



New Hermite-Hadamard Type Inequalities for Convex Mappings Utilizing Generalized Fractional Integrals

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Abstract. In this work, we first establish new Hermite-Hadamard inequalities for convex function utilizing fractional integrals with respect to another function which are generalization of some important fractional integrals such as the Riemann-Liouville fractional integrals and the Hadamard fractional integrals. Moreover, we obtain some generalized midpoint and trapezoid type inequalities for these kinds of fractional integrals. The inequalities given in this paper provide generalizations of several results obtained in earlier works.

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., [10], [14], [29, p.137]). These inequalities state that if $f : I \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \quad (1)$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics. A number of mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions such as using convex mappings.

Over the last twenty years, the numerous studies have focused on to establish generalization of the inequality (1) and to obtain new bounds for left hand side and right hand side of the inequality (1). For some examples, please refer to ([1], [3], [7], [11], [22], [27], [28], [32])

The overall structure of the study takes the form of four sections with introduction. The remainder of this work is organized as follows: we first give some kinds of fractional integrals and then we mention some works which focus on fractional version of Hermite-Hadamard inequality. In Section 2 new Hermite-Hadamard type inequalities for generalized fractional integrals are proved. In Section 3 and Section 4, using

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generalized fractional integrals, we present some midpoint and trapezoid type inequalities for functions whose first derivatives in absolute value are convex, respectively.

In the following we present the definitions of the Riemann-Liouville fractional integrals:

Definition 1.1. Let $f \in L_1[a, b]$. The Riemann-Liouville fractional integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

The definition of Hadamard fractional integrals is given as follows:

Definition 1.2. Let $f \in L_1[a, b]$. The Hadamard fractional integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad x < b$$

respectively.

Now, we give the following generalized fractional integrals:

Definition 1.3. Let $w : [a, b] \rightarrow \mathbb{R}$ be an increasing and positive monotone function on (a, b) , having a continuous derivative $w'(x)$ on (a, b) . The left-sides $(I_{a+;w}^\alpha f(x))$ and right-sides $(I_{b-;w}^\alpha f(x))$ fractional integral of f with respect to the function g on $[a, b]$ of order $\alpha < 0$ are defined by

$$I_{a+;w}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{w'(t)f(t)}{[w(x) - w(t)]^{1-\alpha}} dt, \quad x > a$$

and

$$I_{b-;w}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{w'(t)f(t)}{[w(t) - w(x)]^{1-\alpha}} dt, \quad x < b$$

respectively.

In [34], Sarikaya et al. first proved the following important Hermite-Hadamard type utilizing Riemann-Liouville fractional integrals.

Theorem 1.4. Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{2} \tag{2}$$

with $\alpha > 0$.

On the other hand, Sarikaya and Yildirim give the following Hermite-Hadamard type inequality for the Riemann-Liouville fractional integrals in [33].

Theorem 1.5. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1 [a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) + J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2}. \tag{3}$$

Whereupon Sarikaya et al. obtain the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals, many authors have studied to generalize this inequality and to establish Hermite-Hadamard inequality other fractional integrals such as k -fractional integral, Hadamard fractional integrals, Katugampola fractional integrals, Conformable fractional integrals, etc. For some of them, please see ([2], [4]-[6], [8], [9], [12], [15]-[17], [19], [21], [24]-[26], [31], [35]- [45]). More details for fractional calculus, one can refer to ([13], [20], [23], [30]).

In [18], Jleli and Samet proved the following Hermite-Hadamard type inequality:

Theorem 1.6. *Let $w : [a, b] \rightarrow \mathbb{R}$ be an increasing and positive monotone function on (a, b) , having a continuous derivative $w'(x)$ on (a, b) and let $\alpha > 0$. If f is a convex on $[a, b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{4[w(b)-w(a)]^\alpha} \left(I_{a^+;w}^\alpha F(b) + I_{b^-;w}^\alpha F(a) \right) \leq \frac{f(a) + f(b)}{2} \tag{4}$$

where

$$F(x) = f(x) + \tilde{f}(x), \text{ and } \tilde{f}(x) = f(a+b-x) \tag{5}$$

for $x \in [a, b]$

The aim of this paper is to establish new Hermite-Hadamard type integral inequalities for convex function utilizing fractional integrals with respect to another function. Furthermore, we present some trapezoid and midpoint type inequalities.

2. Generalized Hermite-Hadamard Type Inequalities

Firstly, let us start with some notations. Let $\alpha > 0$ and let $w : [a, b] \rightarrow \mathbb{R}$ be an increasing and positive monotone function on (a, b) , having a continuous derivative $w'(x)$ on (a, b) . We define the following positive mapping on $[0, 1]$,

$$\Lambda_w^\alpha(t) := \left[w(b) - w\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right]^\alpha + \left[w\left(\frac{2-t}{2}a + \frac{t}{2}b\right) - w(a) \right]^\alpha. \tag{6}$$

Particularly,

$$\Lambda_w^\alpha(1) := \left[w(b) - w\left(\frac{a+b}{2}\right) \right]^\alpha + \left[w\left(\frac{a+b}{2}\right) - w(a) \right]^\alpha.$$

Moreover, if we consider the identity mapping ℓ instead of the mapping w (i.e. $w(t) = \ell(t) = t$), then we get

$$\Lambda_\ell^\alpha(1) = \frac{(b-a)^\alpha}{2^{\alpha-1}}. \tag{7}$$

Furthermore, for $w(t) = \ln t$ we have

$$\Lambda_{\ln}^\alpha(t) := \left[\ln\left(\frac{2b}{ta + (2-t)b}\right) \right]^\alpha + \left[\ln\left(\frac{(2-t)a + tb}{2a}\right) \right]^\alpha \tag{8}$$

and particularly,

$$\Lambda_{\ln}^\alpha(1) := \left[\ln \frac{2b}{a+b} \right]^\alpha + \left[\ln \frac{a+b}{2a} \right]^\alpha.$$

Now, we give the following generalized Hermite-Hadamard inequality:

Theorem 2.1. *Let $\alpha > 0$. Let $w : [a, b] \rightarrow \mathbb{R}$ be an increasing and positive monotone function on (a, b) , having a continuous derivative $w'(x)$ on (a, b) . If f is a convex function on $[a, b]$, then we have the following Hermite-Hadamard type inequalities for generalized fractional integrals*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2\Lambda_w^\alpha(1)} \left[I_{\left(\frac{a+b}{2}\right)^+, w}^\alpha F(b) + I_{\left(\frac{a+b}{2}\right)^-, w}^\alpha F(a) \right] \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned} \tag{9}$$

where the mapping F is defined as in (5) and the mapping Λ_w^α is defined as in (6).

Proof. As f is an convex function on $[a, b]$, we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} \tag{10}$$

for $x, y \in [a, b]$. If we choose $x = \frac{t}{2}a + \frac{2-t}{2}b$ and $y = \frac{2-t}{2}a + \frac{t}{2}b$ for $t \in [0, 1]$, using the onvexity of f , then we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + \frac{1}{2}f\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \leq \frac{f(a) + f(b)}{2}. \tag{11}$$

After multiplying both sides of (11) by

$$\frac{b-a}{2\Gamma(\alpha)} \frac{w'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)}{\left[w(b) - w\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right]^{1-\alpha}}$$

if we integrate the resulting inequality with respect to t over $(0, 1)$, then we get

$$\begin{aligned} &\frac{b-a}{2\Gamma(\alpha)} f\left(\frac{a+b}{2}\right) \int_0^1 \frac{w'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)}{\left[w(b) - w\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right]^{1-\alpha}} dt \\ &\leq \frac{b-a}{4\Gamma(\alpha)} \int_0^1 \frac{w'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)}{\left[w(b) - w\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right]^{1-\alpha}} \left[f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \right] dt \\ &\leq \frac{b-a}{2\Gamma(\alpha)} \left[\frac{f(a) + f(b)}{2} \right] \int_0^1 \frac{w'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)}{\left[w(b) - w\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right]^{1-\alpha}} dt. \end{aligned}$$

By change of variable $u = \frac{t}{2}a + \frac{2-t}{2}b$ with $du = -\frac{b-a}{2}dt$, we obtain the inequalities

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b \frac{w'(u)}{[w(b)-w(u)]^{1-\alpha}} du \\ & \leq \frac{1}{2\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b \frac{w'(u)}{[w(b)-w(u)]^{1-\alpha}} [f(u) + f(a+b-u)] dt \\ & \leq \frac{f(a) + f(b)}{2\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b \frac{w'(u)}{[w(b)-w(u)]^{1-\alpha}} du. \end{aligned}$$

Using the fact that

$$\int_{\frac{a+b}{2}}^b \frac{w'(u)}{[w(b)-w(u)]^{1-\alpha}} du = \frac{1}{\alpha} \left[w(b) - w\left(\frac{a+b}{2}\right) \right]^\alpha, \tag{12}$$

and Definition 1.3, we establish the following inequalities

$$\begin{aligned} \frac{1}{\Gamma(\alpha+1)} \left[w(b) - w\left(\frac{a+b}{2}\right) \right]^\alpha f\left(\frac{a+b}{2}\right) & \leq \frac{1}{2} \left[I_{\left(\frac{a+b}{2}\right)^+, w}^\alpha f(b) + I_{\left(\frac{a+b}{2}\right)^-, w}^\alpha \tilde{f}(b) \right] \\ & \leq \frac{f(a) + f(b)}{2\Gamma(\alpha+1)} \left[w(b) - w\left(\frac{a+b}{2}\right) \right]^\alpha, \end{aligned}$$

i.e.

$$\begin{aligned} & \frac{1}{\Gamma(\alpha+1)} \left[w(b) - w\left(\frac{a+b}{2}\right) \right]^\alpha f\left(\frac{a+b}{2}\right) \\ & \leq \frac{1}{2} I_{\left(\frac{a+b}{2}\right)^+, w}^\alpha F(b) \leq \frac{f(a) + f(b)}{2\Gamma(\alpha+1)} \left[w(b) - w\left(\frac{a+b}{2}\right) \right]^\alpha. \end{aligned} \tag{13}$$

By the similar way, multiplying both sides of (11) by

$$\frac{b-a}{2\Gamma(\alpha)} \frac{w'\left(\frac{2-t}{2}a + \frac{t}{2}b\right)}{\left[w\left(\frac{2-t}{2}a + \frac{t}{2}b\right) - w(a) \right]^{1-\alpha}}$$

and integrating the resulting inequality with respect to t over $(0, 1)$, we obtain

$$\begin{aligned} & \frac{1}{\Gamma(\alpha+1)} \left[w\left(\frac{a+b}{2}\right) - w(a) \right]^\alpha f\left(\frac{a+b}{2}\right) \\ & \leq \frac{1}{2} I_{\left(\frac{a+b}{2}\right)^-, w}^\alpha F(a) \leq \frac{f(a) + f(b)}{2\Gamma(\alpha+1)} \left[w\left(\frac{a+b}{2}\right) - w(a) \right]^\alpha. \end{aligned} \tag{14}$$

Summing the inequalities (13) and (14), we get

$$\frac{\Lambda_w^\alpha(1)}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \left[I_{\left(\frac{a+b}{2}\right)^+, w}^\alpha F(b) + I_{\left(\frac{a+b}{2}\right)^-, w}^\alpha F(a) \right] \leq \frac{\Lambda_w^\alpha(1)}{\Gamma(\alpha+1)} \left[\frac{f(a) + f(b)}{2} \right].$$

which completes the proof of the inequality (9). \square

Remark 2.2. If we choose $w(t) = t$ in (9), then the inequalities (9) reduce to the inequalities (3) proved by Sarikaya and Yildirim in [33].

Corollary 2.3. Under assumption of Theorem 2.1 with $w(t) = \ln t$, we have the following Hermite-Hadamard type inequalities for Hadamard fractional integrals

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2\Lambda_{\ln}^\alpha(1)} \left[J_{\left(\frac{a+b}{2}\right)^+; w}^\alpha F(b) + J_{\left(\frac{a+b}{2}\right)^-; w}^\alpha F(a) \right] \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned}$$

where the mapping Λ_{\ln}^α is given as in (8).

3. Generalized Midpoint Type Inequalities

In this section, we present some generalized midpoint type inequalities for the generalized fractional integrals.

Lemma 3.1. Let $\alpha > 0$ and let the mapping w be as in Theorem 2.1. If $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$, then we have the following identity for generalized fractional integrals

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2\Lambda_w^\alpha(1)} \left[I_{\left(\frac{a+b}{2}\right)^+; w}^\alpha F(b) + I_{\left(\frac{a+b}{2}\right)^-; w}^\alpha F(a) \right] - f\left(\frac{a+b}{2}\right) \\ &= \frac{b-a}{4\Lambda_w^\alpha(1)} \int_0^1 \Lambda_w^\alpha(t) \left[f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) - f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \right] dt \end{aligned} \tag{15}$$

where the mapping F is defined as in (5) and the mapping Λ_w^α is defined as in (6).

Proof. Integrating by parts we have

$$\begin{aligned} I_1 &= \int_0^1 \left[w(b) - w\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right]^\alpha F'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) dt \\ &= -\frac{2}{b-a} \left[w(b) - w\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right]^\alpha F\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \Big|_0^1 \\ &\quad + \alpha \int_0^1 \frac{w'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)}{\left[w(b) - w\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right]^{1-\alpha}} F\left(\frac{t}{2}a + \frac{2-t}{2}b\right) dt \\ &= \frac{2}{b-a} \left[w(b) - w\left(\frac{a+b}{2}\right) \right]^\alpha F\left(\frac{a+b}{2}\right) + \frac{2\alpha}{b-a} \int_{\frac{a+b}{2}}^b \frac{w'(u)}{\left[w(b) - w(u) \right]^{1-\alpha}} F(u) dt \\ &= \frac{2\Gamma(\alpha+1)}{b-a} I_{\left(\frac{a+b}{2}\right)^+; w}^\alpha F(b) - \frac{4}{b-a} \left[w(b) - w\left(\frac{a+b}{2}\right) \right]^\alpha f\left(\frac{a+b}{2}\right). \end{aligned} \tag{16}$$

Similarly integrating by parts, we have

$$\begin{aligned}
 I_2 &= \int_0^1 \left[w\left(\frac{2-t}{2}a + \frac{t}{2}b\right) - w(a) \right]^\alpha F'\left(\frac{2-t}{2}a + \frac{t}{2}b\right) dt \\
 &= \frac{4}{b-a} \left[w\left(\frac{a+b}{2}\right) - w(a) \right]^\alpha f\left(\frac{a+b}{2}\right) - \frac{2\Gamma(\alpha+1)}{b-a} I_{\left(\frac{a+b}{2}\right)^-; w}^\alpha F(a).
 \end{aligned}
 \tag{17}$$

From (16) and (17), we obtain

$$\frac{b-a}{4\Lambda_w^\alpha(1)} (I_1 - I_2) = \frac{\Gamma(\alpha+1)}{2} \left[I_{\left(\frac{a+b}{2}\right)^+; w}^\alpha F(b) + I_{\left(\frac{a+b}{2}\right)^-; w}^\alpha F(a) \right] - f\left(\frac{a+b}{2}\right).
 \tag{18}$$

On the other hand, since $F'(x) = f'(x) - f'(a+b-x)$, we get

$$I_1 = \int_0^1 \left[w(b) - w\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right]^\alpha \left[f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) - f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \right] dt$$

and

$$I_2 = \int_0^1 \left[w\left(\frac{2-t}{2}a + \frac{t}{2}b\right) - w(a) \right]^\alpha \left[f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right) - f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right] dt.$$

Thus we have

$$I_1 - I_2 = \int_0^1 \Lambda_w^\alpha(t) \left[f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) - f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \right] dt.
 \tag{19}$$

Using the equality (19) in (18), we obtain the required identity (15). \square

Remark 3.2. If we choose $w(t) = t$ in (15), then we obtain the following equality for the Riemann-Liouville fractional integrals

$$\begin{aligned}
 &\frac{2^{1-\alpha}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) + J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) \right] - f\left(\frac{a+b}{2}\right) \\
 &= \frac{b-a}{4} \int_0^1 t^\alpha \left[f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) - f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \right] dt
 \end{aligned}$$

which is proved by Sarikaya and Yildirim in [33].

Corollary 3.3. Under assumption of Lemma 3.1 with $w(t) = \ln t$, we have the following identity for Hadamard fractional integrals

$$\begin{aligned}
 &\frac{\Gamma(\alpha+1)}{2\Lambda_{\ln}^\alpha(1)} \left[J_{\left(\frac{a+b}{2}\right)^+; w}^\alpha F(b) + J_{\left(\frac{a+b}{2}\right)^-; w}^\alpha F(a) \right] - f\left(\frac{a+b}{2}\right) \\
 &= \frac{b-a}{4\Lambda_{\ln}^\alpha(1)} \int_0^1 \Lambda_{\ln}^\alpha(t) \left[f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) - f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \right] dt
 \end{aligned}$$

where the mapping Λ_{\ln}^α is given as in (8).

Theorem 3.4. Let $\alpha > 0$ and let the mapping w as in Theorem 2.1. If $|f'|$ is a convex mapping on $[a, b]$, then we have the following trapezoid type inequality for generalized fractional integrals

$$\begin{aligned} & \left| \frac{\Gamma(\alpha + 1)}{2\Lambda_w^\alpha(1)} \left[I_{(\frac{a+b}{2})+;w}^\alpha F(b) + I_{(\frac{a+b}{2})-;w}^\alpha F(a) \right] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{4\Lambda_w^\alpha(1)} \left[|f'(a)| + |f'(b)| \right] \int_0^1 \Lambda_w^\alpha(t) dt \end{aligned} \tag{20}$$

where the mapping F is defined as in (5) and the mapping Λ_w^α is defined as in (6).

Proof. Taking modulus in Lemma 3.1, we have

$$\begin{aligned} & \left| \frac{\Gamma(\alpha + 1)}{2\Lambda_w^\alpha(1)} \left[I_{(\frac{a+b}{2})+;w}^\alpha F(b) + I_{(\frac{a+b}{2})-;w}^\alpha F(a) \right] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{4\Lambda_w^\alpha(1)} \int_0^1 |\Lambda_w^\alpha(t)| \left| f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right| dt + \frac{b-a}{4\Lambda_w^\alpha(1)} \int_0^1 |\Lambda_w^\alpha(t)| \left| f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right| dt. \end{aligned}$$

Since $|f'|$ is an convex function on $[a, b]$, we get

$$\left| f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right| \leq \frac{t}{2} |f'(a)| + \frac{2-t}{2} |f'(b)|$$

and

$$\left| f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \right| \leq \frac{2-t}{2} |f'(a)| + \frac{t}{2} |f'(b)|.$$

Hence,

$$\begin{aligned} & \left| \frac{\Gamma(\alpha + 1)}{2\Lambda_w^\alpha(1)} \left[I_{(\frac{a+b}{2})+;w}^\alpha F(b) + I_{(\frac{a+b}{2})-;w}^\alpha F(a) \right] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{4\Lambda_w^\alpha(1)} \left[|f'(a)| + |f'(b)| \right] \int_0^1 |\Lambda_w^\alpha(t)| dt. \end{aligned}$$

This completes the proof. \square

Remark 3.5. If we choose $w(t) = t$ in (20), then we have the following inequality for Riemann-Liouville fractional integrals

$$\begin{aligned} & \left| \frac{2^{1-\alpha}\Gamma(\alpha + 1)}{(b-a)^\alpha} \left[J_{(\frac{a+b}{2})+}^\alpha f(b) + J_{(\frac{a+b}{2})-}^\alpha f(a) \right] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{4(\alpha + 1)} \left[|f'(a)| + |f'(b)| \right] \end{aligned}$$

which is given by Sarikaya and Yildirim in [33].

Corollary 3.6. Under assumption of Theorem 3.4 with $w(t) = \ln t$, we have the following midpoint type inequality for Hadamard fractional integrals

$$\begin{aligned} & \left| \frac{\Gamma(\alpha + 1)}{2\Lambda_{\ln}^\alpha(1)} \left[J_{\left(\frac{a+b}{2}\right)^+; \gamma w}^\alpha F(b) + J_{\left(\frac{a+b}{2}\right)^-; \gamma w}^\alpha F(a) \right] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{4\Lambda_{\ln}^\alpha(1)} \left[|f'(a)| + |f'(b)| \right] \int_0^1 \Lambda_{\ln}^\alpha(t) dt \end{aligned}$$

where the mapping Λ_{\ln}^α is given as in (8).

Theorem 3.7. Let $\alpha > 0$ and let the mapping w as in Theorem 2.1. If $|f'|^q, q > 1$, is a convex mapping on $[a, b]$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have the following trapezoid type inequality for generalized fractional integrals

$$\begin{aligned} & \left| \frac{\Gamma(\alpha + 1)}{2\Lambda_w^\alpha(1)} \left[I_{\left(\frac{a+b}{2}\right)^+; \gamma w}^\alpha F(b) + I_{\left(\frac{a+b}{2}\right)^-; \gamma w}^\alpha F(a) \right] - f\left(\frac{a+b}{2}\right) \right| \tag{21} \\ & \leq \frac{b-a}{4\Lambda_w^\alpha(1)} \left(\int_0^1 (\Lambda_w^\alpha(t))^p dt \right)^{\frac{1}{p}} \left[\left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right] \\ & \leq \frac{b-a}{4\Lambda_w^\alpha(1)} \left(4 \int_0^1 (\Lambda_w^\alpha(t))^p dt \right)^{\frac{1}{p}} \left[|f'(a)|^q + |f'(b)|^q \right] \end{aligned}$$

where the mapping F is defined as in (5) and the mapping Λ_w^α is defined as in (6).

Proof. Taking modulus in Lemma 3.1 and using the well known Hölder’s inequality, we have

$$\begin{aligned} & \left| \frac{\Gamma(\alpha + 1)}{2\Lambda_w^\alpha(1)} \left[I_{\left(\frac{a+b}{2}\right)^+; \gamma w}^\alpha F(b) + I_{\left(\frac{a+b}{2}\right)^-; \gamma w}^\alpha F(a) \right] - f\left(\frac{a+b}{2}\right) \right| \tag{22} \\ & \leq \frac{b-a}{4\Lambda_w^\alpha(1)} \int_0^1 |\Lambda_w^\alpha(t)| \left| f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right| dt + \frac{b-a}{4\Lambda_w^\alpha(1)} \int_0^1 |\Lambda_w^\alpha(t)| \left| f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right| dt \\ & \leq \frac{b-a}{4\Lambda_w^\alpha(1)} \left(\int_0^1 |\Lambda_w^\alpha(t)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{b-a}{4\Lambda_w^\alpha(1)} \left(\int_0^1 |\Lambda_w^\alpha(t)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

As $|f'|^q$ is a convex mapping on $[a, b]$, we get

$$\begin{aligned} \int_0^1 \left| f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right|^q dt & \leq \int_0^1 \left[\frac{t}{2} |f'(a)|^q + \frac{2-t}{2} |f'(b)|^q \right] dt \tag{23} \\ & = \frac{|f'(a)|^q + 3|f'(b)|^q}{4} \end{aligned}$$

and

$$\int_0^1 \left| f' \left(\frac{2-t}{2}a + \frac{t}{2}b \right) \right| dt \leq \int_0^1 \left[\frac{2-t}{2} |f'(a)|^q + \frac{t}{2} |f'(b)|^q \right] dt \tag{24}$$

$$= \frac{3 |f'(a)|^q + |f'(b)|^q}{4}.$$

Putting the inequalities (23) and (24) in (22), we obtain the first inequality in (21).

For the proof of second inequality, let $a_1 = |f'(a)|^q$, $b_1 = 3 |f'(b)|^q$, $a_2 = 3 |f'(a)|^q$ and $b_2 = |f'(b)|^q$. Using the fact that

$$\sum_{k=1}^n (a_k + b_k)^s \leq \sum_{k=1}^n a_k^s + \sum_{k=1}^n b_k^s, \quad 0 \leq s < 1 \tag{25}$$

and $1 + 3^{\frac{1}{q}} \leq 4$ then the required inequality can be obtained easily. \square

Remark 3.8. If we choose $w(t) = t$ in (21), then we have the following inequalities for Riemann-Liouville fractional integrals

$$\left| \frac{2^{1-\alpha} \Gamma(\alpha + 1)}{(b-a)^\alpha} \left[J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) + J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) \right] - f\left(\frac{a+b}{2}\right) \right|$$

$$\leq \frac{b-a}{4} \left(\frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left[\left(\frac{|f'(a)|^q + 3 |f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3 |f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right]$$

$$\leq \frac{b-a}{4} \left(\frac{4}{\alpha p + 1} \right)^{\frac{1}{p}} [|f'(a)| + |f'(b)|]$$

which are given by Sarikaya and Yildirim in [33].

Corollary 3.9. Under assumption of Theorem 3.7 with $w(t) = \ln t$, we have the following midpoint type inequality for Hadamard fractional integrals

$$\left| \frac{\Gamma(\alpha + 1)}{2\Lambda_{\ln}^\alpha(1)} \left[J_{\left(\frac{a+b}{2}\right)^+;w}^\alpha F(b) + J_{\left(\frac{a+b}{2}\right)^-;w}^\alpha F(a) \right] - f\left(\frac{a+b}{2}\right) \right|$$

$$\leq \frac{b-a}{4\Lambda_{\ln}^\alpha(1)} \left(\int_0^1 (\Lambda_{\ln}^\alpha(t))^p dt \right)^{\frac{1}{p}}$$

$$\times \left[\left(\frac{|f'(a)|^q + 3 |f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3 |f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right] \leq \frac{b-a}{4\Lambda_{\ln}^\alpha(1)} \left(4 \int_0^1 (\Lambda_{\ln}^\alpha(t))^p dt \right)^{\frac{1}{p}} [|f'(a)|^q + |f'(b)|^q]$$

where the mapping Λ_{\ln}^α is given as in (8).

4. Generalized Trapezoid Type Inequalities

In this section, we establish some generalized trapezoid type inequalities for the generalized fractional integrals.

Lemma 4.1. Let $\alpha > 0$ and let the mapping w be as in Theorem 2.1. If $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$, then have the following identity for generalized fractional integrals

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2\Lambda_w^\alpha(1)} \left[I_{\left(\frac{a+b}{2}\right)^+; w}^\alpha F(b) + I_{\left(\frac{a+b}{2}\right)^-; w}^\alpha F(a) \right] \\ &= \frac{b-a}{4\Lambda_w^\alpha(1)} \int_0^1 (\Lambda_w^\alpha(1) - \Lambda_w^\alpha(t)) \left[f' \left(\frac{t}{2}a + \frac{2-t}{2}b \right) - f' \left(\frac{2-t}{2}a + \frac{t}{2}b \right) \right] dt \end{aligned} \tag{26}$$

where the mapping F is defined as in (5) and the mapping Λ_w^α is defined as in (6).

Proof. Using the integration by parts, we have

$$\begin{aligned} J_1 &= \int_0^1 \left(\left[w(b) - w \left(\frac{a+b}{2} \right) \right]^\alpha - \left[w(b) - w \left(\frac{t}{2}a + \frac{2-t}{2}b \right) \right]^\alpha \right) F' \left(\frac{t}{2}a + \frac{2-t}{2}b \right) dt \\ &= -\frac{2}{b-a} \left(\left[w(b) - w \left(\frac{a+b}{2} \right) \right]^\alpha - \left[w(b) - w \left(\frac{t}{2}a + \frac{2-t}{2}b \right) \right]^\alpha \right) F \left(\frac{t}{2}a + \frac{2-t}{2}b \right) \Big|_0^1 \\ &\quad - \alpha \int_0^1 \frac{w' \left(\frac{t}{2}a + \frac{2-t}{2}b \right)}{\left[w(b) - w \left(\frac{t}{2}a + \frac{2-t}{2}b \right) \right]^{1-\alpha}} F \left(\frac{t}{2}a + \frac{2-t}{2}b \right) dt \\ &= \frac{2}{b-a} \left[w(b) - w \left(\frac{a+b}{2} \right) \right]^\alpha F(b) - \frac{2\alpha}{b-a} \int_{\frac{a+b}{2}}^b \frac{w'(u)}{\left[w(b) - w(u) \right]^{1-\alpha}} F(u) du \\ &= \frac{2}{b-a} \left[w(b) - w \left(\frac{a+b}{2} \right) \right]^\alpha [f(a) + f(b)] - \frac{2\Gamma(\alpha + 1)}{b-a} I_{\left(\frac{a+b}{2}\right)^+; w}^\alpha F(b). \end{aligned}$$

By the similar way, we obtain

$$\begin{aligned} J_2 &= \int_0^1 \left(\left[w \left(\frac{2-t}{2}a + \frac{t}{2}b \right) - w(a) \right]^\alpha - \left[w \left(\frac{a+b}{2} \right) - w(a) \right]^\alpha \right) F' \left(\frac{2-t}{2}a + \frac{t}{2}b \right) dt \\ &= \frac{2}{b-a} \left[w \left(\frac{a+b}{2} \right) - w(a) \right]^\alpha [f(a) + f(b)] - \frac{2\Gamma(\alpha + 1)}{b-a} I_{\left(\frac{a+b}{2}\right)^-; w}^\alpha F(a). \end{aligned}$$

Then it follows that

$$\begin{aligned} & \frac{b-a}{4\Lambda_w^\alpha(1)} (J_1 + J_2) \\ &= \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2\Lambda_w^\alpha(1)} \left[I_{\left(\frac{a+b}{2}\right)^+; w}^\alpha F(b) + I_{\left(\frac{a+b}{2}\right)^-; w}^\alpha F(a) \right]. \end{aligned} \tag{27}$$

On the other hand, using the fact that $F'(x) = f'(x) - f'(a + b - x)$, we have

$$\begin{aligned} J_1 &= \int_0^1 \left(\left[w(b) - w \left(\frac{a+b}{2} \right) \right]^\alpha - \left[w(b) - w \left(\frac{t}{2}a + \frac{2-t}{2}b \right) \right]^\alpha \right) \\ &\quad \times \left[f' \left(\frac{t}{2}a + \frac{2-t}{2}b \right) - f' \left(\frac{2-t}{2}a + \frac{t}{2}b \right) \right] dt \end{aligned} \tag{28}$$

and

$$\begin{aligned}
 J_2 &= \int_0^1 \left(\left[w\left(\frac{2-t}{2}a + \frac{t}{2}b\right) - w(a) \right]^\alpha - \left[w\left(\frac{a+b}{2}\right) - w(a) \right]^\alpha \right) \\
 &\quad \times \left[f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right) - f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right] dt.
 \end{aligned}
 \tag{29}$$

By substituting the equalities (28) and (29) in (27), we establish the desired result (26) \square

Remark 4.2. If we choose $w(t) = t$ in (26), then we have the following equality for Riemann-Liouville fractional integrals

$$\begin{aligned}
 &\frac{f(a) + f(b)}{2} - \frac{2^{1-\alpha}\Gamma(\alpha + 1)}{(b - a)^\alpha} \left[J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) + J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) \right] \\
 &= \frac{b - a}{4} \int_0^1 (1 - t^\alpha) \left[f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) - f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \right] dt
 \end{aligned}$$

which is proved by Özdemir et al. in [26, Lemma 2 (for $x = \frac{a+b}{2}$)].

Corollary 4.3. Under assumption of Lemma 4.1 with $w(t) = \ln t$, we have the following identity for Hadamard fractional integrals

$$\begin{aligned}
 &\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2\Lambda_{\ln}^\alpha(1)} \left[J_{\left(\frac{a+b}{2}\right)^+;w}^\alpha F(b) + J_{\left(\frac{a+b}{2}\right)^-;w}^\alpha F(a) \right] \\
 &= \frac{b - a}{4\Lambda_{\ln}^\alpha(1)} \int_0^1 (\Lambda_{\ln}^\alpha(1) - \Lambda_{\ln}^\alpha(t)) \left[f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) - f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \right] dt
 \end{aligned}$$

where the mapping Λ_{\ln}^α is given as in (8).

Theorem 4.4. Let $\alpha > 0$ and let the mapping w as in Theorem 2.1. If $|f'|$ is a convex mapping on $[a, b]$, then we have the following trapezoid type inequality for generalized fractional integrals

$$\begin{aligned}
 &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2\Lambda_w^\alpha(1)} \left[I_{\left(\frac{a+b}{2}\right)^+;w}^\alpha F(b) + I_{\left(\frac{a+b}{2}\right)^-;w}^\alpha F(a) \right] \right| \\
 &\leq \frac{b - a}{4\Lambda_w^\alpha(1)} \left[|f'(a)| + |f'(b)| \right] \int_0^1 |\Lambda_w^\alpha(1) - \Lambda_w^\alpha(t)| dt
 \end{aligned}
 \tag{30}$$

where the mapping F is defined as in (5) and the mapping Λ_w^α is defined as in (6).

Proof. Taking modulus in Lemma 4.1, we have

$$\begin{aligned}
 &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2\Lambda_w^\alpha(1)} \left[I_{\left(\frac{a+b}{2}\right)^+;w}^\alpha F(b) + I_{\left(\frac{a+b}{2}\right)^-;w}^\alpha F(a) \right] \right| \\
 &\leq \frac{b - a}{4\Lambda_w^\alpha(1)} \int_0^1 |\Lambda_w^\alpha(1) - \Lambda_w^\alpha(t)| \left| f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right| dt \\
 &\quad + \frac{b - a}{4\Lambda_w^\alpha(1)} \int_0^1 |\Lambda_w^\alpha(1) - \Lambda_w^\alpha(t)| \left| f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \right| dt.
 \end{aligned}$$

Since $|f'|$ is a convex function on $[a, b]$, we get

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2\Lambda_w^\alpha(1)} \left[I_{\left(\frac{a+b}{2}\right)^+; w}^\alpha F(b) + I_{\left(\frac{a+b}{2}\right)^-; w}^\alpha F(a) \right] \right| \\ & \leq \frac{b-a}{4\Lambda_w^\alpha(1)} \left[|f'(a)| + |f'(b)| \right] \int_0^1 |\Lambda_w^\alpha(1) - \Lambda_w^\alpha(t)| dt \end{aligned}$$

which completes the proof. \square

Remark 4.5. If we choose $w(t) = t$ in (30), then we have the following inequality for Riemann-Liouville fractional integrals

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{2^{1-\alpha}\Gamma(\alpha + 1)}{(b-a)^\alpha} \left[J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) + J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) \right] \right| \\ & \leq \frac{b-a}{4} \left(\frac{\alpha}{\alpha + 1} \right) \left[|f'(a)| + |f'(b)| \right] \end{aligned}$$

which is proved by Budak et al. in [5].

Corollary 4.6. Under assumption of Theorem 4.4 with $w(t) = \ln t$, we have the following trapezoid type inequalities for Hadamard fractional integrals

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2\Lambda_{\ln}^\alpha(1)} \left[J_{\left(\frac{a+b}{2}\right)^+; w}^\alpha F(b) + J_{\left(\frac{a+b}{2}\right)^-; w}^\alpha F(a) \right] \right| \\ & \leq \frac{b-a}{4\Lambda_{\ln}^\alpha(1)} \left[|f'(a)| + |f'(b)| \right] \int_0^1 |\Lambda_{\ln}^\alpha(1) - \Lambda_{\ln}^\alpha(t)| dt \end{aligned}$$

where the mapping Λ_{\ln}^α is given as in (8).

Theorem 4.7. Let $\alpha > 0$ and let the mapping w as in Theorem 2.1. If $|f'|^q, q > 1$, is a convex mapping on $[a, b]$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have the following trapezoid type inequality for generalized fractional integrals

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2\Lambda_w^\alpha(1)} \left[I_{\left(\frac{a+b}{2}\right)^+; w}^\alpha F(b) + I_{\left(\frac{a+b}{2}\right)^-; w}^\alpha F(a) \right] \right| \tag{31} \\ & \leq \frac{b-a}{4\Lambda_w^\alpha(1)} \left(\int_0^1 |\Lambda_w^\alpha(1) - \Lambda_w^\alpha(t)|^p dt \right)^{\frac{1}{p}} \left[\left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right] \\ & \leq \frac{b-a}{4\Lambda_w^\alpha(1)} \left(4 \int_0^1 |\Lambda_w^\alpha(1) - \Lambda_w^\alpha(t)|^p dt \right)^{\frac{1}{p}} \left[|f'(a)| + |f'(b)| \right] \end{aligned}$$

where the mapping F is defined as in (5) and the mapping Λ_w^α is defined as in (6).

Proof. Taking modulus in Lemma 4.1 and using Hölder’s inequality, we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2\Lambda_w^\alpha(1)} \left[I_{\left(\frac{a+b}{2}\right)^+; w}^\alpha F(b) + I_{\left(\frac{a+b}{2}\right)^-; w}^\alpha F(a) \right] \right| \\ & \leq \frac{b-a}{4\Lambda_w^\alpha(1)} \left(\int_0^1 |\Lambda_w^\alpha(1) - \Lambda_w^\alpha(t)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f' \left(\frac{t}{2}a + \frac{2-t}{2}b \right) \right|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{b-a}{4\Lambda_w^\alpha(1)} \left(\int_0^1 |\Lambda_w^\alpha(1) - \Lambda_w^\alpha(t)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f' \left(\frac{2-t}{2}a + \frac{t}{2}b \right) \right|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{b-a}{4\Lambda_w^\alpha(1)} \left(\int_0^1 |\Lambda_w^\alpha(1) - \Lambda_w^\alpha(t)|^p dt \right)^{\frac{1}{p}} \left[\left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

This completes the proof of the first inequality in (31)

The proof of second inequality in (31) is obvious from the inequality (25). \square

Remark 4.8. If we choose $w(t) = t$ in (31), then we have the following inequality for Riemann-Liouville fractional integrals

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{2^{1-\alpha}\Gamma(\alpha + 1)}{(b-a)^\alpha} \left[J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) + J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) \right] \right| \\ & \leq \frac{b-a}{4} \left(\frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left[\left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right] \\ & \leq \frac{b-a}{4} \left(\frac{4}{\alpha p + 1} \right)^{\frac{1}{p}} [|f'(a)| + |f'(b)|] \end{aligned}$$

which is proved by Budak et al. in [5].

Proof. For $w(t) = t$, we get

$$\frac{b-a}{4\Lambda_w^\alpha(1)} \left(\int_0^1 |\Lambda_w^\alpha(1) - \Lambda_w^\alpha(t)|^p dt \right)^{\frac{1}{p}} = \frac{b-a}{4} \left(\int_0^1 |1 - t^\alpha|^p dt \right)^{\frac{1}{p}}.$$

Using the fact that $|t_1^\lambda - t_2^\lambda| \leq |t_1 - t_2|^\lambda$ for $\lambda \in (0, 1]$ and $\forall t_1, t_2 \in [0, 1]$, we have

$$\frac{b-a}{4} \left(\int_0^1 |1 - t^\alpha|^p dt \right)^{\frac{1}{p}} \leq \frac{b-a}{4} \left(\int_0^1 |1 - t|^{\alpha p} dt \right)^{\frac{1}{p}} = \frac{b-a}{4} \left(\frac{1}{\alpha p + 1} \right)^{\frac{1}{p}}$$

which completes the proof. \square

Corollary 4.9. *Under assumption of Theorem 4.7 with $w(t) = \ln t$, we have the following trapezoid type inequalities for Hadamard fractional integrals*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2\Lambda_{\ln}^{\alpha}(1)} \left[\mathbf{J}_{\left(\frac{a+b}{2}\right)^{+}; \tau w}^{\alpha} F(b) + \mathbf{J}_{\left(\frac{a+b}{2}\right)^{-}; \tau w}^{\alpha} F(a) \right] \right| \\ & \leq \frac{b-a}{4\Lambda_{\ln}^{\alpha}(1)} \left(\int_0^1 |\Lambda_{\ln}^{\alpha}(1) - \Lambda_{\ln}^{\alpha}(t)|^p dt \right)^{\frac{1}{p}} \left[\left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right] \\ & \leq \frac{b-a}{4\Lambda_{\ln}^{\alpha}(1)} \left(4 \int_0^1 |\Lambda_{\ln}^{\alpha}(1) - \Lambda_{\ln}^{\alpha}(t)|^p dt \right)^{\frac{1}{p}} [|f'(a)| + |f'(b)|] \end{aligned}$$

where the mapping Λ_{\ln}^{α} is given as in (8).

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