an involute of α. We also give some relations between involutes of orders 1 and 2 of a null Cartan curve and related examples. Finally, as an application we show that involutes of order 1 of a null Cartan curve in $\mathbb{E}^3_1$, evolving according to null Betchov-Da Rios vortex filament equation, generate the timelike Hasimoto surfaces.

2. Preliminaries

Minkowski space $\mathbb{E}^3_1$ is the real vector space $\mathbb{E}^3$ equipped with the standard indefinite flat metric $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3,$$

for any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{E}^3_1$. Since $\langle \cdot, \cdot \rangle$ is an indefinite metric, an arbitrary vector $x \in \mathbb{E}^3_1 \setminus \{0\}$ can have one of three causal characters: it can be *spacelike*, *timelike* or *null* (*lightlike*), if $\langle x, x \rangle$ is positive, negative or zero, respectively. In particular, the vector $x = 0$ is a spacelike. The *norm* (length) of a vector $x \in \mathbb{E}^3_1$ is given by $||x|| = \sqrt{|\langle x, x \rangle|}$. An arbitrary curve $\alpha : I \to \mathbb{E}^3_1$ can locally be *spacelike*, *timelike* or *null (lightlike)*, if all of its velocity vectors $\alpha'(s)$ satisfy $\langle \alpha'(s), \alpha'(s) \rangle > 0$, $\langle \alpha'(s), \alpha'(s) \rangle < 0$ or $\langle \alpha'(s), \alpha'(s) \rangle = 0$ and $\alpha'(s) \neq 0$, respectively ([20]).

Let $\{T, N, B\}$ be the moving Frenet frame along a non-null curve α in $\mathbb{E}^3_1$, consisting of the tangent, principal normal and binormal vector field, respectively. If N is a non-null vector field, the Frenet equations of α have the form ([18])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2\kappa_1 & 0 \\ -\epsilon_1\kappa_1 & 0 & -\epsilon_1\epsilon_2\kappa_2 \\ 0 & -\epsilon_2\kappa_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{3}$$

where κ_1 and κ_2 are the *curvature* and the *torsion* of α respectively and $\epsilon_1 = \langle T, T\rangle = \pm 1$, $\epsilon_2 = \langle N, N\rangle = \pm 1$ and $\langle B, B\rangle = -\epsilon_1\epsilon_2$.

A null curve $\beta : I \to \mathbb{E}^3_1$ is called a *null Cartan curve*, if it is parameterized by the pseudo-arc function s given by ([5])

$$s(t) = \int_0^t \sqrt{\|\beta''(u)\|}\, du. \tag{4}$$

There exists a unique Cartan frame $\{T, N, B\}$ along a non-geodesic null Cartan curve β in $\mathbb{E}^3_1$ satisfying the Cartan equations ([8])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ -\kappa_2 & 0 & \kappa_1 \\ 0 & -\kappa_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{5}$$

where $\kappa_1(s) = 1$ and $\kappa_2(s)$ are the *curvature* and the *torsion* of β, respectively. In particular, if $\kappa_2(s) = 0$ the null Cartan curve $\beta(s)$ is called a *null Cartan cubic*.

The Cartan's frame vectors of β satisfy the relations

$$\begin{aligned} &\langle T, T\rangle = \langle B, B\rangle = 0,\ \langle N, N\rangle = 1, \\ &\langle T, N\rangle = \langle N, B\rangle = 0,\ \langle T, B\rangle = -1, \end{aligned} \tag{6}$$

$$T \times N = -T,\ N \times B = -B,\ B \times T = N. \tag{7}$$

The Cartan frame $\{T, N, B\}$ is *positively oriented*, if $det(T, N, B) = 1$.

Definition 2.1. *A null Cartan curve $\alpha : I \mapsto \mathbb{E}^3_1$ is called Bertrand null Cartan curve, if there exists a non-null or null Cartan curve $\alpha^\star : I^\star \mapsto \mathbb{E}^3_1$ and a bijection $\varphi : \alpha \mapsto \alpha^\star$ such that at the corresponding points of the curves the principal normal vector fields N and $N^\star$ of α and $\alpha^\star$ respectively are linearly dependent.*

A timelike ruled surface in $\mathbb{E}^3_1$ with parametrization $x(s, t) = \beta(s) + tB(s)$ is called a *B-scroll*, if β is a null Cartan base curve and the rulings $B(s)$ are the binormal vectors along β.

Throughout the next sections, let $\mathbb{R}_0$ denote $\mathbb{R}\backslash\{0\}$.

3. Involutes and evolving involutes of order k of a null Cartan curve in $\mathbb{E}^n_1$

In this section, we firstly define the k-dimensional osculating space of a null Cartan curve in $\mathbb{E}^n_1$ for $1 \leq k \leq n-1$ and $n > 2$. We also define an involute of order k and an evolving involute of order k of a null Cartan curve α with the Cartan frame $\{T, N, B_1, \ldots, B_{n-2}\}$ in Minkowski space $\mathbb{E}^n_1$ for $n \geq 3$ and $1 \leq k \leq n-1$. If α is a null Cartan curve in $\mathbb{E}^3_1$, then its osculating plane $B^\perp$ is spanned by $\{N, B\}$ and thus can be regarded as the 2-dimensional osculating space of α. In general case, we give the following definition of the k-dimensional osculating space.

Definition 3.1. *The 1-dimensional and the 2-dimensional osculating space of a null Cartan curve α with the Cartan frame $\{T, N, B_1, \ldots, B_{n-2}\}$ in $\mathbb{E}^n_1$ for $n \geq 3$ are respectively given by*

$$V_1 = span\{N\}, \quad V_2 = span\{N, B_1\}.$$

The k-dimensional osculating space of α in $\mathbb{E}^n_1$ for $n \geq 4$ and $3 \leq k \leq n-1$ is given by

$$V_k = span\{T, N, B_1, \ldots, B_{k-2}\}.$$

In the next definition, we introduce an *involute of order k* of a null Cartan curve.

Definition 3.2. *A non-null or a null Cartan curve $\alpha^\star$ is called an involute of order k of a null Cartan curve α in $\mathbb{E}^n_1$ for $n \geq 3$ and $1 \leq k \leq n-1$, if $\alpha^\star$ is orthogonal to the k-dimensional osculating space of α.*

According to Definitions 3.1 and 3.2, the curve $\alpha^\star$ is an involute of order 1 or 2 of a null Cartan curve α in $\mathbb{E}^3_1$, if respectively holds

$$\langle \alpha^{\star\prime}, N\rangle = 0, \tag{8}$$

$$\langle \alpha^{\star\prime}, N\rangle = 0, \quad \langle \alpha^{\star\prime}, B\rangle = 0. \tag{9}$$

The condition (8) implies that an involute of order 1 of α can be a spacelike, a timelike or a null Cartan curve. Similarly, the condition (9) implies that an involute of order 2 of α can only be a null Cartan curve.

For $n \geq 4$ and $3 \leq k \leq n-1$, according to Definitions 3.1 and 3.2 it follows that $\alpha^\star$ is an involute of order k of a null Cartan curve α in $\mathbb{E}^n_1$, if the following conditions hold

$$\langle \alpha^{\star\prime}, T\rangle = 0, \quad \langle \alpha^{\star\prime}, N\rangle = 0, \quad \langle \alpha^{\star\prime}, B_j\rangle = 0,$$

where $1 \leq j \leq k-2$. We also define an *evolving involute* $\beta(s^\star, t^\star)$ of a null Cartan curve as follows.

Definition 3.3. *A non-null or a null Cartan curve $\beta(s^\star, t^\star)$ parameterized by the arc-length or pseudo-arc $s^\star$ respectively for all time $t^\star$, is called an evolving involute of order k of a null Cartan curve α in $\mathbb{E}^n_1$ for $n \geq 3$ and $1 \leq k \leq n-1$, if $\beta(s^\star, t^\star)$ is orthogonal to the k-dimensional osculating space of α for all time $t^\star$.*

Throughout the next sections, by an involute (evolving involute) we will mean an involute (evolving involute) of order 1.

4. Involutes of order 1 or 2 of a null Cartan curve in $\mathbb{E}^3_1$

In this section, we characterize involutes of order 1 or 2 of a null Cartan curve in $\mathbb{E}^3_1$. In relation to that, we obtain a new characterization of Bertrand null Cartan curves in terms of involutes. Namely, we prove that if a null Cartan helix has a null Cartan involute of order 1 or 2, then it is Bertrand null Cartan curve and its involute is its Bertrand mate curve. In particular, we find the relationship between the Cartan frame of a null Cartan curve and the Frenet or the Cartan frame of its non-null or null Cartan involute, respectively. We also show that among all null Cartan curves in $\mathbb{E}^3_1$, only null Cartan cubic has two families of involutes of order 1, one of which lies on B-scroll. More precisely, we prove that every non-null curve lying on B-scroll whose base curve is null Cartan cubic α is an involute of α. We also give some relations between involutes of orders 1 and 2 of a null Cartan curve and related examples.

In the first theorem, we characterize null Cartan involute $\alpha^\star$ of a null Cartan curve α as Bertrand mate curve of α.

Theorem 4.1. *Let α and $\alpha^\star$ be null Cartan curves in $\mathbb{E}^3_1$ respectively parameterized by the pseudo-arcs s and $s^\star$ with the torsions $\kappa_2(s) \neq 0$ and $\kappa^\star_2(s^\star) \neq 0$. Then $\alpha^\star$ is an involute of α if and only if $(\alpha, \alpha^\star)$ is Bertrand pair of curves having equal constant torsions.*

Proof. Assume that $\alpha^\star$ is an involute of α. Then $\alpha^\star$ can be parameterized by

$$\alpha^\star(s) = \alpha(s) + \lambda(s)N(s), \tag{10}$$

where $\lambda(s)$ is some differentiable function in s. Differentiating the previous equation with respect to s and using (5), we obtain

$$\alpha^{\star\prime}(s) = (1 - \lambda\kappa_2)T + \lambda' N + \lambda B. \tag{11}$$

Since $\alpha^\star$ is an involute of α, according to Definition 3.2 it holds

$$\langle \alpha^{\star\prime}, N\rangle = 0. \tag{12}$$

Relations (6), (11) and (12) give $\lambda' = 0$. Hence $\lambda = \lambda_0 \in \mathbb{R}_0$ and thus $\alpha^\star = \alpha + \lambda_0 N$. Substituting $\lambda = \lambda_0$ in (11) and using the condition $\langle \alpha^{\star\prime}, \alpha^{\star\prime}\rangle = 0$, we find

$$\kappa_2 = \frac{1}{\lambda_0}. \tag{13}$$

Relations (11) and (13) yield $\alpha^{\star\prime} = \lambda_0 B$. Differentiating the last relation with respect to s, we get $\alpha^{\star\prime\prime} = -N$. Consequently, $\langle \alpha^{\star\prime\prime}, \alpha^{\star\prime\prime}\rangle = 1$, which means that s is pseudo-arc parameter of $\alpha^\star$. Therefore, the relation between the Cartan frames of α and $\alpha^\star$ reads

$$T^\star = \alpha^{\star\prime} = \lambda_0 B, \quad N^\star = T^{\star\prime} = -N, \quad B^\star = \frac{1}{\lambda_0}T. \tag{14}$$

According to Definition 2.1, $(\alpha, \alpha^\star)$ is a Bertrand pair of curves. Differentiating the relation $B^\star = \frac{1}{\lambda_0}T$ with respect to s and using (5), (11) and (14), we get

$$B^{\star\prime} = \frac{1}{\lambda_0}N = -\kappa_2^\star N^\star = \kappa_2^\star N.$$

The previous relation together with (13) gives $\kappa_2^\star(s) = \kappa_2 = 1/\lambda_0$. Consequently, the curves α and $\alpha^\star$ have equal constant torsions.

Conversely, assume that $(\alpha, \alpha^\star)$ is Bertrand pair of null Cartan curves having equal constant torsions $\kappa_2 = \kappa_2^* \neq 0$. Then Bertrand mate curve $\alpha^\star$ of α is given by

$$\alpha^\star = \alpha + \frac{1}{\kappa_2}N.$$

It can be easily verified that $\langle \alpha^{\star\prime}(s), N(s)\rangle = 0$, so $\alpha^\star$ is an involute of α. □

Example 4.2. *Let us consider a null Cartan helix in $\mathbb{E}^3_1$ with parametric equation*

$$\alpha(s) = (\sinh s, \cosh s, s).$$

Differentiating the previous equation three times with respect to s and using (5), we find

$$\alpha'''(s) = N'(s) = (\cosh s, \sinh s, 0).$$

By using (5), (6) and the last relation, we obtain $\langle N'(s), N'(s)\rangle = 2\kappa_2(s) = -1$. Therefore, the torsion $\kappa_2(s) = -\frac{1}{2}$. According to the proof of Theorem 4.1, an involute $\alpha^\star$ has parametric equation of the form

$$\alpha^\star(s) = \alpha(s) + \frac{1}{\kappa_2(s)}N(s).$$

It can be easily checked that $\alpha^\star$ is a null Cartan helix with the torsion $\kappa_2^\star(s) = \kappa_2(s) = -\frac{1}{2}$. Moreover, $N = -N^\star$ which means that $(\alpha, \alpha^\star)$ is Bertrand pair of null Cartan helices.

Theorem 4.3. *Let α be null Cartan curve in $\mathbb{E}^3_1$ parameterized by pseudo-arc s with the torsion $\kappa_2(s)$ and $\alpha^\star$ a non-null curve with a non-null principal normal parameterized by arc-length $s^\star$ with the torsion $\kappa_2^\star(s^\star) \neq 0$. Then $\alpha^\star$ is an involute of α if and only if the Cartan frame $\{T, N, B\}$ of α and the Frenet frame $\{T^\star, N^\star, B^\star\}$ of $\alpha^\star$ are related by*

$$\begin{aligned}
T^\star &= \Big(\frac{1-\lambda_0\kappa_2}{|f|}\Big)T + \frac{\lambda_0}{|f|}B,\\
N^\star &= \frac{\epsilon_2}{\kappa_1^\star}\Big[\frac{1}{|f|}\Big(\frac{1-\lambda_0\kappa_2}{|f|}\Big)'T + \Big(\frac{1-2\lambda_0\kappa_2}{f^2}\Big)N + \frac{1}{|f|}\Big(\frac{\lambda_0}{|f|}\Big)'B\Big],\\
B^\star &= -\frac{\epsilon_1}{\kappa_1^\star}\Big(\frac{(\lambda_0\kappa_2-1)(1-2\lambda_0\kappa_2)}{|f|f^2}T + \frac{1}{|f|}\Big(\Big(\frac{1-\lambda_0\kappa_2}{|f|}\Big)'\Big(\frac{\lambda_0}{|f|}\Big)\\
&\quad -\Big(\frac{1-\lambda_0\kappa_2}{|f|}\Big)\Big(\frac{\lambda_0}{|f|}\Big)'\Big)N + \frac{\lambda_0(1-2\lambda_0\kappa_2)}{|f|f^2}B\Big),
\end{aligned} \tag{15}$$

where $\lambda_0 \in \mathbb{R}_0$ *and* $|f(s)| = \sqrt{|2\epsilon_1\lambda_0(\lambda_0\kappa_2(s)-1)|} \neq 0$.

Proof. Assume that $\alpha^\star$ is a spacelike or a timelike involute of α. Then $\alpha^\star$ can be parameterized by

$$\alpha^\star(s) = \alpha(s) + \lambda(s)N(s), \tag{16}$$

where $\lambda(s)$ is some differentiable function in s. Differentiating relation (16) with respect to s and using (5), we obtain

$$\alpha^{\star\prime}(s) = (1-\lambda\kappa_2)T + \lambda' N + \lambda B. \tag{17}$$

Since $\alpha^\star$ is an involute of α, it holds

$$\langle \alpha^{\star\prime}, N\rangle = 0. \tag{18}$$

Relations (6), (17) and (18) give $\lambda' = 0$. Hence $\lambda = \lambda_0 \in \mathbb{R}_0$ and thus

$$\alpha^{\star\prime} = (1-\lambda_0\kappa_2)T + \lambda_0 B.$$

By taking the scalar product of the last equation with $\alpha^{\star\prime}$ and using (6), we get

$$\langle \alpha^{\star\prime}, \alpha^{\star\prime}\rangle = 2\lambda_0(\lambda_0\kappa_2 - 1) \neq 0.$$

Let us put $\langle \alpha^{\star\prime}(s), \alpha^{\star\prime}(s)\rangle = \epsilon_1 f^2(s)$, where $\epsilon_1 = 1$ is $\alpha^\star$ is a spacelike curve, or $\epsilon_1 = -1$ is $\alpha^\star$ is a timelike curve. Therefore,

$$|f(s)| = \|\alpha^{\star\prime}(s)\| = \sqrt{|2\epsilon_1\lambda_0(\lambda_0\kappa_2(s)-1)|}. \tag{19}$$

Thus the unit tangent vector of $\alpha^\star$ has the form

$$T^\star(s) = \frac{\alpha^{\star\prime}(s)}{\|\alpha^{\star\prime}(s)\|} = \Big(\frac{1-\lambda_0\kappa_2}{|f|}\Big)T + \frac{\lambda_0}{|f|}B. \tag{20}$$

The arc-length parameter of $\alpha^\star$ is given by

$$s^\star(s) = \int_0^s \|\alpha^{\star\prime}(u)\|\, du.$$

By using (19) and the last relation, we obtain $\dfrac{ds^\star}{ds} = |f(s)|$. Differentiating relation (20) with respect to $s^\star$, we find

$$\frac{dT^\star}{ds^\star} = \frac{dT^\star}{ds}\frac{ds}{ds^\star} = \frac{1}{|f|}\Big(\frac{1-\lambda_0\kappa_2}{|f|}\Big)'T + \Big(\frac{1-2\lambda_0\kappa_2}{f^2}\Big)N + \frac{1}{|f|}\Big(\frac{\lambda_0}{|f|}\Big)'B. \tag{21}$$

On the other hand, by using (3) we have

$$\frac{dT^\star}{ds^\star} = \epsilon_2\kappa_1^\star N^\star. \tag{22}$$

Relations (21) and (22) give

$$N^\star = \frac{\epsilon_2}{\kappa_1^\star}\Big[\frac{1}{|f|}\Big(\frac{1-\lambda_0\kappa_2}{|f|}\Big)'T + \Big(\frac{1-2\lambda_0\kappa_2}{f^2}\Big)N + \frac{1}{|f|}\Big(\frac{\lambda_0}{|f|}\Big)'B\Big]. \tag{23}$$

By using the relations $T^\star \times N^\star = -\epsilon_1\epsilon_2 B^\star$, (7), (20) and (23), we get

$$\begin{aligned} B^\star = -\frac{\epsilon_1}{\kappa_1^\star}\Big(&\frac{(\lambda_0\kappa_2-1)(1-2\lambda_0\kappa_2)}{|f|f^2}T + \frac{1}{|f|}\Big(\Big(\frac{1-\lambda_0\kappa_2}{|f|}\Big)'\Big(\frac{\lambda_0}{|f|}\Big) \\ &-\Big(\frac{1-\lambda_0\kappa_2}{f}\Big)\Big(\frac{\lambda_0}{f}\Big)'\Big)N + \frac{\lambda_0(1-2\lambda_0\kappa_2)}{|f|f^2}B\Big), \end{aligned} \tag{24}$$

Finally, relations (20), (23) and (24) give relation (15). □

Corollary 4.4. *Let α be a null Cartan helix in $\mathbb{E}^3_1$ parameterized by the pseudo-arc s and $\alpha^\star$ a non-null curve with non-null principal normal parameterized by arc-length $s^\star$. Then $\alpha^\star$ is an involute of α if and only if $(\alpha, \alpha^\star)$ is a Bertrand pair of curves.*

Remark 4.5. *According to Corollary 4.4, Bertrand mate curve $\alpha^\star$ of Bertrand null Cartan curve can be a spacelike or a timelike curve. On the other hand, in [3] Bertrand mate curve of Bertrand null curve is defined as null (Cartan) curve. Therefore, the mentioned definition is not completely correct. Hence we have redefined Bertrand null Cartan curves (Definition 2.1 in Section 2) with respect to possible causal characters of their Bertrand mate curves.*

Example 4.6. *Let us consider the null Cartan helix α in $\mathbb{E}^3_1$ with parametric equation*

$$\alpha(s) = (\sinh s, \cosh s, s)$$

and the Cartan frame

$$T(s) = (\cosh s, \sinh s, 1), \quad N(s) = (\sinh s, \cosh s, 0), \quad B(s) = \frac{1}{2}(\cosh s, \sinh s, -1).$$

Let us define the curve $\alpha^\star$ by

$$\alpha^\star(s) = \alpha(s) - \frac{1}{2}N(s) = (\frac{1}{2}\sinh s, \frac{1}{2}\cosh s, s).$$

Then $\alpha^\star$ is a spacelike hyperbolic helix with the Frenet frame

$$\begin{aligned} T^\star(s) &= \tfrac{\sqrt{3}}{3}(\cosh s, \sinh s, 2),\\ N^\star(s) &= (\sinh s, \cosh s, 0),\\ B^\star(s) &= -\tfrac{\sqrt{3}}{3}(2\cosh s, 2\sinh s, 1). \end{aligned}$$

Since the vectors $N(s)$ and $N^\star(s)$ are collinear, by Definition 2.1 it follows that $(\alpha, \alpha^\star)$ is a Bertrand pair of curves. It can be easily verified that $\langle \alpha^{\star\prime}(s), N(s)\rangle = 0$, so $\alpha^\star$ is a spacelike involute of α.

According to the proofs of Theorems 4.1 and 4.3, a family of involutes of a null Cartan curve is given by

$$\alpha^\star(s) = \alpha(s) + \lambda_0 N(s), \tag{25}$$

where $\lambda_0 \in R_0$. We will show that among all null Cartan curves in $\mathbb{E}^3_1$, only the null Cartan cubic has two different families of involutes. The first family of its involutes is given by relation (25). In order to obtain the second family, recall that an involute of a regular curve α in Euclidean space $\mathbb{E}^3$ is a curve lying on the tangent surface $x(s,t) = \alpha(s) + t\alpha'(s)$ of α which meet the generators (rulings) of $x(s,t)$ at right angles ([13]). The next theorem shows that a similar property holds for null Cartan cubic in $\mathbb{E}^3_1$.

Theorem 4.7. *Every non-null curve lying on B-scroll in $\mathbb{E}^3_1$ whose base curve is null Cartan cubic α, is an involute of α.*

Proof. Assume that $\beta(s,t)$ is a B-scroll with parameter equation

$$\beta(s,t) = \alpha(s) + tB(s),$$

where $\alpha(s)$ is a null Cartan cubic parameterized by pseudo-arc s and $B(s)$ is the binormal vector of α. Let γ be an arbitrary curve lying on B-scroll with parametric equation

$$\gamma(s) = \alpha(s) + t(s)B(s). \tag{26}$$

Differentiating the relation (26) with respect to s and using (5), we find

$$\gamma'(s) = T(s) + t'(s)B(s). \tag{27}$$

Relations (6) and (27) give $\langle \gamma', \gamma'\rangle = -2t'(s) \neq 0$, which means that γ is a non-null curve. By using (6) and (27), we get

$$\langle \gamma', N\rangle = 0.$$

Consequently, γ is an involute of α. □

Relation (26) gives the second family of involutes of a null Cartan cubic. By using the similar methods as in the proof of Theorem 4.3, the next theorem can be proved.

Theorem 4.8. *If $\alpha^\star$ is a timelike helix lying on B-scroll whose base curve is a null Cartan cubic α, then $\alpha^\star$ is an involute of α and $(\alpha, \alpha^\star)$ is a Bertrand pair of curves.*

Example 4.9. *Let us consider the curve $\alpha^\star$ in $\mathbb{E}^3_1$ with parameter equation*

$$\alpha^\star(s) = \alpha(s) + sB(s), \tag{28}$$

where $\alpha(s)$ is a null Cartan cubic with parameter equation

$$\alpha(s) = (\frac{s^3}{4} + \frac{s}{3}, \frac{s^2}{2}, \frac{s^3}{4} - \frac{s}{3}) \tag{29}$$

and $B(s)$ is the binormal vector of α. Clearly, $\alpha^\star$ lies on B-scroll with parameter equation $x(s,t) = \alpha(s) + tB(s)$. The Cartan frame of α reads

$$\begin{aligned} T(s) &= (\frac{3s^2}{4} + \frac{1}{3}, s, \frac{3s^2}{4} - \frac{1}{3}), \\ N(s) &= (\frac{3s}{2}, 1, \frac{3s}{2}), \\ B(s) &= (\frac{3}{2}, 0, \frac{3}{2}). \end{aligned} \tag{30}$$

By using (28), (29) and (30), we obtain

$$\alpha^\star(s) = (\frac{s^3}{4} + \frac{11s}{6}, \frac{s^2}{2}, \frac{s^3}{4} + \frac{7s}{6}).$$

It can be easily checked that $\alpha^\star$ is a timelike helix. In particular, it holds $\langle \alpha^{\star\prime}(s), N(s)\rangle = 0$, so $\alpha^\star$ is a timelike involute of α. Moreover, $N^\star(s) = N(s)$ which means that $(\alpha, \alpha^\star)$ is a Bertrand pair of curves.

The next two Corollaries give the relations between involutes of order 1 and 2 of a null Cartan curve in $\mathbb{E}^3_1$.

Corollary 4.10. *Every null Cartan involute of a null Cartan curve α in $\mathbb{E}^3_1$ is an involute of order 2 of α.*

Corollary 4.11. *There are no non-null involutes of a null Cartan curve α with the torsion $\tau(s) \neq$ constant in $\mathbb{E}^3_1$, which are involutes of order 2 of α.*

Corollary 4.12. *There are no involutes of a null Cartan cubic α lying on B-scroll in $\mathbb{E}^3_1$ which are involutes of order 2 of α.*

According to Definition 3.2, if $\alpha^\star$ is an involute of order 2 of null Cartan curve α, then $\alpha^\star$ is orthogonal to the 2-dimensional osculating space of α. Hence relation (9) implies that the tangent vector field $T^\star$ of $\alpha^\star$ is collinear with the binormal vector field B of α, so $\alpha^\star$ is a null Cartan curve. In the next theorem, we show that there are no involutes of order 2 of null Cartan cubic. In relation to that, recall that the null straight lines are not null Cartan curves, since their frame is not a unique.

Theorem 4.13. *There are no involutes of order 2 of a null Cartan cubic in $\mathbb{E}^3_1$.*

Proof. Assume that there exists involute $\alpha^\star$ of order 2 of a null Cartan cubic α. According to relation (9), the tangent vector field $T^\star$ of $\alpha^\star$ is collinear with the constant binormal vector field B of α. Thus $\alpha^\star$ is a null straight line, which is a contradiction. □

Theorem 4.14. *Let α and $\alpha^\star$ be the null Cartan curves in $\mathbb{E}^3_1$, parameterized by the pseudo-arcs s and $s^\star$ and with the torsions $\kappa_2(s) \neq 0$ and $\kappa_2^\star(s^\star) \neq 0$, respectively. Then $\alpha^\star$ is an involute of order 2 of α if and only if the Cartan frames $\{T, N, B\}$ and $\{T^\star, N^\star, B^\star\}$ of α and $\alpha^\star$ respectively are related by*

$$\begin{aligned} T^\star &= \frac{f}{\sqrt{|f\kappa_2|}}B,\\ N^\star &= \frac{1}{\sqrt{|f\kappa_2|}}\Big(\frac{f}{\sqrt{|f\kappa_2|}}\Big)'B - sgn(f\kappa_2)N,\\ B^\star &= \frac{\sqrt{|f\kappa_2|}}{f}T - sgn(f\kappa_2)\frac{1}{f}\Big(\frac{f}{\sqrt{|f\kappa_2|}}\Big)'N + \frac{\Big(\frac{f}{\sqrt{|f\kappa_2|}}\Big)'}{2f\sqrt{|f\kappa_2|}}B, \end{aligned} \tag{31}$$

where

$$f(s) = \frac{1}{\kappa_2(s)} + \Big(\frac{1}{\kappa_2(s)}\Big(\frac{1}{\kappa_2(s)}\Big)'\Big)' \neq 0.$$

Proof. Assume that $\alpha^\star$ is a null Cartan involute of order 2 of α. Then $\alpha^\star$ can be parameterized by

$$\alpha^\star(s) = \alpha(s) + \lambda(s)N(s) + \mu(s)B(s), \tag{32}$$

where $\lambda(s)$ and $\mu(s)$ are some differentiable functions in s. Differentiating (32) with respect to s and using (5), we obtain

$$\alpha^{\star\prime}(s) = (1 - \lambda\kappa_2)T + (\lambda' - \mu\kappa_2)N + (\lambda + \mu')B. \tag{33}$$

Since $\alpha^\star$ is an involute of order 2 of α, there hold the equations

$$\langle \alpha^{\star\prime}, N\rangle = 0, \quad \langle \alpha^{\star\prime}, B\rangle = 0. \tag{34}$$

The relations (6), (33) and (34) give

$$\lambda = \frac{1}{\kappa_2}, \quad \mu = \frac{1}{\kappa_2}\Big(\frac{1}{\kappa_2}\Big)'.$$

Substituting this in (33), we get

$$\alpha^{\star\prime} = \Big(\frac{1}{\kappa_2} + \Big(\frac{1}{\kappa_2}\Big(\frac{1}{\kappa_2}\Big)'\Big)'\Big)B.$$

Let us put

$$f(s) = \frac{1}{\kappa_2(s)} + \Big(\frac{1}{\kappa_2(s)}\Big(\frac{1}{\kappa_2(s)}\Big)'\Big)'.$$

The last two relations give

$$\alpha^{\star\prime\prime} = f'B - f\kappa_2 N,$$

which implies $\|\alpha^{\star\prime\prime}\| = |f\kappa_2|$. The pseudo-arc parameter of $\alpha^\star$ is given by

$$s^\star(s) = \int_0^s \sqrt{\|\alpha^{\star\prime}(u)\|}\, du.$$

Consequently, $\dfrac{ds^\star}{ds} = \sqrt{|f\kappa_2|}$ so the tangent vector field of $\alpha^\star$ reads

$$T^\star = \frac{d\alpha^\star}{ds^\star} = \frac{d\alpha^\star}{ds}\frac{ds}{ds^\star} = \frac{f}{\sqrt{|f\kappa_2|}}B. \tag{35}$$

Differentiating (35) with respect to $s^\star$, using (5) and $\frac{ds^\star}{ds} = \sqrt{|f\kappa_2|}$, we find

$$N^\star = \frac{dT^\star}{ds^\star} = \frac{1}{\sqrt{|f\kappa_2|}}\Big(\frac{f}{\sqrt{|f\kappa_2|}}\Big)' B - sgn(f\kappa_2)N. \tag{36}$$

Finally, by using the conditions $\langle B^\star, B^\star\rangle = 0$, $\langle B^\star, T^\star\rangle = -1$, $\langle B^\star, N^\star\rangle = 0$ and relations (35) and (36), we get

$$B^\star = \frac{\sqrt{|f\kappa_2|}}{f}T - sgn(f\kappa_2)\frac{1}{f}\Big(\frac{f}{\sqrt{|f\kappa_2|}}\Big)' N + \frac{\Big(\frac{f}{\sqrt{|f\kappa_2|}}\Big)'}{2f\sqrt{|f\kappa_2|}}B,$$

which completes the proof. □

Corollary 4.15. *Let α be a null Cartan helix and $\alpha^\star$ a null Cartan curve in $\mathbb{E}^3_1$. Then $\alpha^\star$ is an involute of order 2 of α if and only if $(\alpha, \alpha^\star)$ is a Bertrand pair of curves.*

5. An application

In this section, we give two examples of evolving involutes of the null Cartan curve α in $\mathbb{E}^3_1$, which evolve according to null Betchov-Da Rios vortex filament equation (2) and generate timelike Hasimoto surfaces.

Let us consider evolving curve $\beta(s,t)$ in $\mathbb{E}^3_1$ with evolution equation

$$\beta(s,t) = \alpha(s) - tB(s), \tag{37}$$

where α is null Cartan cubic with parameter equation (29), s is pseudo-arc of α and B is the binormal vector field of α. By taking the partial derivatives of (37) with respect to s and t and using (5), we find

$$\beta_t = -B, \quad \beta_s = T, \quad \beta_{ss} = N, \quad \beta_{sss} = B. \tag{38}$$

From (6) and (38) we obtain $\langle \beta_s, \beta_s\rangle = 0$, $\langle \beta_{ss}, \beta_{ss}\rangle = 1$. Therefore, $\beta(s,t)$ is evolving null Cartan curve parameterized by pseudo-arc s for all time t. It can be easily checked that $\langle \beta_s, N\rangle = 0$ for all time t. According to Definition 3.3, it follows that $\beta(s,t)$ is an evolving involute of α. By using (7) and (38), we easily get $\beta_t = \beta_{ss} \times \beta_{sss}$. Hence $\beta(s,t)$ evolves according to null Betchov-Da Rios vortex filament equation (2). Since the plane $span\{\beta_s, \beta_t\}$ is a timelike, evolving curve $\beta(s,t)$ generates a timelike Hasimoto surface which represents B-scroll (Fig. 1).

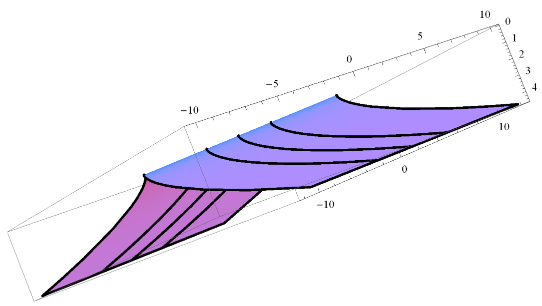

Figure 1: Evolving involute $\beta(s,t)$ and generated timelike Hasimoto surface

Next, let us consider an evolving curve $\gamma(s,t)$ in $\mathbb{E}^3_1$ with evolution equation

$$\gamma(s,t) = \alpha(s) - t(\tau(s)T(s) + B(s)), \tag{39}$$

where α is a null Cartan helix parameterized by the pseudo-arc s with torsion $\tau(s) = constant \neq 0$, T is the tangent vector field of α and B is the binormal vector field of α. By taking the partial derivatives of (39) with respect to s and t and using (5), we find

$$\gamma_t = -\tau T - B, \quad \gamma_s = T, \quad \gamma_{ss} = N, \quad \gamma_{sss} = -\tau T + B. \tag{40}$$

By using (6) and (40) we obtain $\langle \gamma_s, \gamma_s \rangle = 0$ and $\langle \gamma_{ss}, \gamma_{ss} \rangle = 1$. Thus $\gamma(s,t)$ is evolving null Cartan curve parameterized by pseudo-arc s for all time t. Since $\langle \gamma_s, N \rangle = 0$ for all time t, Definition 3.3 implies that $\gamma(s,t)$ is an evolving involute of α. From (7) and (40) we get $\gamma_t = -\tau T - B = \gamma_{ss} \times \gamma_{sss}$, which means that $\gamma(s,t)$ evolves according to null Betchov-Da Rios vortex filament equation (2). It generates timelike Hasimoto surface, since the plane spanned by $\{\gamma_s, \gamma_t\}$ is a timelike (Fig. 2).

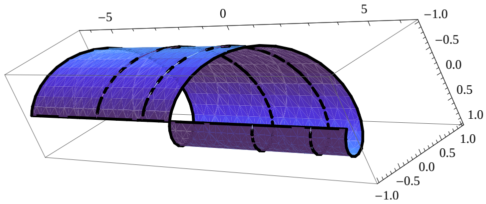

Figure 2: Evolving involute $\gamma(s,t)$ and generated timelike Hasimoto surface

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