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Reeb Flow Symmetry on 3-Dimensional Almost Paracosymplectic Manifolds

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Abstract. Mainly, we prove that the Ricci operator Q of an 3-dimensional almost paracosymplectic manifold M is invariant along the Reeb flow, that is M satisfies $\mathcal{L}_\xi Q = 0$ if and only if M is an almost paracosymplectic κ-manifold with $\kappa \neq -1$.

1. Introduction

Almost (para)contact metric structure is given by a pair (η, Φ), where η is a 1-form, Φ is a 2-form and $\eta \wedge \Phi^n$ is a volume element. It is well known that then there exists a unique vector field ξ, called the characteristic (Reeb) vector field, such that $i_\xi \eta = 1$, $i_\xi \Phi = 0$. The Riemannian or pseudo-Riemannian geometry appears if we try to introduce *a compatible* structure which is a metric or pseudo-metric g and an affinor φ ((1,1)-tensor field), such that

$$\Phi(X,Y) = g(\varphi X, Y), \quad \varphi^2 = \epsilon(Id - \eta \otimes \xi). \tag{1}$$

We have almost paracontact metric structure for $\epsilon = +1$ and almost contact metric for $\epsilon = -1$. Then, the triple (φ, ξ, η) is called almost paracontact structure or almost contact structure, resp.

Combining the assumption concerning the forms η and Φ, we obtain many different types of almost (para)contact manifolds, e.g. (para)contact if η is contact form and $d\eta = \Phi$, almost (para)cosymplectic if $d\eta = 0$, $d\Phi = 0$, almost (para)Kenmotsu if $d\eta = 0$, $d\Phi = 2\eta \wedge \Phi$.

Almost paracosymplectic manifolds were studied by [6], [7]. Later, İ. Küpeli Erken et al. study almost α-paracosymplectic manifolds in [11].

A paracontact metric manifold whose characteristic vector field ξ is a harmonic vector field is called an *H-paracontact* manifold. In [1], G. Calvaruso and D. Perrone proved that ξ is *harmonic* if and only if ξ is an eigenvector of the Ricci operator for contact semi-Riemannian manifolds. G. Calvaruso and D. Perrone [2] proved that all 3-dimensional homogeneous paracontact metric manifolds are H-paracontact. Recently, İ. Küpeli Erken, P. Dacko and C. Murathan in [11] study the harmonicity of the characteristic vector field of 3-dimensional almost α-paracosymplectic manifolds. It is proved that characteristic (Reeb) vector field ξ is harmonic on almost α-para-Kenmotsu manifold if and only if it is an eigenvector of the Ricci operator. 3-dimensional almost α-para-Kenmotsu manifolds are also classified.

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A symmetry in general relativity is a smooth vector field whose local flow diffeomorphisms preserve certain mathematical or physical quantities ([8], [9]). So, one can regard it as vector fields preserving certain geometric quantities like the metric tensor, the curvature tensor or the Ricci tensor in general relativity.

In [3–5], J.T. Cho study Reeb flow symmetry on almost contact and almost cosymplectic three-manifolds. Ricci collineations on 3-dimensional paracontact metric manifolds were studied in [12]. But no effort has been made to investigate Reeb flow symmetry on 3-dimensional almost paracosymplectic manifolds.

The class of almost paracontact manifolds with which we concerned holds the properties $\mathcal{L}_\xi \xi = \mathcal{L}_\xi \eta = 0$, that is, the Reeb vector field and its associated 1-form are invariant along the Reeb flow, or the Reeb flow yields a contact transformation, which means a diffeomorphism preserving a contact form. In the present work, we study such a class of almost paracontact metric three-manifolds whose Ricci operator Q is invariant along the Reeb flow ξ, that is, $\mathcal{L}_\xi Q = 0$.

The paper is organized in the following way.

Section 2 is preliminary section, where we recall the definition of almost paracontact metric manifold and the class of almost paracontact metric manifolds which are called almost α-paracosymplectic. Section 3 is focused on harmonicity of the characteristic vector field of 3-dimensional almost paracosymplectic manifolds. In Section 4, we proved that for any 3-dimensional almost paracosymplectic κ-manifold is η-Einstein and satisfies the condition $\xi(r) = 0$, where r denotes the scalar curvature. Also we proved that the Ricci operator Q on a 3-dimensional almost paracosymplectic manifold is invariant along the Reeb vector field if and only if the manifold is an almost paracosymplectic κ-manifold with $\kappa \neq -1$.

For the case $\kappa = -1$, we proved that $\mathcal{L}_\xi Q = 0$ if and only if $\nabla_\xi Q = 0$.

2. Preliminaries

An $(2n+1)$-dimensional smooth manifold M is said to have an *almost paracontact structure* if it admits a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η satisfying the following conditions:

(i) $\eta(\xi) = 1,\ \varphi^2 = I - \eta \otimes \xi$,

(ii) the tensor field φ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, i.e. the ± 1-eigendistributions, $\mathcal{D}^{\pm} := \mathcal{D}_\varphi(\pm 1)$ of φ have equal dimension n.

From the definition it follows that $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and the endomorphism φ has rank $2n$. If an almost paracontact manifold admits a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2}$$

for all $X, Y \in \Gamma(TM)$, then we say that $(M, \varphi, \xi, \eta, g)$ is an *almost paracontact metric manifold.*

On an almost paracontact metric manifold M, if the Ricci operator satisfies

$$Q = \alpha id + \beta \eta \otimes \xi,$$

where both α and β are smooth functions, then the manifold is said to be an *η-Einstein manifold*.

Moreover, we can define a skew-symmetric tensor field (a 2-form) Φ by

$$\Phi(X,Y) = g(\varphi Y, X), \tag{3}$$

usually called fundamental form. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n+1, n)$. For an almost paracontact metric manifold, there always exists an orthogonal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$ such that $g(X_i, X_j) = \delta_{ij}$, $g(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \varphi X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a φ-basis.

On an almost paracontact manifold, one defines the $(1,2)$-tensor field $N^{(1)}$ by

$$N^{(1)}(X,Y) = [\varphi, \varphi](X,Y) - 2d\eta(X,Y)\xi,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ

$$[\varphi, \varphi](X, Y) = \varphi^2 [X, Y] + [\varphi X, \varphi Y] - \varphi [\varphi X, Y] - \varphi [X, \varphi Y].$$

If $N^{(1)}$ vanishes identically, then the almost paracontact manifold (structure) is said to be *normal* [13]. The normality condition says that the almost paracomplex structure J defined on $M \times \mathbb{R}$

$$J(X, \lambda \frac{d}{dt}) = (\varphi X + \lambda \xi, \eta(X) \frac{d}{dt}),$$

is integrable.

An almost paracontact metric manifold M^{2n+1}, with a structure (φ, ξ, η, g) is said to be an *almost α-paracosymplectic manifold*, if

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi, \tag{4}$$

where α may be a constant or function on M.

For a particular choices of the function α we have the following subclasses,

• *almost α-para-Kenmotsu manifolds*, $\alpha = const. \neq 0$,

• *almost paracosymplectic manifolds*, $\alpha = 0$.

If additionaly normality conditon is fulfilled, then manifolds are called α-para-Kenmotsu or paracosymplectic, resp.

İ. Küpeli Erken et al. proved the following results in [11]. We will use them in our original results.

Proposition 2.1. *[11] For an almost α-paracosymplectic manifold M^{2n+1}, we have*

$$\begin{aligned} i)\ \mathcal{L}_\xi \eta &= 0,\ ii)\ g(\mathcal{A}X, Y) = g(X, \mathcal{A}Y),\ iii)\ \mathcal{A}\xi = 0, \\ iv)\ \mathcal{L}_\xi \Phi &= 2\alpha\Phi,\ v)\ (\mathcal{L}_\xi g)(X, Y) = -2g(\mathcal{A}X, Y), \\ vi)\ \eta(\mathcal{A}X) &= 0,\ vii)\ d\alpha = f\eta \text{ if } n \geqslant 2 \end{aligned} \tag{5}$$

where $\mathcal{L}$ indicates the operator of the Lie differentiation, X, Y are arbitrary vector fields on M^{2n+1} and $f = i_\xi d\alpha$.

Proposition 2.2. *[11] For an almost α-paracosymplectic manifold, we have*

$$\mathcal{A}\varphi + \varphi\mathcal{A} = -2\alpha\varphi, \tag{6}$$

$$\nabla_\xi \varphi = 0. \tag{7}$$

Let define $h = \frac{1}{2}\mathcal{L}_\xi \varphi$. In the following proposition we establish some properties of the tensor field h.

Proposition 2.3. *[11]For an almost α-paracosymplectic manifold, we have the following relations*

$$g(hX, Y) = g(X, hY), \tag{8}$$

$$h \circ \varphi + \varphi \circ h = 0, \tag{9}$$

$$h\xi = 0, \tag{10}$$

$$\nabla \xi = \alpha\varphi^2 + \varphi \circ h = -\mathcal{A}. \tag{11}$$

Corollary 2.4. *[11]All the above Propositions imply the following formulas for the traces*

$$\begin{aligned} tr(\mathcal{A}\varphi) &= tr(\varphi\mathcal{A}) = 0,\ tr(h\varphi) = tr(\varphi h) = 0, \\ tr(\mathcal{A}) &= -2\alpha n,\ \ tr(h) = 0. \end{aligned} \tag{12}$$

Theorem 2.5. *[11]Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold. Then, for any $X, Y \in \chi(M^{2n+1})$,*

$$R(X, Y)\xi = \alpha\eta(X)(\alpha Y + \varphi hY) - \alpha\eta(Y)(\alpha X + \varphi hX) + (\nabla_X \varphi h)Y - (\nabla_Y \varphi h)X. \tag{13}$$

Theorem 2.6. *[11]Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold. Then, for any $X \in \chi(M^{2n+1})$ we have*

$$\begin{aligned} R(\xi, X)\xi &= \alpha^2\varphi^2 X + 2\alpha\varphi hX - h^2X + \varphi(\nabla_\xi h)X, && (14)\\ (\nabla_\xi h)X &= -\alpha^2\varphi X - 2\alpha hX + \varphi h^2 X - \varphi R(X, \xi)\xi, && (15)\\ \frac{1}{2}(R(\xi, X)\xi + \varphi R(\xi, \varphi X)\xi) &= \alpha^2\varphi^2 X - h^2 X, && (16)\\ S(X, \xi) &= -2n\alpha^2\eta(X) + g(div(\varphi h), X), && (17)\\ S(\xi, \xi) &= -2n\alpha^2 + trh^2 && (18) \end{aligned}$$

where $S(X, Y) = g(QX, Y)$.

Henceforward, we denote $S_{ij} = S(e_i, e_j)$ for $i, j = 1, 2, 3$.

3. Classification of the 3-Dimensional Almost Paracosymplectic Manifolds

In this section, we will give the summary of the classification of 3-dimensional almost paracosymplectic manifolds. 3-dimensional almost paracosymplectic manifolds under assumption that the curvature satisfies (κ, μ, ν)-nullity condition

$$R(X, Y)\xi = \eta(Y)BX - \eta(X)BY, \tag{19}$$

where B is Jacobi operator of ξ, $BX = R(X, \xi)\xi$, and

$$BX = \kappa\varphi^2 X + \mu hX + \nu\varphi hX,$$

for all $X, Y \in \Gamma(TM)$, where κ, μ, ν are smooth functions on M. Particularly $B\xi = 0$.

If an almost paracosymplectic manifold satisfies (19), then the manifold is said to be *almost paracosymplectic (κ, μ, ν)-space.*

A 3-dimensional almost paracosymplectic manifold κ-manifold satisfies [11]

$$Q\xi = 2\kappa\xi. \tag{20}$$

Theorem 3.1. *[11]Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold. Characteristic vector field ξ is harmonic if and only if it is an eigenvector of the Ricci operator.*

Beside the other results, the different possibilities for the tensor field h are analyzed in [11].

The tensor h has the canonical form (I). Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional almost α-paracosymplectic manifold. Then operator h has following types.

$$\begin{aligned} U_1 &= \{p \in M \mid h(p) \neq 0\} \subset M\\ U_2 &= \{p \in M \mid h(p) = 0, \text{in a neighborhood of p}\} \subset M \end{aligned}$$

That h is a smooth function on M implies $U_1 \cup U_2$ is an open and dense subset of M, so any property satisfied in $U_1 \cup U_2$ is also satisfied in M. For any point $p \in U_1 \cup U_2$ there exists a local orthonormal φ-basis $\{e, \varphi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p, where $-g(e, e) = g(\varphi e, \varphi e) = g(\xi, \xi) = 1$. On U_1 we put $he = \lambda e$, where λ is a non-vanishing smooth function. Since $trh = 0$, we have $h\varphi e = -\lambda\varphi e$. The eigenvalue function λ is continuous on M and smooth on $U_1 \cup U_2$. So, h has following form

$$\begin{pmatrix} \lambda & 0 & 0\\ 0 & -\lambda & 0\\ 0 & 0 & 0 \end{pmatrix} \tag{21}$$

respect to local orthonormal φ-basis $\{e, \varphi e, \xi\}$. In this case, we will say the operator h is of $\mathfrak{h}_1$ *type.*

Lemma 3.2. *[11]Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\mathfrak{h}_1$ type. Then for the covariant derivative on $\mathcal{U}_1$ the following equations are valid*

$$\begin{aligned}
&i)\ \nabla_e e = \frac{1}{2\lambda}\left[\sigma(e) - (\varphi e)(\lambda)\right]\varphi e + \alpha\xi, \\
&ii)\ \nabla_e \varphi e = \frac{1}{2\lambda}\left[\sigma(e) - (\varphi e)(\lambda)\right] e - \lambda\xi, \\
&iii)\ \nabla_e \xi = \alpha e + \lambda\varphi e, \\
&iv)\ \nabla_{\varphi e} e = -\frac{1}{2\lambda}\left[\sigma(\varphi e) + e(\lambda)\right]\varphi e - \lambda\xi, \\
&v)\ \nabla_{\varphi e}\varphi e = -\frac{1}{2\lambda}\left[\sigma(\varphi e) + e(\lambda)\right]\varphi e - \alpha\xi, \\
&vi)\ \nabla_{\varphi e}\xi = \alpha\varphi e - \lambda e, \\
&vii)\ \nabla_\xi e = a_1\varphi e,\ \ viii)\ \nabla_\xi \varphi e = a_1 e, \\
&ix)\ [e, \xi] = \alpha e + (\lambda - a_1)\varphi e, \\
&x)\ [\varphi e, \xi] = -(\lambda + a_1)e + \alpha\varphi e, \\
&xi)\ [e, \varphi e] = \frac{1}{2\lambda}\left[\sigma(e) - (\varphi e)(\lambda)\right] e + \frac{1}{2\lambda}\left[\sigma(\varphi e) + e(\lambda)\right]\varphi e, \\
&xii)\ h^2 - \alpha^2\varphi^2 = \frac{1}{2}S(\xi, \xi)\varphi^2
\end{aligned} \tag{22}$$

where

$$a_1 = g(\nabla_\xi e, \varphi e),\ \sigma = S(\xi, .)_{\ker\eta}\ .$$

Proposition 3.3. *[11]Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\mathfrak{h}_1$ type. Then we have*

$$\nabla_\xi h = -2a_1 h\varphi + \xi(\lambda)s, \tag{23}$$

where s is the $(1, 1)$-type tensor defined by $s\xi = 0$, $se = e$, $s\varphi e = -\varphi e$.

Lemma 3.4. *[11]Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\mathfrak{h}_1$ type. Then the Ricci operator Q is given by*

$$\begin{aligned}
Q &= \left(\frac{r}{2} + \alpha^2 - \lambda^2\right)I + \left(-\frac{r}{2} + 3(\lambda^2 - \alpha^2)\right)\eta\otimes\xi - 2\alpha\varphi h - \varphi(\nabla_\xi h) \\
&\quad + \sigma(\varphi^2)\otimes\xi - \sigma(e)\eta\otimes e + \sigma(\varphi e)\eta\otimes\varphi e
\end{aligned} \tag{24}$$

where r denotes scalar curvature.

Moreover from (24) the components of the Ricci operator Q are can be given by

$$\begin{aligned}
Q\xi &= 2(\lambda^2 - \alpha^2)\xi - \sigma(e)e + \sigma(\varphi e)\varphi e, \\
Qe &= \sigma(e)\xi + \left(\frac{r}{2} + \alpha^2 - \lambda^2 - 2a_1\lambda\right)e - (2\alpha\lambda + \xi(\lambda))\varphi e, \\
Q\varphi e &= \sigma(\varphi e)\xi + (2\alpha\lambda + \xi(\lambda))e + \left(\frac{r}{2} + \alpha^2 - \lambda^2 + 2a_1\lambda\right)\varphi e.
\end{aligned} \tag{25}$$

From (25), we get

$$S_{11} = -\left(\frac{r}{2} + \alpha^2 - \lambda^2 - 2a_1\lambda\right), S_{12} = -(2\alpha\lambda + \xi(\lambda)), S_{22} = \left(\frac{r}{2} + \alpha^2 - \lambda^2 + 2a_1\lambda\right), S_{11} + S_{22} = 4a_1\lambda. \tag{26}$$

The tensor h has the canonical form (II). Using same methods in [10], one can construct a local pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ in a neighborhood of p where $g(e_1, e_1) = g(e_2, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0$ and

$g(e_1, e_2) = g(e_3, e_3) = 1$. Let $\mathcal{U}$ be the open subset of M where $h \neq 0$. For every $p \in \mathcal{U}$ there exists an open neighborhood of p such that $he_1 = e_2, he_2 = 0, he_3 = 0$ and $\varphi e_1 = \pm e_1, \varphi e_2 = \mp e_2, \varphi e_3 = 0$ and also $\xi = e_3$. Thus the tensor h has the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{27}$$

relative a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$. In this case, we call h is of $\mathfrak{h}_2$ *type*.

Remark 3.5. *Without loss of generality, we can assume that* $\varphi e_1 = e_1$ $\varphi e_2 = -e_2$. *Moreover one can easily get* $h^2 = 0$ *but* $h \neq 0$.

Lemma 3.6. *[11]Let* $(M, \varphi, \xi, \eta, g)$ *be a 3-dimensional almost α-para-Kenmotsu manifold with h of* $\mathfrak{h}_2$ *type. Then for the covariant derivative on* $\mathcal{U}$ *the following equations are valid*

$$\begin{aligned} i)\ \nabla_{e_1} e_1 &= -b_1 e_1 + \xi, \ \ ii)\ \nabla_{e_1} e_2 = b_1 e_2 - \alpha\xi, \ \ iii)\ \nabla_{e_1}\xi = \alpha e_1 - e_2, \\ iv)\ \nabla_{e_2} e_1 &= -b_2 e_1 - \alpha\xi, \ \ v)\ \nabla_{e_2} e_2 = b_2 e_2, \ \ vi)\ \nabla_{e_2}\xi = \alpha e_2, \\ vii)\ \nabla_{\xi} e_1 &= a_2 e_1, \ \ viii)\ \nabla_{\xi} e_2 = -a_2 e_2, \\ ix)\ [e_1, \xi] &= (\alpha - a_2)e_1 - e_2, \ \ x)\ [e_2, \xi] = (\alpha + a_2)e_2, \\ xi)\ [e_1, e_2] &= b_2 e_1 + b_1 e_2, \\ xii)\ h^2 &= 0. \end{aligned} \tag{28}$$

where $a_2 = g(\nabla_\xi e_1, e_2)$, $b_1 = g(\nabla_{e_1} e_2, e_1)$ *and* $b_2 = g(\nabla_{e_2} e_2, e_1) = -\frac{1}{2}\sigma(e_1)$.

Proposition 3.7. *[11]Let* $(M, \varphi, \xi, \eta, g)$ *be a 3-dimensional almost α-para-Kenmotsu manifold with h of* $\mathfrak{h}_2$ *type. Then we have*

$$\nabla_\xi h = 2a_2 \varphi h, \tag{29}$$

on $\mathcal{U}$.

Lemma 3.8. *[11]Let* $(M, \varphi, \xi, \eta, g)$ *be a 3-dimensional almost α-para-Kenmotsu manifold with h of* $\mathfrak{h}_2$ *type. Then the Ricci operator Q is given by*

$$Q = (\frac{r}{2} + \alpha^2)I - (\frac{r}{2} + 3\alpha^2)\eta \otimes \xi - 2\alpha\varphi h - \varphi(\nabla_\xi h) + \sigma(\varphi^2) \otimes \xi + \sigma(e_1)\eta \otimes e_2. \tag{30}$$

A consequence of Lemma 3.8, we can give the components of the Ricci operator Q by following,

$$\begin{aligned} Q\xi &= \sigma(e_1)e_2 - 2\alpha^2\xi, \\ Qe_1 &= \sigma(e_1)\xi + (\frac{r}{2} + \alpha^2)e_1 - 2(a_2 - \alpha)e_2, \\ Qe_2 &= (\frac{r}{2} + \alpha^2)e_2. \end{aligned} \tag{31}$$

The tensor h has the canonical form (III). We can find a local orthonormal φ-basis $\{e, \varphi e, \xi\}$ in a neighborhood of p where $-g(e, e) = g(\varphi e, \varphi e) = g(\xi, \xi) = 1$. Now, let $\mathcal{U}_1$ be the open subset of M where $h \neq 0$ and let $\mathcal{U}_2$ be the open subset of points $p \in M$ such that $h = 0$ in a neighborhood of p. $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open subset of M. For every $p \in \mathcal{U}_1$ there exists an open neighborhood of p such that $he = \lambda\varphi e, h\varphi e = -\lambda e$ and $h\xi = 0$ where λ is a non-vanishing smooth function. Since $trh = 0$, the matrix form of h is given by

$$\begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{32}$$

with respect to local orthonormal basis $\{e, \varphi e, \xi\}$. In this case, we say that h is of $\mathfrak{h}_3$ *type*.

Lemma 3.9. *[11]Let (M,φ,ξ,η,g) be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\mathfrak{h}_3$ type. Then for the covariant derivative on $\mathcal{U}_1$ the following equations are valid*

$$\begin{aligned} i)\ \nabla_e e &= b_3\varphi e + (\alpha+\lambda)\xi,\ \ ii)\ \nabla_e\varphi e = b_3 e,\ \ iii)\ \nabla_e\xi = (\alpha+\lambda)e,\\ iv)\ \nabla_{\varphi e}e &= b_4\varphi e,\ \ v)\ \nabla_{\varphi e}\varphi e = b_4 e + (\lambda-\alpha)\xi,\ \ vi)\ \nabla_{\varphi e}\xi = -(\lambda-\alpha)\varphi e,\\ vii)\ \nabla_\xi e &= a_3\varphi e,\ \ viii)\ \nabla_\xi\varphi e = a_3 e,\\ ix)\ [e,\xi] &= (\alpha+\lambda)e - a_3\varphi e,\ \ x)\ [\varphi e,\xi] = -a_3 e - (\lambda-\alpha)\varphi e,\\ xi)\ [e,\varphi e] &= b_3 e - b_4\varphi e,\\ xii) h^2 - \alpha^2\varphi^2 &= \frac{1}{2}S(\xi,\xi)\varphi^2, \end{aligned} \tag{33}$$

where $a_3 = g(\nabla_\xi e,\varphi e), b_3 = -\frac{1}{2\lambda}\left[\sigma(\varphi e) + (\varphi e)(\lambda)\right]$ *and* $b_4 = \frac{1}{2\lambda}\left[\sigma(e) - e(\lambda)\right].$

Proposition 3.10. *[11]Let (M,φ,ξ,η,g) be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\mathfrak{h}_3$ type. So, on U_1 we have*

$$\nabla_\xi h = -2a_3 h\varphi + \xi(\lambda)s, \tag{34}$$

where s is the $(1,1)$-type tensor defined by $s\xi = 0$, $se = \varphi e$, $s\varphi e = -e$.

Lemma 3.11. *[11]Let (M,φ,ξ,η,g) be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\mathfrak{h}_3$ type. Then the Ricci operator Q is given by*

$$Q = a\,I + b\eta\otimes\xi - 2\alpha\varphi h - \varphi(\nabla_\xi h) + \sigma(\varphi^2)\otimes\xi - \sigma(e)\eta\otimes e + \sigma(\varphi e)\eta\otimes\varphi e, \tag{35}$$

where a and b are smooth functions defined by $a = \alpha^2 + \lambda^2 + \frac{r}{2}$ and $b = -3(\lambda^2+\alpha^2) - \frac{r}{2}$, respectively.

Moreover from the above Lemma the components of the Ricci operator Q are given by

$$\begin{aligned} Q\xi &= -2(\alpha^2+\lambda^2)\xi - \sigma(e)e + \sigma(\varphi e)\varphi e,\\ Qe &= \sigma(e)\xi + (\alpha^2+\lambda^2+\frac{r}{2} - \xi(\lambda))e - 2a_3\lambda\varphi e,\\ Q\varphi e &= \sigma(\varphi e)\xi + 2a_3\lambda e + (\alpha^2+\lambda^2+\frac{r}{2}+\xi(\lambda))\varphi e. \end{aligned} \tag{36}$$

From (36), we get

$$S_{11} = -(\alpha^2+\lambda^2+\frac{r}{2}-\xi(\lambda)),\ S_{12} = -2a_3\lambda,\ S_{22} = (\alpha^2+\lambda^2+\frac{r}{2}+\xi(\lambda)),\ S_{11}+S_{22} = 2\xi(\lambda). \tag{37}$$

Theorem 3.12. *[11]Let (M,φ,ξ,η,g) be a 3-dimensional almost α-para-Kenmotsu manifold. If the characteristic vector field ξ is harmonic map then almost α-paracosymplectic (κ,μ,ν)-manifold always exist on every open and dense subset of M. Conversely, if M is an almost α-paracosymplectic (κ,μ,ν)-manifold with constant α then the characteristic vector field ξ is harmonic map.*

4. Reeb Flow Symmetry on 3-Dimensional Almost Paracosymplectic Manifolds

In this section, we will study reeb flow symmetry on 3-dimensional almost paracosymplectic manifolds. So, we will take $\alpha = 0$ in results which were given in Section 3.

We recall that the curvature tensor of a 3-dimensional pseudo-Riemannian manifold satisfies

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y) \tag{38}$$

for all vector fields X,Y,Z, where r denotes the scalar curvature

First of all, we will investigate three possibilities according to canonical form h of 3-dimensional almost paracosymplectic manifold.

Case1: We suppose that h is $\mathfrak{h}_1$ type $(\kappa > -1)$.

Lemma 4.1. *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional almost paracosymplectic manifold. If h is $\mathfrak{h}_1$ type on U_1, then we have,*

$\mathcal{L}_\xi Q = 0$ *if and only if* $\nabla_\xi Q = 0$ *and* $Q\xi = \rho\xi$*, where* ρ *is a function.*

Proof. Assume that M satisfies $\mathcal{L}_\xi Q = 0$. In this case, we have

$$\begin{aligned} \mathcal{L}_\xi(QX) - Q(\pounds_\xi X) &= 0 \\ [\xi, QX] - Q[\xi, X] &= 0. \end{aligned}$$

From (11), we obtain an equivalent equation to $\mathcal{L}_\xi Q = 0$ as follows

$$(\nabla_\xi Q)X = (\varphi h Q - Q\varphi h)X. \tag{39}$$

Since $\nabla_\xi Q$ is self-adjoint operator, it follows that

$$Q\varphi h - \varphi h Q = Qh\varphi - h\varphi Q.$$

Using the anti-commutative property h with φ in the last equation, we have

$$Q\varphi h = \varphi h Q. \tag{40}$$

Hence, from (39) and (40), we get $\nabla_\xi Q = 0$ on U_1. Applying ξ to both sides of (40), we get $hQ\xi = 0$.
Using this in the first equation of (25), we obtain $Q\xi = \rho\xi$, $\rho = 2\lambda^2$ on U_1.
Conversely, we assume that $\nabla_\xi Q = 0$ and $Q\xi = \rho\xi$, on U_1. By (18), we find that $\rho = 2\lambda^2$ and

$$S_{13} = S_{31} = 0, \quad S_{23} = S_{32} = 0. \tag{41}$$

After some calculations using the fact that $(\nabla_\xi S)(\xi, \xi) = 0$ and $\nabla_\xi \xi = 0$, one can get

$$\xi(\lambda) = 0. \tag{42}$$

Using the second equation of (25), we obtain $S_{12} = g(Qe_1, e_2) = -\xi(\lambda) = 0$. So we have

$$S_{12} = S_{21} = 0. \tag{43}$$

If we take the covariant derivative of (43) according to ξ and use (22) and $\nabla_\xi Q = 0$, we obtain

$$a_1(S_{22} + S_{11}) = 0. \tag{44}$$

By the help of (26) and (44) we find $a_1 = 0$ and

$$S_{11} = -S_{22}. \tag{45}$$

From the assumption of $Q\xi = \rho\xi$ and the equations (41), (43) and (45) we get

$$\begin{aligned} Qe &= (\frac{r}{2} - \lambda^2)e \\ Q\varphi e &= (\frac{r}{2} - \lambda^2)\varphi e. \end{aligned} \tag{46}$$

So, we can see $Q\varphi h = \varphi h Q$ by using (46). Hence, $\mathcal{L}_\xi Q = 0$ comes from (39). □

Remark 4.2. *In Lemma 4.1, for a 3-dimensional almost paracosymplectic manifold with h is $\mathfrak{h}_1$ type on U_1, we proved that if $\nabla_\xi Q = 0$ and $Q\xi = \rho\xi$, then $\xi(\lambda) = 0$. Now, accept $\mathcal{L}_\xi Q = 0$ on U_1. Using $(\nabla_\xi S)(\xi, \xi) = 0$ and $\nabla_\xi \xi = 0$, one can get $\xi(\lambda) = 0$. Also, by definition of Ricci curvature S, we have $S_{12} = S_{21} = 0$. From (40) we have $S_{22} = -S_{11} = 0$, $a_1 = 0$. Moreover, one can write $r = -S(e, e) + S(\varphi e, \varphi e) + S(\xi, \xi) = 2(S_{22} + \lambda^2) = 2(-S_{11} + \lambda^2)$.*

We now check whether λ is constant or not.

In view of (38), Lemma 4.1 and Remark 4.2, the following formulas hold in U_1

$$\begin{aligned} R(e,\varphi e)\varphi e &= Qe-\lambda^2 e,\\ R(e,\varphi e)e &= Q\varphi e-\lambda^2\varphi e,\\ R(\varphi e,\xi)\varphi e &= -\lambda^2\xi,\\ R(e,\xi)e &= \lambda^2\xi,\\ R(e,\xi)\xi &= \lambda^2 e,\\ R(\varphi e,\xi)\xi &= \lambda^2\varphi e, \end{aligned} \tag{47}$$

where $R(e_i,e_j)e_k=0$, for $i\neq j\neq k$.

On the other hand, taking into account, (22) and (47), direct calculations give

$$\begin{aligned} (\nabla_e R)(\varphi e,\xi)\varphi e &= -e(\lambda^2)\xi,\\ (\nabla_{\varphi e} R)(\xi,e)\varphi e &= 0,\\ (\nabla_\xi R)(e,\varphi e)\varphi e &= \xi(\frac{r}{2}-\lambda^2)e,\\ (\nabla_{\varphi e} R)(e,\xi)e &= \varphi e(\lambda^2)\xi,\\ (\nabla_e R)(\xi,\varphi e)e &= 0,\\ (\nabla_\xi R)(\varphi e,e)e &= -\xi(\frac{r}{2}-\lambda^2)\varphi e. \end{aligned} \tag{48}$$

With the help of second bianchi identity and (48), we find $e(\lambda)=0$ and $\varphi e(\lambda)=0$. Regarding $\xi(\lambda)=0$, we can conclude that λ is constant on M.

So we can state following

Lemma 4.3. *λ is constant.*

Using Lemma 4.3, (22) returns to

$$\begin{aligned} i)\ \nabla_e e &= 0,\ \ ii)\ \nabla_e\varphi e=-\lambda\xi,\\ iii)\ \nabla_e\xi &= \lambda\varphi e,\\ iv)\ \nabla_{\varphi e}e &= -\lambda\xi,\ \ v)\ \nabla_{\varphi e}\varphi e=0,\\ vi)\ \nabla_{\varphi e}\xi &= -\lambda e,\\ vii)\ \nabla_\xi e &= 0,\ \ viii)\ \nabla_\xi\varphi e=0,\\ ix)\ [e,\xi] &= \lambda\varphi e,\ \ x)\ [\varphi e,\xi]=-\lambda e,\\ xi)\ [e,\varphi e] &= 0. \end{aligned} \tag{49}$$

In view of (47) and (49), we have

$$Qe=0,\ Q\varphi e=0,\ Q\xi=2\lambda^2\xi. \tag{50}$$

From (50) we can easily see that $(\mathcal{L}_\xi Q)e=(\mathcal{L}_\xi Q)\varphi e=0$.

Case2: We suppose that h is $\mathfrak{h}_3$ type ($\kappa<-1$).

As the proof of the following lemma is similar to Lemma 4.1, we don't give its proof.

Lemma 4.4. *Let (M,φ,ξ,η,g) be a 3-dimensional almost paracosymplectic manifold. If h is $\mathfrak{h}_3$ type on U_1, then we have,*

$\mathcal{L}_\xi Q=0$ if and only if $\nabla_\xi Q=0$ and $Q\xi=\rho\xi$, where ρ is a function.

We now check whether λ is constant or not.

In view of (38) and Lemma 4.4, the following formulas hold in U_1

$$\begin{aligned} R(e,\varphi e)\varphi e &= Qe+\lambda^2 e,\\ R(e,\varphi e)e &= Q\varphi e+\lambda^2\varphi e,\\ R(\varphi e,\xi)\varphi e &= \lambda^2\xi,\\ R(e,\xi)e &= -\lambda^2\xi,\\ R(e,\xi)\xi &= -\lambda^2 e,\\ R(\varphi e,\xi)\xi &= -\lambda^2\varphi e, \end{aligned} \tag{51}$$

where $R(e_i,e_j)e_k=0$, for $i\neq j\neq k$.

On the other hand, taking into account, (33) and (51), direct calculations give

$$\begin{aligned} (\nabla_e R)(\varphi e,\xi)\varphi e &= \lambda(\frac{r}{2}+3\lambda^2)e+e(\lambda^2)\xi,\\ (\nabla_{\varphi e} R)(\xi,e)\varphi e &= -\lambda(\frac{r}{2}+3\lambda^2)e,\\ (\nabla_\xi R)(e,\varphi e)\varphi e &= \xi(\frac{r}{2}+\lambda^2)e,\\ (\nabla_{\varphi e} R)(e,\xi)e &= -\varphi e(\lambda^2)\xi+\lambda(\frac{r}{2}+3\lambda^2)\varphi e,\\ (\nabla_e R)(\xi,\varphi e)e &= -\lambda(\frac{r}{2}+3\lambda^2)\varphi e,\\ (\nabla_\xi R)(\varphi e,e)e &= -\xi(\frac{r}{2}+\lambda^2)\varphi e. \end{aligned} \tag{52}$$

With the help of second bianchi identity and (52), we find $e(\lambda)=0$ and $\varphi e(\lambda)=0$. Regarding $\xi(\lambda)=0$, we can conclude that λ is constant on M.

So we can state following

Lemma 4.5. *λ is constant.*

Using Lemma 4.5, (33) returns to

$$\begin{aligned} i)\ \nabla_e e &= \lambda\xi,\ \ ii)\ \nabla_e\varphi e=0,\\ iii)\ \nabla_e\xi &= \lambda e,\\ iv)\ \nabla_{\varphi e}e &= 0,\ \ v)\ \nabla_{\varphi e}\varphi e=\lambda\xi,\\ vi)\ \nabla_{\varphi e}\xi &= -\lambda\varphi e,\\ vii)\ \nabla_\xi e &= 0,\ \ viii)\ \nabla_\xi\varphi e=0,\\ ix)\ [e,\xi] &= \lambda e,\ \ x)\ [\varphi e,\xi]=-\lambda\varphi e,\\ xi)\ [e,\varphi e] &= 0. \end{aligned} \tag{53}$$

In view of (51) and (53), we have

$$Qe=0,\ Q\varphi e=0,\ Q\xi=-2\lambda^2\xi. \tag{54}$$

From (54) we can easily see that $(\mathcal{L}_\xi Q)e=(\mathcal{L}_\xi Q)\varphi e=0$.

Theorem 4.6. *Any 3-dimensional almost paracosymplectic κ-manifold is η-Einstein and also we have*

$$\xi(r)=0. \tag{55}$$

Proof. If we replace $Y = Z$ by ξ in (38) and use (19), (20) we get

$$QX = \left(\frac{r}{2} - \kappa\right)X + \left(-\frac{r}{2} + 3\kappa\right)\eta(X)\xi \tag{56}$$

for any vector field $X \in \chi(M)$. So, the manifold is η-Einstein. If we use (56), (11) and (20) in the following well known formula for semi-Riemannian manifolds

$$trace\,\{Y \to (\nabla_Y Q)X\} = \frac{1}{2}\nabla_X r$$

we obtain $\xi(r) = 0$. □

Theorem 4.7. *Let M be a 3-dimensional almost paracosymplectic manifold. Then $\mathcal{L}_\xi Q = 0$ if and only if M is an almost paracosymplectic κ-manifold with $\kappa \neq -1$.*

Proof. Assume that M is a 3-dimensional almost paracosymplectic manifold with h of $\mathfrak{h}_1$ type whose Ricci operator Q satisfies $\mathcal{L}_\xi Q = 0$. If we take into account Theorem 3.1, Theorem 3.12 and Lemma 4.1, together we obtain that M is an almost paracosymplectic κ-manifold with $\kappa = \lambda^2$. Conversely, let M is an almost paracosymplectic κ-manifold with $\kappa \neq -1$. Using Lemma 4.3 and (55), if we take the Lie derivative of (56) according to ξ, we get $\mathcal{L}_\xi Q = 0$. The proof for a 3-dimensional almost paracosymplectic manifold with h of $\mathfrak{h}_3$ type is similar to this proved case. So, we complete the proof of the theorem. □

Case3: We suppose that h is $\mathfrak{h}_2$ type ($\kappa = -1$).

The proof of following theorem is similar to the Case1(h is $\mathfrak{h}_1$ type). But in this case, one can should be careful while computing because of $g(e_1, e_1) = g(e_2, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0$ and $g(e_1, e_2) = g(e_3, e_3) = 1$.

Theorem 4.8. *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional almost paracosymplectic manifold. If h is $\mathfrak{h}_2$ type on U, then we have,*

$$\mathcal{L}_\xi Q = 0 \textit{ if and only if } \nabla_\xi Q = 0.$$

Remark 4.9. *For a 3-dimensional almost paracosymplectic manifold with h is $\mathfrak{h}_2$ type on U. Then $\mathcal{L}_\xi Q = 0$ if and only if $\xi(\sigma(e_1)) - a_2\sigma(e_1) = 0$, $\xi(r) = 0$ and $\xi(a_2) - 2a_2^2 = 0$. Using (28) and (31), one can calculate these relations.*

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