



Stability of the Equilibria in a Discrete-Time Sivs Epidemic Model with Standard Incidence

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Abstract. A discrete-time SIS epidemic model with vaccination is presented and studied. The model includes deaths due to disease and the total population size is variable. First, existence and positivity of the solutions are discussed and equilibria of the model and basic reproduction number are obtained. Next, the stability of the equilibria is studied and conditions of stability are obtained in terms of the basic reproduction number \mathcal{R}_0 . Also, occurrence of the fold bifurcation, the flip bifurcation, and the Neimark-Sacker bifurcation is investigated at equilibria. In addition, obtained results are numerically discussed and some diagrams for bifurcations, Lyapunov exponents, and solutions of the model are presented.

1. Introduction

Most of epidemic models has been formulated and analyzed as differential equations that yields to continuous-time models[1–4]. However, in recent years the interest to discrete-time models which are based on difference equations has been increased[5–8]. It may be for some reasons such as reacher dynamics of a discrete model in compare with a continuous one, availability of discrete data in time increments, necessity of discretization continuous models when a numerical solution is required, etc. The probably most well-known epidemic model, the SIS model, has been formulated in discrete formulation and analyzed in many works[9, 10]. Among them some models include vaccination as a efficient strategy to control and eliminate the disease[11–15]. These models consider the vaccinated individuals as a separate component in the model. The vaccination may be permanent or temporal, perfect (with efficiency %100) or imperfect (with efficiency less than %100). In this paper we introduce and analyze a discrete-time SIS epidemic model with a temporary vaccination program, namely SIVS model. The structure of the paper is as follows: In the next section the SIVS epidemic model is developed and after describing its components, parameters, and how to transmit individuals between components, we obtain equilibria of the model and the basic reproduction number. Section 3 is devoted to study the stability of the equilibria and in section 4 the bifurcations of the model are investigated. Finally, after a numerical discussion about the theoretical results obtained in other sections, we end the paper with conclusions.

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2. The SIS model with vaccination

2.1. Description of the model

Consider a population with $N(t)$ individuals at time t from them $S(t)$ individuals are susceptible, $I(t)$ individuals are infected, and $V(t)$ individuals are vaccinated. In a simple SIS epidemic model (an SIS model without vital dynamics) the susceptible individuals are be infected and then after a period of time they will be recovered from infection and become susceptible again. The number of cases that become infected per unit time is $\beta(N)SI/N$ in which $\beta(N)$ is the contact rate, i.e., the number of successful contacts for each individual per unit time. Thus the average number of successful contact of an infected individual with a susceptible individual is $\lambda = \beta(N)I/N$ per unit time. The expression λ is called force of infection. Therefore with S susceptible individuals, the all number of successful contacts due to I infected individuals will be $\lambda S = \beta(N)SI/N$ per unit time that is called the incidence rate and we denote it by Υ . When $\beta(N) = \beta$ the incidence is called standard incidence and when $\beta(N) = \beta N$ it is called mass-action incidence or bilinear incidence[16]. The under study model considers not only natural deaths, but also deaths due to disease. Moreover, the recruitment in population consists of newborns and immigrants. The vaccination program is performed on both new individuals added to the population and susceptible individuals. The vaccination is assumed to be perfect and thus the vaccinated individuals don't become infected. However, the immunity caused by vaccination is temporary and after a period of time it is lost and individuals become susceptible again. Figure 1 shows the SIVS epidemic model and transmission rates (probabilities) between its compartments. All changes in population and transmissions of individuals between compartments take place per unit time (during time interval) Δt and it is assumed that follow a uniform distribution. The probability that a susceptible individual becomes infected during time interval Δt is $\Upsilon = \frac{\beta IS}{N}$ that is assumed as the standard incidence rate. The SIVS epidemic model is given by the following system of difference equations:

$$\begin{aligned} I(t + \Delta t) &= \Upsilon(t) + [1 - (\mu + \gamma + \alpha)]I(t), \\ S(t + \Delta t) &= (1 - \sigma)\Lambda - \Upsilon(t) + [1 - (\mu + \varphi)]S(t) + \gamma I(t) + \psi V(t), \\ V(t + \Delta t) &= \sigma\Lambda + \varphi S(t) + [1 - (\mu + \psi)]V(t), \end{aligned} \tag{1}$$

where $\Upsilon(t) = \beta S(t)I(t)/N(t)$. Table 1 explains the parameters of the model.

It is assumed all parameters are in interval $(0, 1)$. Then it can be seen that with positive initial values for

Table 1: Parameters and explanations in the SIVS epidemic model (1).

Parameter	Explanation
$\frac{\beta I}{N}$	probability a susceptible individual becomes infected
γ	probability an infected individual recovers from infection
φ	probability a susceptible individual becomes vaccinated
ψ	probability a vaccinated individual loses immunity
σ	probability a new member becomes vaccinated
μ	probability an individual dies from natural reasons
α	probability an individual dies from infection

$S(0), I(0)$ and $V(0)$, solutions of system (1) are non-negative if the following conditions hold:

$$\begin{aligned} 0 &< \mu + \varphi + \beta < 1, \\ 0 &< \mu + \gamma + \alpha < 1, \\ 0 &< \mu + \varphi + \psi < 1. \end{aligned} \tag{2}$$

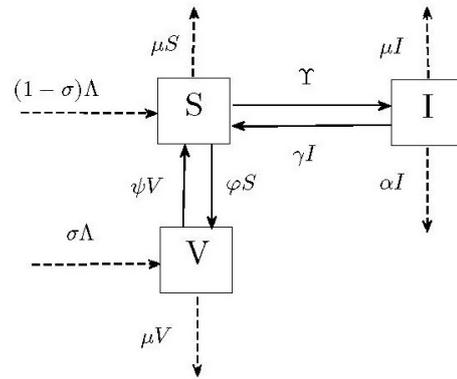


Figure 1: Diagram of the SIVS epidemic model and transmissions there in.

By adding the equations in system (1) we see that the number of all individuals in population obeys from the following difference equation:

$$N(t + \Delta t) = \Lambda + (1 - \mu)N(t) - \alpha I(t). \tag{3}$$

This implies

$$\begin{aligned} N(t + \Delta t) &\leq \Lambda + (1 - \mu)N(t) \\ &\leq \Lambda [1 + (1 - \mu)] + (1 - \mu)^2 N(t - \Delta t) \\ &\leq \Lambda [1 + (1 - \mu) + (1 - \mu)^2] + (1 - \mu)^3 N(t - 2\Delta t) \\ &\vdots \\ &\leq \Lambda \left[\frac{1 - (1 - \mu)^{n+1}}{1 - (1 - \mu)} \right] + (1 - \mu)^{n+1} N(t - n\Delta t) \\ &= \Lambda \left[\frac{1 - (1 - \mu)^{n+1}}{\mu} \right] + (1 - \mu)^{n+1} N(0) \\ &\leq \frac{\Lambda}{\mu} + N(0), \end{aligned}$$

for some $n \in \mathbb{N}$, $t = n\Delta t$. Thus not only $N(t)$ but also $S(t)$, $I(t)$ and $V(t)$ are bounded and moreover $\limsup_{t \rightarrow \infty} N(t) \leq \Lambda/\mu$.

2.2. Equilibria of the model

From system (1) we find two following equilibria for the model:

- (1) The disease-free equilibrium E_{df} in which $I = 0$:

$$E_{df} = (I_{df}, S_{df}, V_{df}) = \left(0, \frac{\Lambda[\mu(1 - \sigma) + \psi]}{\mu(\mu + \psi + \varphi)}, \frac{\Lambda(\mu\sigma + \varphi)}{\mu(\mu + \psi + \varphi)} \right), \tag{4}$$

- (2) The endemic equilibrium E_e in which $I > 0$:

$$E_e = (I_e, S_e, V_e),$$

with

$$\begin{aligned}
 I_e &= \frac{\Lambda[\beta[\mu(1 - \sigma) + \psi] - (\mu + \gamma + \alpha)(\mu + \varphi + \psi)]}{\mu\beta(\mu + \psi) + \alpha[\beta(\mu + \psi) - (\mu + \gamma + \alpha)(\mu + \varphi + \psi)]}, \\
 S_e &= \frac{\mu + \gamma + \alpha}{\beta\mu}(\Lambda - \alpha I_e), \\
 V_e &= \frac{1}{\mu + \psi}(\sigma\Lambda + \varphi S_e).
 \end{aligned}$$

Also, by adding the components of E_{df} and E_e we get $N_{df} = \frac{\Lambda}{\mu}$ and $N_e = \frac{1}{\mu}(\Lambda - \alpha I_e)$, respectively.

2.3. The basic reproduction number

The basic reproduction (reproductive) number is defined as the average number of individuals who become infected by entering one infected individual into a fully susceptible population[2, 16]. We use a method based on the next generation matrix to determine the basic reproduction number for the model. The Jacobian matrix of the system (1) at (I, S, V) is

$$J = J(I, S, V) = \begin{pmatrix} 1 + \Upsilon_I - (\mu + \gamma + \alpha) & \Upsilon_S & \Upsilon_V \\ -\Upsilon_I + \gamma & 1 - \Upsilon_S - (\mu + \varphi) & -\Upsilon_V + \psi \\ 0 & \varphi & 1 - (\mu + \psi) \end{pmatrix}, \tag{5}$$

where Υ_I, Υ_S and Υ_V are partial derivatives of incidence rate $\Upsilon(t)$ with respect to I, S and V respectively and are obtained as

$$\begin{aligned}
 \Upsilon_I &= \frac{\beta SN - \beta SI}{N^2}, \\
 \Upsilon_S &= \frac{\beta IN - \beta SI}{N^2}, \\
 \Upsilon_V &= -\frac{\beta SI}{N^2}.
 \end{aligned}$$

Thus the Jacobian matrix at E_{df} becomes

$$J(E_{df}) = \begin{pmatrix} 1 + \frac{\beta S_{df}}{N_{df}} - (\mu + \gamma + \alpha) & 0 & 0 \\ -\frac{\beta S_{df}}{N_{df}} + \gamma & 1 - (\mu + \varphi) & \psi \\ 0 & \varphi & 1 - (\mu + \psi) \end{pmatrix}. \tag{6}$$

We write $J(E_{df})$ in the block form as

$$J(E_{df}) = \begin{pmatrix} F + T & 0 \\ A & C \end{pmatrix}, \tag{7}$$

in which

$$\begin{aligned}
 F &= \frac{\beta S_{df}}{N_{df}}, \\
 T &= 1 - (\mu + \gamma + \alpha), \\
 A &= \left(-\frac{\beta S_{df}}{N_{df}} + \gamma, 0\right)^\top, \\
 C &= \begin{pmatrix} 1 - (\mu + \varphi) & \psi \\ \varphi & 1 - (\mu + \psi) \end{pmatrix}.
 \end{aligned} \tag{8}$$

Now, the basic reproduction number can be determined as[17]

$$\mathcal{R}_0 = \rho(F(1 - T)^{-1}) = \frac{\beta[\mu(1 - \sigma) + \psi]}{(\mu + \gamma + \alpha)(\mu + \varphi + \psi)}. \tag{9}$$

On the other hand, if we take

$$\mathbf{R}(\tau) = \frac{\beta[\mu(1 - \tau) + \psi]}{(\mu + \gamma + \alpha)(\mu + \varphi + \psi)}, \tag{10}$$

then components of E_e can be written as

$$\begin{aligned} I_e &= \frac{\Lambda(\mathcal{R}_0 - 1)}{\mu\mathbf{R}(0) + \alpha[\mathbf{R}(0) - 1]}, \\ S_e &= \frac{\Lambda(\mu + \gamma + \alpha)}{\beta\mu} \left[1 - \frac{\alpha(\mathcal{R}_0 - 1)}{\mu\mathbf{R}(0) + \alpha[\mathbf{R}(0) - 1]} \right], \\ V_e &= \frac{\sigma\Lambda}{\mu + \psi} \left[1 + \frac{\varphi(\mu + \alpha\sigma + \psi)/[\sigma(\mu + \varphi + \psi)]}{\mu\mathbf{R}(0) + \alpha[\mathbf{R}(0) - 1]} \right]. \end{aligned}$$

Here we must notice that $\mathbf{R}(\tau)$ is a decreasing function and thus $\mathcal{R}_0 = \mathbf{R}(\sigma) < \mathbf{R}(0)$. Therefore when $\mathcal{R}_0 > 1$, we have $\mathbf{R}(0) - 1 > 0$. This implies $I_e > 0$ exists if $\mathcal{R}_0 > 1$ and we can state the following:

Theorem 2.1. *The SIVS epidemic model described by system (1) has only equilibrium E_{df} when $\mathcal{R}_0 \leq 1$ and it has also equilibrium E_e when $\mathcal{R}_0 > 1$.*

3. Stability of the equilibria

We first consider the stability of the equilibrium E_{df} in the following subsection.

3.1. Stability of the disease-free equilibrium

From relations (8) it is found that the summations of columns in matrix C are all less than one thus its matrix norm is less than one:

$$\begin{aligned} \|C\|_1 &= \max\{1 - (\mu + \varphi) + \varphi, \psi + 1 - (\mu + \psi)\} \\ &= 1 - \mu < 1. \end{aligned}$$

Since $\rho(C) < \|C\|_1$, thus the spectral radius of C is less than one, $\rho(C) < 1$.

On the other hand

$$\rho(F + T) < 1 \iff \mathcal{R}_0 < 1.$$

Therefore the Jacobian matrix $J(E_{df})$ in (7) has the spectral radius less than one if and only if $\mathcal{R}_0 < 1$. Then the following theorem has been proven:

Theorem 3.1. *The disease-free equilibrium E_{df} is stable if and only if $\mathcal{R}_0 < 1$ and it is unstable if $\mathcal{R}_0 > 1$.*

3.2. Stability of the endemic equilibrium

At the endemic equilibrium E_e we have

$$\begin{aligned} \frac{\beta S_e}{N_e} &= \mu + \gamma + \alpha, \\ \sigma\Lambda + \varphi(N_e - I_e) - (\mu + \varphi + \psi)V_e &= 0, \\ \Lambda - \mu N_e - \alpha I_e &= 0, \end{aligned}$$

then we can write

$$\begin{aligned} \Upsilon_{I_e} &= (\mu + \gamma + \alpha)\left(1 - \frac{I_e}{N_e}\right), \\ \Upsilon_{S_e} &= [\beta - (\mu + \gamma + \alpha)]\frac{I_e}{N_e}, \\ \Upsilon_{V_e} &= -(\mu + \gamma + \alpha)\frac{I_e}{N_e}, \end{aligned} \tag{11}$$

and

$$(\mu + \gamma + \alpha) - \Upsilon_{I_e} = -\Upsilon_{V_e} = (\mu + \gamma + \alpha)\frac{I_e}{N_e}. \tag{12}$$

For Jacobian matrix at the endemic equilibrium ($J(E_e)$), we have

$$\begin{aligned} tr(J(E_e)) &= 3 - ((\mu + \gamma + \alpha) - \Upsilon_{I_e} + \Upsilon_{S_e} + (\mu + \varphi) + (\mu + \psi)), \\ (J^2(E_e))_{(1,1)} &= \left(1 - [(\mu + \gamma + \alpha) - \Upsilon_{I_e}]\right)^2 + \Upsilon_{S_e}(-\Upsilon_{I_e} + \gamma), \\ (J^2(E_e))_{(2,2)} &= \Upsilon_{S_e}(-\Upsilon_{I_e} + \gamma) + \left(1 - [\Upsilon_{S_e} + (\mu + \varphi)]\right)^2 + \varphi(-\Upsilon_{V_e} + \psi), \\ (J^2(E_e))_{(3,3)} &= \varphi(-\Upsilon_{V_e} + \psi) + \left(1 - (\mu + \psi)\right)^2. \end{aligned}$$

Therefore we have

$$\begin{aligned} &tr(J^2(E_e)) - tr^2(J(E_e)) \\ &= 2\Upsilon_{S_e}(-\Upsilon_{I_e} + \gamma) + 2\varphi(-\Upsilon_{V_e} + \psi) \\ &\quad - 2\{1 - [(\mu + \gamma + \alpha) - \Upsilon_{I_e}] - [\Upsilon_{S_e} + (\mu + \varphi)] + [(\mu + \gamma + \alpha) - \Upsilon_{I_e}][\Upsilon_{S_e} + (\mu + \varphi)]\} \\ &\quad - 2\{1 - [(\mu + \gamma + \alpha) - \Upsilon_{I_e}] - (\mu + \psi) + [(\mu + \gamma + \alpha) - \Upsilon_{I_e}](\mu + \psi)\} \\ &\quad - 2\{1 - [\Upsilon_{S_e} + (\mu + \varphi)] - (\mu + \psi) + [\Upsilon_{S_e} + (\mu + \varphi)](\mu + \psi)\} \\ &= -6 + 4\{[(\mu + \gamma + \alpha) - \Upsilon_{I_e}] + [\Upsilon_{S_e} + (\mu + \varphi)] + (\mu + \psi)\} \\ &\quad - 2\{[(\mu + \gamma + \alpha) - \Upsilon_{I_e}][\Upsilon_{S_e} + (\mu + \varphi)] + [(\mu + \gamma + \alpha) - \Upsilon_{I_e}](\mu + \psi) \\ &\quad \quad + [\Upsilon_{S_e} + (\mu + \varphi)](\mu + \psi) - \Upsilon_{S_e}(-\Upsilon_{I_e} + \gamma) - \varphi(-\Upsilon_{V_e} + \psi)\}. \end{aligned}$$

Also, by expanding the determinant of $J(E_e)$ according to the first column we get

$$\begin{aligned} det(J(E_e)) &= \\ &1 - \{[\Upsilon_{S_e} + (\mu + \varphi)] + (\mu + \psi) + [(\mu + \gamma + \alpha) - \Upsilon_{I_e}]\} \\ &\quad + \{[\Upsilon_{S_e} + (\mu + \varphi)](\mu + \psi) - \varphi(-\Upsilon_{V_e} + \psi) + [(\mu + \gamma + \alpha) - \Upsilon_{I_e}][\Upsilon_{S_e} + (\mu + \varphi)] \\ &\quad \quad + [(\mu + \gamma + \alpha) - \Upsilon_{I_e}](\mu + \psi) - \Upsilon_{S_e}(-\Upsilon_{I_e} + \gamma)\} \\ &\quad - \{[(\mu + \gamma + \alpha) - \Upsilon_{I_e}][\Upsilon_{S_e} + (\mu + \varphi)](\mu + \psi) - \varphi[(\mu + \gamma + \alpha) - \Upsilon_{I_e}](-\Upsilon_{V_e} + \psi) \\ &\quad \quad - \Upsilon_{S_e}(-\Upsilon_{I_e} + \gamma)(\mu + \psi) - \varphi(-\Upsilon_{I_e} + \gamma)\Upsilon_{V_e}\}. \end{aligned}$$

Now assuming

$$\begin{aligned}
 b_1 &= [(\mu + \gamma + \alpha) - \Upsilon_{I_e}] + [\Upsilon_{S_e} + (\mu + \varphi)] + (\mu + \psi), \\
 b_2 &= [(\mu + \gamma + \alpha) - \Upsilon_{I_e}][\Upsilon_{S_e} + (\mu + \varphi)] + [(\mu + \gamma + \alpha) - \Upsilon_{I_e}](\mu + \psi) \\
 &\quad + [\Upsilon_{S_e} + (\mu + \varphi)](\mu + \psi) - \Upsilon_{S_e}(-\Upsilon_{I_e} + \gamma) - \varphi(-\Upsilon_{V_e} + \psi), \\
 b_3 &= [(\mu + \gamma + \alpha) - \Upsilon_{I_e}][\Upsilon_{S_e} + (\mu + \varphi)](\mu + \psi) - \varphi[(\mu + \gamma + \alpha) - \Upsilon_{I_e}](-\Upsilon_{V_e} + \psi) \\
 &\quad - \Upsilon_{S_e}(-\Upsilon_{I_e} + \gamma)(\mu + \psi) - \varphi(-\Upsilon_{I_e} + \gamma)\Upsilon_{V_e},
 \end{aligned} \tag{13}$$

we can write

$$\begin{aligned}
 \text{tr}(J(E_e)) &= 3 - b_1, \\
 \text{tr}(J^2(E_e)) - \text{tr}^2(J(E_e)) &= -6 + 4b_1 - 2b_2, \\
 \det(J(E_e)) &= 1 - b_1 + b_2 - b_3.
 \end{aligned}$$

The characteristic equation for the Jacobian matrix $J(E_e)$ is then as

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3,$$

in which

$$\begin{aligned}
 a_1 &= -\text{tr}(J(E_e)) = -3 + b_1, \\
 a_2 &= -\frac{1}{2}[\text{tr}(J^2(E_e)) - \text{tr}^2(J(E_e))] = 3 - 2b_1 + b_2, \\
 a_3 &= -\det(J(E_e)) = -1 + b_1 - b_2 + b_3.
 \end{aligned} \tag{14}$$

The following theorem considers conditions under which the endemic equilibrium is stable.

Theorem 3.2. *When $\mathcal{R}_0 > 1$, the endemic equilibrium E_e is locally asymptotically stable.*

Proof. The Jury conditions[5] state that roots of characteristic equation $P(\lambda)$ (i.e. eigenvalues of $J(E_e)$) lie inside the unite circle if and only if

- (i) $P(1) > 0$,
- (ii) $-P(-1) > 0$,
- (iii) $|a_2 - a_1a_3| < 1 - a_3^2$.

We see that $P(1) = 1 + a_1 + a_2 + a_3 = b_3$ and thus the condition (i) is equivalent to $b_3 > 0$. By replacing partial derivatives of Υ at endemic equilibrium from (11) and (12) into relations (13) we will have

$$\begin{aligned}
 P(1) &= b_3 \\
 &= \underbrace{(\mu + \gamma + \alpha) \frac{I_e}{N_e} [\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} (\mu + \psi)}_I + \underbrace{(\mu + \varphi)(\mu + \psi)(\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{II} \\
 &\quad - \underbrace{\varphi(\mu + \gamma + \alpha) \frac{I_e}{N_e} (\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{III} - \underbrace{\varphi\psi(\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{II} \\
 &\quad + \underbrace{[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} (\mu + \gamma + \alpha) (1 - \frac{I_e}{N_e}) (\mu + \psi)}_I - \underbrace{[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} \gamma (\mu + \psi)}_I \\
 &\quad - \underbrace{\varphi(\mu + \gamma + \alpha) (1 - \frac{I_e}{N_e}) (\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{III} + \varphi\gamma(\mu + \gamma + \alpha) \frac{I_e}{N_e},
 \end{aligned}$$

and by considering

$$I = [\beta - (\mu + \gamma + \alpha)](\mu + \gamma + \alpha)(\mu + \psi) \frac{I_e}{N_e},$$

$$II = (\mu + \gamma + \alpha) \frac{I_e}{N_e} [\mu(\mu + \varphi + \psi)],$$

$$III = -\varphi(\mu + \gamma + \alpha)^2 \frac{I_e}{N_e},$$

we get

$$b_3 = \underbrace{[\beta - (\mu + \gamma + \alpha)](\mu + \gamma + \alpha)(\mu + \psi) \frac{I_e}{N_e}}_{\star} + \underbrace{(\mu + \gamma + \alpha) \frac{I_e}{N_e} \mu(\mu + \varphi + \psi)}_{\star\star} \\ + \underbrace{\varphi\gamma(\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{\star\star} - \underbrace{\varphi(\mu + \gamma + \alpha)^2 \frac{I_e}{N_e}}_{\star} - \underbrace{[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} \gamma(\mu + \psi)}_{\star}$$

and observing

$$\star = [\beta - (\mu + \gamma + \alpha)](\mu + \alpha)(\mu + \psi) \frac{I_e}{N_e},$$

$$\star\star = \underbrace{\varphi\mu(\mu + \gamma + \alpha) \frac{I_e}{N_e} + \mu(\mu + \psi)(\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{\star\star}$$

$$\star\star\star = \varphi(\mu + \gamma)(\mu + \gamma + \alpha) \frac{I_e}{N_e},$$

we obtain

$$b_3 = -\varphi(\mu + \gamma + \alpha)\alpha \frac{I_e}{N_e} + \mu(\mu + \psi)(\mu + \gamma + \alpha) \frac{I_e}{N_e} + [\beta - (\mu + \gamma + \alpha)](\mu + \alpha)(\mu + \psi) \frac{I_e}{N_e} \\ = \left[-\varphi\alpha(\mu + \gamma + \alpha) + \beta(\mu + \alpha)(\mu + \psi) - \alpha(\mu + \psi)(\mu + \gamma + \alpha) \right] \frac{I_e}{N_e} \\ = \left[-\alpha(\mu + \gamma + \alpha)(\mu + \varphi + \psi) + \beta(\mu + \alpha)(\mu + \psi) \right] \frac{I_e}{N_e} \\ = (\mu + \gamma + \alpha)(\mu + \varphi + \psi) \left[-\alpha + (\mu + \alpha)\mathbf{R}(0) \right] \frac{I_e}{N_e} \\ = (\mu + \gamma + \alpha)(\mu + \varphi + \psi) \underbrace{\left[\frac{(\mu + \alpha)\mathbf{R}(0) - \alpha}{(\mu + \alpha)\mathbf{R}(0) - \alpha\mathcal{R}_0} \right]}_{>1} \mu(\mathcal{R}_0 - 1) > 0, \tag{15}$$

because $\mathcal{R}_0 > 1$ and

$$\frac{I_e}{N_e} = \frac{\mu(\mathcal{R}_0 - 1)}{\mu\mathbf{R}(0) + \alpha(\mathbf{R}(0) - \mathcal{R}_0)}.$$

Also, we can write condition (ii) as

$$-P(-1) = 1 - a_1 + a_2 - a_3 \\ = 8 - 4b_1 + 2b_2 - b_3$$

$$\begin{aligned}
 &= 8 - \underbrace{4(\mu + \gamma + \alpha) \frac{I_e}{N_e} - 4\left\{[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} + (\mu + \varphi)\right\}}_I - \underbrace{4(\mu + \psi)}_{II} \\
 &\quad + \underbrace{2(\mu + \gamma + \alpha) \frac{I_e}{N_e} \varphi}_{III} + \underbrace{2(\mu + \gamma + \alpha) \frac{I_e}{N_e} [\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} + 2(\mu + \gamma + \alpha) \frac{I_e}{N_e} \mu}_{\diamond} \\
 &\quad + 2(\mu + \gamma + \alpha) \frac{I_e}{N_e} (\mu + \psi) + \underbrace{2\left\{[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} + (\mu + \varphi)\right\}(\mu + \psi)}_{IV} \\
 &\quad + \underbrace{2[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} (\mu + \gamma + \alpha) \left(1 - \frac{I_e}{N_e}\right)}_V \\
 &\quad - \underbrace{2\gamma[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} - 2\varphi(\mu + \gamma + \alpha) \frac{I_e}{N_e} - 2\varphi\psi}_{\diamond} \\
 &\quad - \underbrace{(\mu + \gamma + \alpha) \frac{I_e}{N_e} \left\{[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} + (\mu + \varphi)\right\}(\mu + \psi)}_{IV} \\
 &\quad + \varphi(\mu + \gamma + \alpha) \frac{I_e}{N_e} (\mu + \gamma + \alpha) \frac{I_e}{N_e} + \varphi\psi(\mu + \gamma + \alpha) \frac{I_e}{N_e} \\
 &\quad - \underbrace{[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} (\mu + \gamma + \alpha) \left(1 - \frac{I_e}{N_e}\right) (\mu + \psi)}_{\star} \\
 &\quad + \underbrace{[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} \gamma (\mu + \psi) + \varphi(\mu + \gamma + \alpha) \left(1 - \frac{I_e}{N_e}\right) (\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{\star} \\
 &\quad - \underbrace{\varphi\gamma(\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{\star}
 \end{aligned}$$

Now, considering the sufficient conditions for positive solutions stated in (2), and inequalities

$$\begin{aligned}
 &\beta > \mu + \gamma + \alpha, \\
 &\beta(\mu + \psi) > (\mu + \gamma + \alpha)(\mu + \psi + \varphi),
 \end{aligned}$$

which are concluded from $\mathcal{R}_0 > 1$, we will have

$$\begin{aligned}
 I &= -4\left[\beta \frac{I_e}{N_e} + (\mu + \varphi)\right] = -4(\beta + \mu + \varphi) + 4\beta\left(1 - \frac{I_e}{N_e}\right) \\
 &> -4 + \underbrace{4\beta\left(1 - \frac{I_e}{N_e}\right)}_{\diamond},
 \end{aligned}$$

$$II = -4(\mu + \varphi + \psi) + 4\varphi > -4 \underbrace{+ 2\varphi}_{\star} \underbrace{+ \varphi}_{\star} + \varphi,$$

$$III = 0,$$

$$IV > \left\{ [\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} + (\mu + \varphi) \right\} (\mu + \psi),$$

$$V > [\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} (\mu + \gamma + \alpha) \left(1 - \frac{I_e}{N_e} \right)$$

$$+ [\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} (\mu + \gamma + \alpha) \left(1 - \frac{I_e}{N_e} \right) \varphi$$

$$+ \underbrace{[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} (\mu + \gamma + \alpha) \left(1 - \frac{I_e}{N_e} \right) (\mu + \psi)}_{\star},$$

$$\star = 0,$$

$$\spadesuit > 0,$$

$$\clubsuit > 0,$$

$$\diamond > -2\gamma [\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e}$$

$$+ [\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} \left\{ 2(\mu + \gamma + \alpha) \frac{I_e}{N_e} + 4\beta \left(1 - \frac{I_e}{N_e} \right) \right\}$$

$$> -2\gamma [\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e}$$

$$+ [\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} \left\{ 2(\mu + \gamma + \alpha) \frac{I_e}{N_e} + 2(\mu + \gamma + \alpha) \left(1 - \frac{I_e}{N_e} \right) + 2\beta \left(1 - \frac{I_e}{N_e} \right) \right\}$$

$$= 2[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} \left[\mu + \alpha + \beta \left(1 - \frac{I_e}{N_e} \right) \right].$$

Therefore all negative parts in $-P(-1)$ are covered by positive parts and thus $-P(-1) > 0$.

Condition (iii) holds if and only if $a_2 - a_1a_3 < 1 - a_3^2$ and $a_2 - a_1a_3 > -(1 - a_3^2)$. From (14), $a_2 - a_1a_3 < 1 - a_3^2$ is equivalent to $b_3 < (b_2 - b_3)(b_1 - b_2 + b_3)$. Consider the following nonlinear programming problem

$$\text{maximization } b_3 - (b_2 - b_3)(b_1 - b_2 + b_3)$$

subject to

$$\mu > 0, \varphi > 0, \beta > 0,$$

$$\psi > 0, \alpha > 0, \gamma > 0,$$

$$\eta > 0, \eta < 1,$$

$$\mu + \varphi + \beta < 1,$$

$$\mu + \varphi + \psi < 1,$$

$$\mu + \gamma + \alpha < 1,$$

$$\beta > \mu + \gamma + \alpha,$$

$$\beta(\mu + \psi) > (\mu + \gamma + \alpha)(\mu + \varphi + \psi),$$

(16)

where $b_i, i = 1, 2, 3$, are as in (13) and $\eta = \frac{I_e}{N_e}$. By solving this problem we see that maximum value is negative and thus $b_3 < (b_2 - b_3)(b_1 - b_2 + b_3)$. Besides, $a_2 - a_1a_3 > -(1 - a_3^2)$ is equivalent to $b_3 > (b_1 - b_2 + b_3)(-b_2 + b_3)$ and by solving the nonlinear programming problem

$$\text{minimization } b_3 - (b_1 - b_2 + b_3)(-b_2 + b_3),$$

subject to the constrains as problem (16), we find that the minimum value is positive and thus we have $b_3 > (b_1 - b_2 + b_3)(-b_2 + b_3)$. Therefore condition (iii) holds also.

4. Bifurcations

In this section we consider bifurcations at the equilibria of the model. Bifurcation occurs when the eigenvalue of the Jacobian matrix has module one. When a real eigenvalue is either 1 or -1, the fold bifurcation or the flip bifurcation occurs, respectively and the Neimark-Sacker bifurcation occurs if there is a pair of complex conjugate eigenvalues with module one [18].

4.1. Bifurcations at the disease-free equilibrium

As we saw in (6), the eigenvalues of the Jacobian matrix at E_{df} are as

$$\begin{aligned} \lambda_1 &= 1 + \frac{\beta S_{df}}{N_{df}} - (\mu + \gamma + \alpha), \\ \lambda_2 &= 1 - \mu, \\ \lambda_3 &= 1 - (\mu + \varphi + \psi). \end{aligned}$$

$|\lambda_2| < 1$ and $|\lambda_3| < 1$. Moreover, $\lambda_1 = 1$ if and only if $\frac{\beta S_{df}}{N_{df}} = (\mu + \gamma + \alpha)$ if and only if $\mathcal{R}_0 = 1$. Thus the fold bifurcation occurs at E_{df} if and only if $\mathcal{R}_0 = 1$. As the same way we see that $\lambda = -1$ if and only if $\mathcal{R}_0 = 1 - \frac{2}{\mu + \gamma + \alpha}$. This implies $\mathcal{R}_0 < 0$ since $\mu + \gamma + \alpha < 1$. This is a contradiction and shows that the flip bifurcation does not appear at E_{df} . In addition, all eigenvalues are real and therefore the Neimark-Sacker bifurcation does not appear also. By these discussions we can state the following theorem:

Theorem 4.1. *At the disease-free equilibrium of the model (1) the flip bifurcation and the Neimark-Sacker bifurcation don't appear while the fold bifurcation occurs if $\mathcal{R}_0 = 1$.*

4.2. bifurcations at the endemic equilibrium

Here we consider the bifurcations of the model at E_e in the following theorem:

Theorem 4.2. *At the endemic equilibrium of the model (1) the fold bifurcation occurs when $\mathcal{R}_0 = 1$ while the flip bifurcation and the Neimark-Sacker bifurcation don't take place.*

Proof. We notice that $\lambda = 1$ is an eigenvalue of the Jacobian matrix at the endemic equilibrium E_e if and only if it is a root of characteristic equation i.e., $p(1) = 0$ that is equivalent to $b_3 = 0$. From (15) we have

$$b_3 = (\mu + \gamma + \alpha)(\mu + \varphi + \psi) \left[\frac{(\mu + \alpha)\mathbf{R}(0) - \alpha}{(\mu + \alpha)\mathbf{R}(0) - \alpha\mathcal{R}_0} \right] \mu(\mathcal{R}_0 - 1),$$

which implies $b_3 = 0$ if and only if $\mathcal{R}_0 = 1$. Therefore the fold bifurcation happens if $\mathcal{R}_0 = 1$.

Using similar argument, $\lambda = -1$ if and only if $p(-1) = 0$ or equivalently $-8 + 4b_1 - 2b_2 + b_3 = 0$. After some manipulations we get

$$\begin{aligned} p(-1) &= \left\{ \beta[2 - (\mu + \psi)][2 - (\mu + \alpha)] + \alpha(\mu + \gamma + \alpha)[2 - (\mu + \varphi + \psi)] \right\} \frac{I_e}{N_e} \\ &\quad - 2(2 - \mu)[2 - (\mu + \varphi + \psi)]. \end{aligned} \tag{17}$$

In proof of Theorem 3.2 we found that $-p(-1) > 0$ when $\mathcal{R}_0 > 1$. On the other hand if $\mathcal{R}_0 = 1$, then $\frac{I_e}{N_e} = 0$ and so we must have $2(2 - \mu)[2 - (\mu + \varphi + \psi)] = 0$ which is impossible. Therefore the flip bifurcation does not appear at E_e .

For investigating the Neimark-Sacker bifurcation we notice that characteristic equation $p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ has a pair of complex roots on the unit circle and a root inside the unit circle if and only if the following conditions hold[19]:

(i) $p(1) > 0$,

(ii) $-p(-1) > 0$,

(iii) $|a_3| < 1$,

(iv) $a_2 - a_1a_3 = 1 - a_3^2$.

In proof of theorem (3.2) it was proven that conditions (i) and (ii) hold when $\mathcal{R}_0 > 1$. Condition (iii) holds if and only if $0 < b_1 - b_2 + b_3 < 2$. Firstly, we have

$$\begin{aligned}
 b_1 - b_2 + b_3 = & \underbrace{(\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{III} + \underbrace{[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e}}_I + \underbrace{(\mu + \varphi)}_{VII} + \underbrace{(\mu + \psi)}_{VI} \\
 & + \underbrace{[\beta - (\mu + \gamma + \alpha)] \gamma \frac{I_e}{N_e}}_V + \underbrace{\varphi(\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{II} + \underbrace{\varphi\psi}_{IV} \\
 & + \underbrace{(\mu + \gamma + \alpha) \frac{I_e}{N_e} [\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} (\mu + \psi)}_{IIIX} + \underbrace{(\mu + \gamma + \alpha) \frac{I_e}{N_e} (\mu + \varphi)(\mu + \psi)}_{IIIX} \\
 & + \underbrace{(\mu + \gamma + \alpha) \frac{I_e}{N_e} [\beta - (\mu + \gamma + \alpha)] \left(1 - \frac{I_e}{N_e}\right) (\mu + \psi)}_{IIIX} + \varphi\gamma(\mu + \gamma + \alpha) \frac{I_e}{N_e} \\
 & - \left\{ \underbrace{(\mu + \gamma + \alpha) \frac{I_e}{N_e} [\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e}}_I + \underbrace{(\mu + \gamma + \alpha) \frac{I_e}{N_e} (\mu + \varphi)}_{III} + \underbrace{(\mu + \gamma + \alpha) \frac{I_e}{N_e} (\mu + \psi)}_{VI} \right. \\
 & + \underbrace{[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} (\mu + \psi)}_{VI} + \underbrace{(\mu + \varphi)(\mu + \psi)}_{VII} + \underbrace{[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} (\mu + \gamma + \alpha) \left(1 - \frac{I_e}{N_e}\right)}_I \\
 & + \underbrace{\varphi(\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{II} + \underbrace{(\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{II} + \underbrace{\varphi\psi(\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{IV} \\
 & \left. + \underbrace{[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} \gamma (\mu + \psi)}_V + \underbrace{\varphi(\mu + \gamma + \alpha) \left(1 - \frac{I_e}{N_e}\right) (\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{II} \right\}
 \end{aligned}$$

and thus

$$\begin{aligned}
 b_1 - b_2 + b_3 > & \underbrace{\psi(\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{III} + \underbrace{\varphi\psi(\mu + \gamma + \alpha) \left(1 - \frac{I_e}{N_e}\right)}_{IV} + \underbrace{\varphi[\beta - (\mu + \gamma + \alpha)] \gamma \frac{I_e}{N_e}}_V \\
 & + \underbrace{(\mu + \gamma + \alpha) \frac{I_e}{N_e} [\beta - (\mu + \gamma + \alpha)] (\mu + \psi)}_{IIIX} + \underbrace{(\mu + \gamma + \alpha) \frac{I_e}{N_e} (\mu + \varphi)(\mu + \psi)}_{IIIX} \\
 & + \varphi\gamma(\mu + \gamma + \alpha) \frac{I_e}{N_e} > 0.
 \end{aligned}$$

Notice that from the right hand side of $b_1 - b_2 + b_3$ in the previous expression we have

$$IV = \varphi\psi - \varphi\psi(\mu + \gamma + \alpha) \frac{I_e}{N_e}.$$

But $(\mu + \gamma + \alpha) < 1$ and thus $\varphi\psi > \varphi\psi(\mu + \gamma + \alpha)$. Therefore,

$$IV > \varphi\psi(\mu + \gamma + \alpha) - \varphi\psi(\mu + \gamma + \alpha) \frac{I_e}{N_e} = \varphi\psi(\mu + \gamma + \alpha) \left(1 - \frac{I_e}{N_e}\right).$$

Secondly,

$$\begin{aligned} 2 - (b_1 - b_2 + b_3) &= \\ &\underbrace{2}_I + \underbrace{(\mu + \gamma + \alpha)[\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e}}_{II} + \underbrace{(\mu + \gamma + \alpha)(\mu + \varphi) \frac{I_e}{N_e}}_{III} \\ &+ \underbrace{(\mu + \gamma + \alpha)(\mu + \psi) \frac{I_e}{N_e}}_V + \underbrace{[\beta - (\mu + \gamma + \alpha)](\mu + \psi) \frac{I_e}{N_e}}_{IV} + (\mu + \varphi)(\mu + \psi) \\ &+ \underbrace{\varphi(\mu + \gamma + \alpha)(\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{VI} + \varphi\psi(\mu + \gamma + \alpha) \frac{I_e}{N_e} + [\beta - (\mu + \gamma + \alpha)]\gamma(\mu + \psi) \frac{I_e}{N_e} \\ &- \left\{ \underbrace{(\mu + \gamma + \alpha) \frac{I_e}{N_e} + [\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} + (\mu + \varphi)}_{<\mu+\varphi+\beta<1, \quad I} \right. \\ &\quad + \underbrace{(\mu + \psi) + [\beta - (\mu + \gamma + \alpha)]\gamma \frac{I_e}{N_e}}_I + \underbrace{\varphi(\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{III} + \underbrace{\varphi\psi}_I \\ &\quad + \underbrace{(\mu + \gamma + \alpha)[\beta - (\mu + \gamma + \alpha)](\mu + \psi) \frac{I_e}{N_e}}_{IV} + \underbrace{(\mu + \gamma + \alpha)(\mu + \varphi)(\mu + \psi) \frac{I_e}{N_e}}_V \\ &\quad \left. + \underbrace{\varphi\gamma(\mu + \gamma + \alpha) \frac{I_e}{N_e}}_{VI} \right\} \\ &> (\mu + \varphi)(\mu + \psi) + \varphi\psi(\mu + \gamma + \alpha) \frac{I_e}{N_e} + [\beta - (\mu + \gamma + \alpha)]\gamma(\mu + \psi) \frac{I_e}{N_e} > 0. \end{aligned}$$

For expression I notice that Firstly, $(\mu + \gamma + \alpha) \frac{I_e}{N_e} + [\beta - (\mu + \gamma + \alpha)] \frac{I_e}{N_e} + (\mu + \varphi) = \beta \frac{I_e}{N_e} + (\mu + \varphi) < \beta + \mu + \varphi$, since $\frac{I_e}{N_e} < 1$, and therefore by first inequality in (2), it is less than one. Secondly, $(\mu + \psi) + \varphi\psi < \mu + \psi + \varphi < 1$ by last inequality in (2). Thus the summation of these two expressions is less than 2 i.e., $I < 2$. Therefore condition (iii) holds. Also, condition (iv) holds if and only if

$$b_3 = (b_1 - b_2 + b_3)(b_2 - b_3).$$

But we saw that the nonlinear programming problem (16) implies

$$b_3 - (b_2 - b_3)(b_1 - b_2 + b_3) < 0.$$

This shows that $a_2 - a_1 a_3 < 1 - a_3^2$ and condition (iv) is not satisfied. Therefore the Neimark-Sacker bifurcation does not take place at E_e .

Remark 4.3. The condition $\beta < 1$ with other conditions stated in (2) are sufficient condition to the model (1) has positive solutions but not a necessary condition for this. If we omit this constraint from β and allow it to take greater values also (as it may take place in real world) , then it becomes possible that $p(-1) = 0$ while $\mathcal{R}_0 > 1$ and thus the flip bifurcation may occur. In this case, some sufficient conditions can be extracted from (17).

5. Numerical discussions

In this section we simulate the model and challenge numerically the theoretical results obtained in preceding sections. Take the parameters of the model as

$$\sigma = 0.4, \mu = 0.1, \varphi = 0.2, \alpha = 0.05, \gamma = 0.15, \text{ and } \psi = 0.25. \tag{18}$$

Also, suppose that the unit of time is one day and unit of population is one million persons and initial population of susceptible, infected and vaccinated individuals are $S_0 = 0.8, I_0 = 0.4,$ and $V_0 = 0.5$ while a constant number of new members $\Lambda = 0.2$ enters the population per unit time. We take β as the bifurcation parameter and get the bifurcation diagram as in Figure 2. From (9) we have $\mathcal{R}_0 = 1$ if $\beta = 0.5323$ and $\mathcal{R}_0 < (>)1$ when $\beta < (>)0.5323$. As it was stated in Theorem 3.1 Figure 2 shows that the disease-free equilibrium E_{df} is stable when $\beta < 0.5323$ and it is unstable for $\beta > 0.5323$. In this case the endemic equilibrium E_e becomes stable as it was stated in Theorem 3.2. We see that at $\beta = 0.5323$ the stability of the model is changed and a fold bifurcation occurs as it was stated in Theorem 4.1 and Theorem 4.2. Also, Theorem 4.1 and Theorem 4.2 state that when $\beta < 1$ the flip bifurcation and the Neimark-Sacker bifurcation can not be occurred but if we let it to take values greater than one, then the flip bifurcation may take place as it was stated in Remark 4.3. From (17) we find that for $\beta = 2.6810, p(-1) = 0$ and thus the flip bifurcation happens. This is observable also in Figure 2. In Figure 3 we graph the Lyapunov exponent as a function of

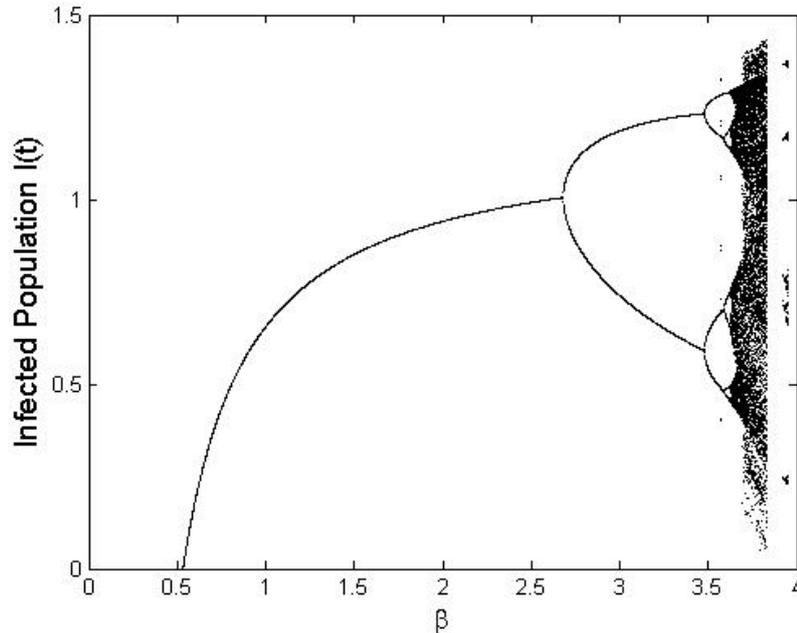


Figure 2: Bifurcation diagram for infected population I_t in terms of β as bifurcation parameter.

β . The Lyapunov exponent at equilibrium $X = (I, S, V)$ is calculated by[2]

$$\lambda_0(X_0) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \rho \left(\prod_{k=0}^{t-1} J(X_k) \right),$$

where X_0 is the under consideration equilibrium and $X_k, k = 1, \dots, t-1$ are its next $t-1$ solutions after each time interval Δt . For those values of β at which the Lyapunov exponent has not negative value the model is not stable. It is seen that at $\beta = 0.5323, 2.6810, \text{ and } 3.479$ this quantity is not negative and bifurcations occur. Moreover, for $\beta > 3.6$ the most Lyapunov exponents are positive and thus the solution has chaotic behavior. Figure 4 illustrates solutions of the model for various values for β and parameters value as in

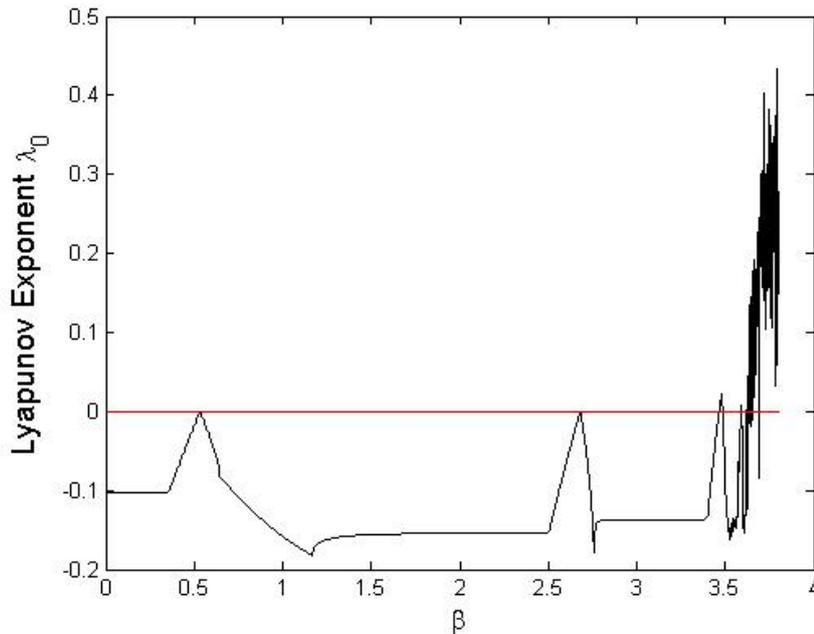


Figure 3: Lyapunov exponent of the Jacobian matrix in terms of β .

(18). At $\beta = 0.5$, the basic reproduction number is $\mathcal{R}_0 = 0.9394 < 1$ and as Theorem 3.1 states the disease-free equilibrium is stable and it is seen that the disease is extinct. However, at $\beta = 0.6$, $\mathcal{R}_0 = 1.1273 > 1$ and according to Theorem 3.2 the endemic equilibrium is stable. In this case, the disease persists in population. For parameter values $\beta = 2.7$ and $\beta = 3.6$ the system is unstable as Figure 2 shows and has oscillatory behavior.

6. Conclusions

In this paper, a discrete-time epidemic model for infectious diseases was presented and studied. The vital dynamics such as natural deaths, deaths due to disease, newborns, and immigrants are included in the model and number of individuals in population is not fixed. The model includes a perfect but temporary vaccination program which is performed on both newcomers and susceptibles. For under study model, the SIVS epidemic model, some basic properties such as two equilibria of the model and basic reproduction number \mathcal{R}_0 were obtained. Next, the stability of the equilibria were investigated and proved that disease-free equilibrium E_{df} and endemic equilibrium E_e are stable if $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$, respectively. The bifurcations of the model were studied also and it was shown that under assumptions on the model that guaranties positivity of the solutions, the model has only the fold (transcritical) bifurcation when $\mathcal{R}_0 = 1$ and the flip (period-doubling) bifurcation and the Neimark-Sacker bifurcation don't appear. However, as it was shown in simulations if we omit some restrictions on bifurcation parameter, the flip bifurcation may also be appeared. Moreover, the theoretical results of the paper were investigated and confirmed in some numerical experiments via diagrams of bifurcations, Lyapunov exponents, and solutions of the model.

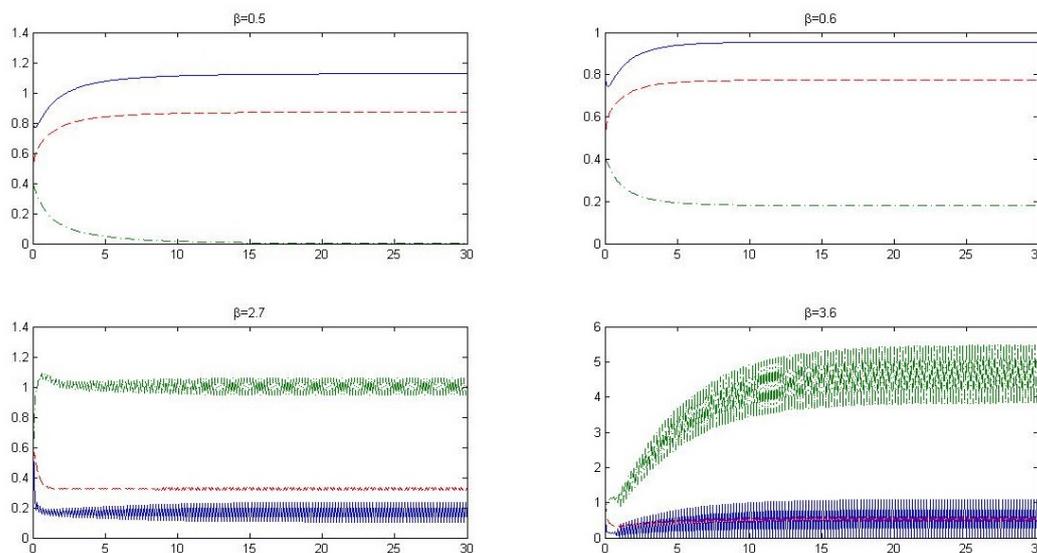


Figure 4: Solutions of the model for various values of β , $I(t)$:'- green line, $S(t)$:'- blue line, $V(t)$:'- red line.

References

- [1] Fred Brauer and Carlos Castillo-Chavez. *Mathematical models in population biology and epidemiology*, volume 1. Springer, 2001.
- [2] Linda JS Allen. *Introduction to mathematical biology*. Pearson/Prentice Hall, 2007.
- [3] Rahman Farnoosh and Mahmood Parsamanesh. Stochastic differential equation systems for an SIS epidemic model with vaccination and immigration. *Communications in Statistics-Theory and Methods*, 46(17):8723–8736, 2017.
- [4] Mahmood Parsamanesh. Global stability analysis of a VEISV model for network worm attack. *University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics*, 79(4):179–188, 2017.
- [5] Saber Elaydi. *An introduction to difference equations*. Springer Science & Business Media, 2005.
- [6] Keith E Emmert and Linda JS Allen. Population persistence and extinction in a discrete-time, stage-structured epidemic model. *Journal of Difference Equations and Applications*, 10(13-15):1177–1199, 2004.
- [7] Zengyun Hu, Zhidong Teng, and Haijun Jiang. Stability analysis in a class of discrete SIRS epidemic models. *Nonlinear Analysis: Real World Applications*, 13(5):2017–2033, 2012.
- [8] Qianqian Cui and Qiang Zhang. Global stability of a discrete SIR epidemic model with vaccination and treatment. *Journal of Difference Equations and Applications*, 21(2):111–117, 2015.
- [9] Linda JS Allen. Some discrete-time SI, SIR, and SIS epidemic models. *Mathematical biosciences*, 124(1):83–105, 1994.
- [10] Abdul-Aziz Yakubu. Allee effects in a discrete-time SIS epidemic model with infected newborns. *Journal of Difference Equations and Applications*, 13(4):341–356, 2007.
- [11] Christopher M Kribs-Zaleta and Jorge X Velasco-Hernández. A simple vaccination model with multiple endemic states. *Mathematical biosciences*, 164(2):183–201, 2000.
- [12] Wei Yang, Chengjun Sun, and Julien Arino. Global analysis for a general epidemiological model with vaccination and varying population. *Journal of Mathematical Analysis and Applications*, 372(1):208–223, 2010.
- [13] Rahman Farnoosh and Mahmood Parsamanesh. Disease extinction and persistence in a discrete-time SIS epidemic model with vaccination and varying population size. *Filomat*, 31(15):4735–4747, 2017.
- [14] Mahmood Parsamanesh. Global dynamics of an sivs epidemic model with bilinear incidence rate. *Italian Journal of Pure and Applied Mathematics*, 40:544–557, 2018.
- [15] Mahmood Parsamanesh and Majid Erfanian. Global dynamics of an epidemic model with standard incidence rate and vaccination strategy. *Chaos, Solitons & Fractals*, 117:192–199, 2018.
- [16] Herbert W Hethcote. The mathematics of infectious diseases. *SIAM review*, 42(4):599–653, 2000.
- [17] Linda JS Allen and P Van Den Driessche. The basic reproduction number in some discrete-time epidemic models. *Journal of Difference Equations and Applications*, 14(10-11):1127–1147, 2008.
- [18] Yuri A Kuznetsov. *Elements of applied bifurcation theory*, volume 112. Springer Science & Business Media, 2013.
- [19] Guilin Wen, Daolin Xu, and Xu Han. On creation of hopf bifurcations in discrete-time nonlinear systems. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 12(2):350–355, 2002.