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The Left Conformable Fractional Hermite-Hadamard Type Inequalities for Convex Functions

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Abstract. In this paper, a new fractional Hermite-Hadamard type inequality for convex functions is obtained by using only the left conformable fractional integral. Also, to have new fractional trapezoid and midpoint type inequalities for the differentiable convex functions, two new equalities are proved.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is well known in the literature as Hermite-Hadamard's inequality [3, 4].

In [2, 8], the authors used the following equality to obtain trapezoid type inequalities and some applications:

Lemma 1.1. *Let $a, b \in I$ with $a < b$ and $f : I^\circ \to \mathbb{R}$ is a differentiable mapping (I° is the interior of I). If $f' \in L[a, b]$, then we have*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)\,du = \frac{b-a}{2}\int_0^1 (1-2t)\, f'(ta+(1-t)b)\,dt. \tag{2}$$

In [5], Kırmacı used the following equality to obtain midpoint type inequalities and some applications:

Lemma 1.2. *Let $a, b \in I$ with $a < b$ and $f : I^\circ \to \mathbb{R}$ is a differentiable mapping (I° the interior of I). If $f' \in L[a, b]$, then we have*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right) &= (b-a)\int_0^{1/2} t f'(ta+(1-t)b)\,dt \\ &+ \int_{1/2}^1 (t-1)\, f'(ta+(1-t)b)\,dt. \end{aligned} \tag{3}$$

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Definition 1.3. *[9, page 12]. A function f defined on I has a support at $x_0 \in I$ if there exist an affine functions $A(x) = f(x_0) + m(x - x_0)$ such that $A(x) \leq f(x)$ for all $x \in I$. The graph of the support function A is called a line of support for f at x_0.*

Theorem 1.4. *[9, page 12] $f : (a, b) \to \mathbb{R}$ is a convex function if and only if there is at least one line of support for f at each $x_0 \in (a, b)$.*

Following definitions of the left and right side Riemann-Liouville fractional integrals are well known in the literature.

Definition 1.5. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f \in L[a, b]$. The left and right Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t) dt, \; x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha - 1} f(t) dt, \; x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-t} t^{\alpha - 1} dt$ (see [7, page 69] and [11, page 4]).

The beta function and incomplete beta function defined as follows:

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u + v)} = \int_0^1 t^{u-1} (1 - t)^{v-1} dt, \quad u, v > 0,$$

$$B_w(u, v) = \int_0^w t^{u-1} (1 - t)^{v-1} dt \quad u, v > 0 \text{ and } 0 \leq w \leq 1.$$

In [6], Kunt et al. proved the following fractional Hermite-Hadamard type inequality via the left Riemann-Liouville fractional integral and next equalities:

Theorem 1.6. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a convex function. If $f \in L[a, b]$, then the following inequality for the left Riemann-Liouville fractional integral holds:*

$$f\left(\frac{\alpha a + b}{\alpha + 1}\right) \leq \frac{\Gamma(\alpha + 1)}{(b - a)^{\alpha}} J_{a+}^{\alpha} f(b) \leq \frac{\alpha f(a) + f(b)}{\alpha + 1} \tag{4}$$

with $\alpha > 0$.

Lemma 1.7. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $f' \in L[a, b]$, then the following equality for the left Riemann-Liouville fractional integrals holds:*

$$\frac{\alpha f(a) + f(b)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b - a)^{\alpha}} J_{a+}^{\alpha} f(b) = \frac{b - a}{\alpha + 1} \int_0^1 [1 - (\alpha + 1) t^{\alpha}] f'(ta + (1 - t) b) dt \tag{5}$$

with $\alpha > 0$.

Lemma 1.8. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $f' \in L[a, b]$, then the following equality for the left Riemann-Liouville fractional integrals holds:*

$$\frac{\Gamma(\alpha + 1)}{(b - a)^{\alpha}} J_{a+}^{\alpha} f(b) - f\left(\frac{\alpha a + b}{\alpha + 1}\right) \tag{6}$$

$$= (b - a) \left[\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha} f'(ta + (1 - t) b) dt + \int_{\frac{\alpha}{\alpha+1}}^1 (t^{\alpha} - 1) f'(ta + (1 - t) b) dt \right]$$

with $\alpha > 0$.

Following definitions of the left and right side conformable fractional integrals given in [1] (see also [10]):

Definition 1.9. *Let $\alpha \in (n, n+1]$, $n = 0, 1, 2, \ldots$, $\beta = \alpha - n$, $a, b \in \mathbb{R}$ with $a < b$ and $f \in L[a, b]$. The left and right conformable fractional integrals $I_\alpha^a f$ and ${}^bI_\alpha f$ of order $\alpha > 0$ are defined by*

$$I_\alpha^a f(t) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx, \; t > a$$

and

$${}^bI_\alpha f(t) = \frac{1}{n!} \int_t^b (x-t)^n (b-x)^{\beta-1} f(x)dx, \; t < b$$

respectively.

It is easily seen that if one takes $\alpha = n+1$ in the Definition 1.9 (for the left and right conformable fractional integrals), one has the Definition 1.5 (the left and right Riemann-Liouville fractional integrals) for $\alpha \in \mathbb{N}$.

In [10], Set et al. proved following Hermite-Hadamrd type inequality via conformable fractional integrals:

Theorem 1.10. *Let $f : [a, b] \to \mathbb{R}$ be a function with $0 \le a < b$ and $f \in L[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for conformable fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha \Gamma(\alpha-n)} \left[I_\alpha^a f(b) + {}^bI_\alpha f(a) \right] \le \frac{f(a)+f(b)}{2} \tag{7}$$

with $\alpha \in (n, n+1]$.

In literature, there are hundreds studies for Hermite-Hadamard type inequality by using the left and right fractional integrals (such as Riemann-Liouville fractional integrals, Hadamard fractional integrals, Conformable fractional integrals etc.). In all of them, the left and right fractional integrals are used together. As much as we know, the first study for Hermite-Hadamard type inequality by using only the left Riemann-Liouville fractional integral is given in [6] by Kunt et al.

In this paper, our aim is obtaining new fractional Hermite-Hadamard type inequality by using only the left conformable fractional integral for convex functions. Also we desire proving new equalities to have new conformable fractional trapezoid and midpoint type inequalities for the differentiable convex functions. This study generalises the studies [2, 5, 6, 8].

2. Main Results

Theorem 2.1. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a convex function. If $f \in L[a, b]$, then the following inequality for the left conformable fractional integral holds:*

$$\begin{aligned} f\left(\frac{(n+1)a + (\alpha-n)b}{\alpha+1}\right) &\le \frac{\Gamma(\alpha+1)}{(b-a)^\alpha \Gamma(\alpha-n)} I_\alpha^a f(b) \\ &\le \frac{(n+1)f(a) + (\alpha-n)f(b)}{\alpha+1} \end{aligned} \tag{8}$$

with $n = 0, 1, 2\ldots$ and $\alpha \in (n, n+1]$.

Proof. Let $n = 0, 1, 2...$ and $\alpha \in (n, n+1]$. Since f is convex on $[a,b]$, using Theorem 1.4, there is at least one line of support

$$A(x) = f\left(\frac{(n+1)a + (\alpha - n)b}{\alpha + 1}\right) + m\left(x - \frac{(n+1)a + (\alpha - n)b}{\alpha + 1}\right) \leq f(x) \tag{9}$$

for all $x \in [a,b]$. From (9), we have

$$\begin{aligned} A(ta + (1-t)b) &= f\left(\frac{(n+1)a + (\alpha - n)b}{\alpha + 1}\right) + m\left(ta + (1-t)b - \frac{(n+1)a + (\alpha - n)b}{\alpha + 1}\right) \\ &\leq f(ta + (1-t)b) \end{aligned} \tag{10}$$

for all $t \in [0,1]$. Multiplying both sides of (10) with $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$ and integrating over $[0,1]$ respect to t, we have

$$\begin{aligned}
&\frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1} A(ta + (1-t)b)\,dt \\
&= \frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1}\left[f\left(\frac{(n+1)a + (\alpha - n)b}{\alpha + 1}\right) + m\left(ta + (1-t)b - \frac{(n+1)a + (\alpha - n)b}{\alpha + 1}\right)\right]dt \\
&= f\left(\frac{(n+1)a + (\alpha - n)b}{\alpha + 1}\right)\frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1}\,dt \\
&\quad + \frac{m}{n!}\left[\int_0^1 t^n(1-t)^{\alpha-n-1}[ta + (1-t)b]\,dt - \frac{(n+1)a + (\alpha - n)b}{\alpha + 1}\int_0^1 t^n(1-t)^{\alpha-n-1}\,dt\right] \\
&= f\left(\frac{(n+1)a + (\alpha - n)b}{\alpha + 1}\right)\frac{B(n+1, \alpha - n)}{n!} \\
&\quad + \frac{m}{n!}\left[\frac{(n+1)a + (\alpha - n)b}{\alpha + 1}B(n+1, \alpha - n) - \frac{(n+1)a + (\alpha - n)b}{\alpha + 1}B(n+1, \alpha - n)\right] \\
&= f\left(\frac{(n+1)a + (\alpha - n)b}{\alpha + 1}\right)\frac{B(n+1, \alpha - n)}{n!} \\
&\leq \frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1} f(ta + (1-t)b)\,dt \\
&= \frac{1}{(b-a)^\alpha}\frac{1}{n!}\int_a^b (b-t)^n(t-a)^{\alpha-n-1} f(t)dt = \frac{1}{(b-a)^\alpha}\ I_\alpha^a f(b)\ .
\end{aligned}$$

It means that

$$\begin{aligned} f\left(\frac{(n+1)a + (\alpha - n)b}{\alpha + 1}\right) &\leq \frac{n!}{B(n+1, \alpha - n)}\frac{1}{(b-a)^\alpha}\ I_\alpha^a f(b) \\ &= \frac{\Gamma(\alpha + 1)}{(b-a)^\alpha\,\Gamma(\alpha - n)}\ I_\alpha^a f(b). \end{aligned} \tag{11}$$

On the other hand, using the convexity of f on $[a,b]$, we have

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \tag{12}$$

for all $t \in [0,1]$. Multiplying both sides of (12) with $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$ and integrating over $[0,1]$ respect to t, we have

$$\begin{aligned}
&\frac{1}{n!}\int_0^1 t^n (1-t)^{\alpha-n-1} f(ta+(1-t)b)\,dt\\
&= \frac{1}{(b-a)^\alpha}\frac{1}{n!}\int_a^b (b-t)^n (t-a)^{\alpha-n-1} f(t)dt\\
&= \frac{1}{(b-a)^\alpha}\; I_\alpha^a f(b) \;\; \leq f(a)\frac{1}{n!}\int_0^1 t^{n+1}(1-t)^{\alpha-n-1}\,dt\\
&+f(b)\frac{1}{n!}\int_0^1 t^n (1-t)^{\alpha-n}\,dt = \frac{1}{n!}\frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1}B(n+1,\alpha-n).
\end{aligned}$$

It means that

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\,\Gamma(\alpha-n)}\; I_\alpha^a f(b) \;\; = \frac{n!}{B(n+1,\alpha-n)}\frac{1}{(b-a)^\alpha}\; I_\alpha^a f(b)\\
&\leq \frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1}. \qquad (13)
\end{aligned}$$

By using (11) and (13), we have (8). This completes the proof. □

Remark 2.2. *In Theorem 2.1,*

1. *If one takes $\alpha = n+1$, one has* (4),
2. *If one takes $\alpha = n+1$, after that if one takes $\alpha = 1$, one has* (1) *(Hermite-Hadamard inequality).*

2.1. Lemmas

In this section, we will prove the main equalities related to Lemma 1.1, Lemma 1.2, Lemma 1.7 and Lemma 1.8.

Lemma 2.3. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $f' \in L[a, b]$, then the following equality for the left conformable fractional integrals holds:*

$$\begin{aligned}
&\frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha\,\Gamma(\alpha-n)}\; I_\alpha^a f(b)\\
&= \frac{b-a}{(\alpha+1)B(n+1,\alpha-n)}\int_0^1 [(\alpha-n)B(n+1,\alpha-n)\\
&-(\alpha+1)B_t(n+1,\alpha-n)]\,f'(ta+(1-t)b)\,dt \qquad (14)
\end{aligned}$$

with $n = 0, 1, 2...$ and $\alpha \in (n, n+1]$.

Proof. If we apply the partial integration to the right-hand side of the equation (14), we have

$$\begin{aligned}
&\frac{b-a}{(\alpha+1)B(n+1,\alpha-n)}\int_0^1 \begin{bmatrix} (\alpha-n)B(n+1,\alpha-n)- \\ (\alpha+1)B_t(n+1,\alpha-n) \end{bmatrix} f'(ta+(1-t)b)\,dt\\
&= \frac{b-a}{(\alpha+1)}\int_0^1 \left[(\alpha-n)-(\alpha+1)\frac{B_t(n+1,\alpha-n)}{B(n+1,\alpha-n)}\right] f'(ta+(1-t)b)\,dt\\
&= (b-a)\begin{bmatrix} \frac{\alpha-n}{\alpha+1}\int_0^1 f'(ta+(1-t)b)\,dt \\ -\frac{1}{B(n+1,\alpha-n)}\int_0^1 B_t(n+1,\alpha-n)\,f'(ta+(1-t)b)\,dt \end{bmatrix}\\
&= (b-a)\begin{bmatrix} \frac{\alpha-n}{\alpha+1}\;\frac{f(ta+(1-t)b)}{a-b}\Big|_0^1 \\ -\frac{1}{B(n+1,\alpha-n)}\int_0^1 \left(\int_0^t x^n(1-x)^{\alpha-n-1}\,dx\right) f'(ta+(1-t)b)\,dt \end{bmatrix}
\end{aligned}$$

$$= (b-a)\left[\begin{array}{c} \frac{\alpha-n}{\alpha+1}\frac{f(b)-f(a)}{b-a} \\ -\frac{1}{B(n+1,\alpha-n)}\left(\begin{array}{c} \left(\int_0^t x^n (1-x)^{\alpha-n-1}\,dx\right)\frac{f(ta+(1-t)b)}{a-b}\Big|_0^1 \\ -\int_0^1 t^n (1-t)^{\alpha-n-1}\,\frac{f(ta+(1-t)b)}{a-b}dt \end{array}\right) \end{array}\right]$$

$$= \left[\begin{array}{c} \frac{\alpha-n}{\alpha+1}\left(f(b)-f(a)\right) \\ -\frac{1}{B(n+1,\alpha-n)}\left(\begin{array}{c} -\left(B(n+1,\alpha-n)\right)f(a) \\ +\int_0^1 t^n (1-t)^{\alpha-n-1} f(ta+(1-t)b)\,dt \end{array}\right) \end{array}\right]$$

$$= \left[\begin{array}{c} \frac{\alpha-n}{\alpha+1}\left(f(b)-f(a)\right)+f(a) \\ -\frac{1}{B(n+1,\alpha-n)}\int_0^1 t^n (1-t)^{\alpha-n-1} f(ta+(1-t)b)\,dt \end{array}\right]$$

$$= \left[\begin{array}{c} \frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1} \\ -\frac{1}{B(n+1,\alpha-n)}\int_0^1 t^n (1-t)^{\alpha-n-1} f(ta+(1-t)b)\,dt \end{array}\right]$$

$$= \frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)}\; I_{\alpha}^{a}f(b)\;.$$

This completes the proof. □

Remark 2.4. *In Lemma 2.3,*

1. *If one takes $\alpha = n+1$, one has the Lemma 1.7,*
2. *If one takes $\alpha = n+1$, after that if one takes $\alpha = 1$, one has the Lemma 1.1.*

Lemma 2.5. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b). If $f' \in L[a,b]$, then the following equality for the left conformable fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)}\; I_{\alpha}^{a}f(b)\; - f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) \tag{15}$$
$$= \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} B_t(n+1,\alpha-n)\,f'(ta+(1-t)b)\,dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left(\begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array}\right) f'(ta+(1-t)b)\,dt \end{array}\right]$$

with $n = 0,1,2...$ and $\alpha \in (n, n+1]$.

Proof. If we apply the partial integration to the right-hand side of the equation (15), we have

$$\frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} B_t(n+1,\alpha-n)\,f'(ta+(1-t)b)\,dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left(\begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array}\right) f'(ta+(1-t)b)\,dt \end{array}\right]$$
$$= \quad b-a\left[\begin{array}{c} \frac{1}{B(n+1,\alpha-n)}\int_0^1 B_t(n+1,\alpha-n)\,f'(ta+(1-t)b)\,dt \\ -\int_{\frac{n+1}{\alpha+1}}^1 f'(ta+(1-t)b)\,dt \end{array}\right]$$

$$
\begin{aligned}
&= b-a\left[\begin{array}{c} \frac{1}{B(n+1,\alpha-n)}\int_0^1 B_t\left(n+1,\alpha-n\right) f'\left(ta+(1-t)b\right)dt \\ -\int_{\frac{n+1}{\alpha+1}}^1 f'\left(ta+(1-t)b\right)dt \end{array}\right] \\
&= (b-a)\left[\left(\begin{array}{c} \frac{1}{B(n+1,\alpha-n)} B_t\left(n+1,\alpha-n\right)\frac{f(ta+(1-t)b)}{a-b}\Big|_0^1 \\ -\int_0^1 t^n\left(1-t\right)^{\alpha-n+1}\frac{f(ta+(1-t)b)}{a-b}dt \end{array}\right)-\left(\frac{f\left(ta+(1-t)b\right)}{a-b}\Big|_{\frac{n+1}{\alpha+1}}^1\right)\right] \\
&= \left[\begin{array}{c} \left(-f(a)+\frac{1}{B(n+1,\alpha-n)}\int_0^1 t^n\left(1-t\right)^{\alpha-n+1} f\left(ta+(1-t)b\right)dt\right) \\ +\left(f(a)-f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)\right) \end{array}\right] \\
&= \frac{1}{B\left(n+1,\alpha-n\right)}\int_0^1 t^n\left(1-t\right)^{\alpha-n+1} f\left(ta+(1-t)b\right)dt - f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) \\
&= \frac{\Gamma\left(\alpha+1\right)}{(b-a)^\alpha\,\Gamma\left(\alpha-n\right)}\; I_\alpha^a f(b) \;- f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right).
\end{aligned}
$$

This completes the proof. □

Remark 2.6. *In Lemma 2.5,*

1. *If one takes $\alpha = n+1$, one has the Lemma 1.8,*
2. *If one takes $\alpha = n+1$, after that if one takes $\alpha = 1$, one has the Lemma 1.2.*

2.2. The Left Conformable Fractional Trapezoid and Midpoint Type Inequalities

In this section we will obtain some new left conformable fractional trapezoid and midpoint type inequalities by using Lemma 2.3 and Lemma 2.5.

Theorem 2.7. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $\left|f'\right|$ is convex on $[a, b]$, then the following left conformable fractional integral inequality holds:*

$$
\left|\frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1}-\frac{\Gamma\left(\alpha+1\right)}{(b-a)^\alpha\,\Gamma\left(\alpha-n\right)}\; I_\alpha^a f(b)\right| \tag{16}
$$

$$
\leq \frac{b-a}{\left(\alpha+1\right)B\left(n+1,\alpha-n\right)}\left[\left|f'(a)\right|K_1\left(\alpha,n\right)+\left|f'(b)\right|\right]K_2\left(\alpha,n\right)
$$

where

$$
\begin{aligned}
K_1\left(\alpha,n\right) &= \int_0^1 \left|(\alpha-n)B\left(n+1,\alpha-n\right)-(\alpha+1)B_t\left(n+1,\alpha-n\right)\right| t\,dt, \\
K_2\left(\alpha,n\right) &= \int_0^1 \left|(\alpha-n)B\left(n+1,\alpha-n\right)-(\alpha+1)B_t\left(n+1,\alpha-n\right)\right|(1-t)\,dt,
\end{aligned}
$$

with $n = 0, 1, 2...$ and $\alpha \in (n, n+1]$.

Proof. Using Lemma 2.3 and the convexity of $\left|f'\right|$, we have

$$
\left|\frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1}-\frac{\Gamma\left(\alpha+1\right)}{(b-a)^\alpha\,\Gamma\left(\alpha-n\right)}\; I_\alpha^a f(b)\right|
$$

$$
\leq \frac{b-a}{\left(\alpha+1\right)B\left(n+1,\alpha-n\right)}\int_0^1\left|\begin{array}{c}(\alpha-n)B\left(n+1,\alpha-n\right)\\ -(\alpha+1)B_t\left(n+1,\alpha-n\right)\end{array}\right|\left|f'\left(ta+(1-t)b\right)\right|dt
$$

$$
\leq \frac{b-a}{\left(\alpha+1\right)B\left(n+1,\alpha-n\right)}\int_0^1\left|\begin{array}{c}(\alpha-n)B\left(n+1,\alpha-n\right)\\ -(\alpha+1)B_t\left(n+1,\alpha-n\right)\end{array}\right|\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt
$$

$$\leq \frac{b-a}{(\alpha+1)B(n+1,\alpha-n)}\left[\begin{array}{c} \left|f'(a)\right|\int_0^1 \left|\begin{array}{c}(\alpha-n)B(n+1,\alpha-n)\\-(\alpha+1)B_t(n+1,\alpha-n)\end{array}\right| tdt \\ +\left|f'(b)\right|\int_0^1 \left|\begin{array}{c}(\alpha-n)B(n+1,\alpha-n)\\-(\alpha+1)B_t(n+1,\alpha-n)\end{array}\right|(1-t)\,dt\end{array}\right].$$

This completes the proof. □

Remark 2.8. *In Theorem 2.7,*

1. *If one takes $\alpha = n+1$, one has the inequality proved in [6, Theorem 4].*
2. *If one takes $\alpha = n+1$, after that if one takes $\alpha = 1$, one has the inequality proved in [2, Theorem 2.2].*

Example 2.9. *Under the condition of Theorem 2.7 with $f : [a,b] \subset (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$,*

1. *if we take $n = 0$, $\alpha = 1$, we obtain*

$$\left|H^{-1}(a,b) - L^{-1}(a,b)\right| \leq \frac{b-a}{4}H^{-1}\left(a^2,b^2\right),$$

2. *if we take $n = 1$, $\alpha = 2$, we obtain*

$$\left|\frac{H^{-1}\left(\frac{a}{2},b\right)}{3} - \frac{1}{b-a}\left[A(a,b) - abL^{-1}(a,b)\right]\right| \leq \frac{b-a}{3}\left[\frac{5}{24a^2} + \left(\frac{2\sqrt{3}}{9} - \frac{5}{24}\right)\frac{1}{b^2}\right].$$

Proof. 1. $\left|f'\right|$ is convex function on $[a,b]$ and the coefficients are

$$K_1(1,0) = \int_0^1 |B(1,1) - 2B_t(1,1)|\,tdt = \int_0^1 |1-2t|\,tdt = \frac{1}{4},$$

$$K_2(1,0) = \int_0^1 |B(1,1) - 2B_t(1,1)|\,(1-t)dt = \int_0^1 |1-2t|\,(1-t)dt = \frac{1}{4}.$$

2. $\left|f'\right|$ is convex function on $[a,b]$ and the coefficients are

$$K_1(2,1) = \int_0^1 |B(2,1) - 2B_t(2,1)|\,tdt = \int_0^1 \left|\frac{1}{2} - \frac{3t^2}{2}\right| tdt = \frac{5}{24},$$

$$K_2(2,1) = \int_0^1 |B(2,1) - 2B_t(2,1)|\,(1-t)dt = \int_0^1 \left|\frac{1}{2} - \frac{3t^2}{2}\right|(1-t)dt = \frac{2\sqrt{3}}{9} - \frac{5}{24}.$$

Thus, the example is completed. □

Theorem 2.10. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b). If $\left|f'\right|^q$ is convex on $[a,b]$ for $q \geq 1$, then the following left conformable fractional integral inequality holds:*

$$\left|\frac{(n+1)f(a) + (\alpha-n)f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}\; I_{\alpha}^{a}f(b)\right| \tag{17}$$

$$\leq \frac{b-a}{(\alpha+1)B(n+1,\alpha-n)}K_3^{1-\frac{1}{q}}(\alpha,n)\left(\left|f'(a)\right|^q K_1(\alpha,n) + \left|f'(b)\right|^q K_2(\alpha,n)\right)^{\frac{1}{q}}$$

where $K_1(\alpha,n)$ and $K_2(\alpha,n)$ are same as in Theorem 2.7 and

$$K_3(\alpha,n) = \int_0^1 |(\alpha-n)B(n+1,\alpha-n) - (\alpha+1)B_t(n+1,\alpha-n)|\,dt,$$

with $n = 0,1,2...$ and $\alpha \in (n, n+1]$.

Proof. Using Lemma 2.3, power mean inequality and the convexity of $\left|f'\right|^q$, we have

$$\begin{aligned}
&\left|\frac{(n+1)\,f(a)+(\alpha-n)\,f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)}\;I_{\alpha}^{a}f(b)\;\right| \\
&\leq \frac{b-a}{(\alpha+1)\,B(n+1,\alpha-n)}\int_0^1\left|\begin{array}{c}(\alpha-n)\,B(n+1,\alpha-n)\\-(\alpha+1)\,B_t(n+1,\alpha-n)\end{array}\right|\left|f'(ta+(1-t)\,b)\right|dt \\
&\leq \frac{b-a}{(\alpha+1)\,B(n+1,\alpha-n)}\left(\int_0^1\left|\begin{array}{c}(\alpha-n)\,B(n+1,\alpha-n)\\-(\alpha+1)\,B_t(n+1,\alpha-n)\end{array}\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\left(\int_0^1\left|\begin{array}{c}(\alpha-n)\,B(n+1,\alpha-n)\\-(\alpha+1)\,B_t(n+1,\alpha-n)\end{array}\right|\left|f'(ta+(1-t)\,b)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{(\alpha+1)\,B(n+1,\alpha-n)}\left(\int_0^1\left|\begin{array}{c}(\alpha-n)\,B(n+1,\alpha-n)\\-(\alpha+1)\,B_t(n+1,\alpha-n)\end{array}\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\left(\int_0^1\left|\begin{array}{c}(\alpha-n)\,B(n+1,\alpha-n)\\-(\alpha+1)\,B_t(n+1,\alpha-n)\end{array}\right|\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{(\alpha+1)\,B(n+1,\alpha-n)}\left(\int_0^1\left|\begin{array}{c}(\alpha-n)\,B(n+1,\alpha-n)\\-(\alpha+1)\,B_t(n+1,\alpha-n)\end{array}\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\left(\begin{array}{c}\left|f'(a)\right|^q\int_0^1\left|(\alpha-n)\,B(n+1,\alpha-n)-(\alpha+1)\,B_t(n+1,\alpha-n)\right|t\,dt\\+\left|f'(b)\right|^q\int_0^1\left|(\alpha-n)\,B(n+1,\alpha-n)-(\alpha+1)\,B_t(n+1,\alpha-n)\right|(1-t)\,dt\end{array}\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Remark 2.11. *In Theorem 2.10,*

1. *If one takes $\alpha=n+1$, one has the inequality proved in [6, Theorem 5].*
2. *If one takes $\alpha=n+1$, after that if one takes $\alpha=1$, one has the inequality proved in [8, Theorem 1].*

Example 2.12. *Under the condition of Theorem 2.10 with $f:[a,b]\subset(0,\infty)\to\mathbb{R}\,,f(x)=\frac{1}{x}$,*

1. *if we take $n=0$, $\alpha=1$, we obtain*

$$\left|H^{-1}(a,b)-L^{-1}(a,b)\right|\leq\frac{b-a}{4}\left(H^{-1}\left(a^{2q},b^{2q}\right)\right)^{\frac{1}{q}},$$

2. *if we take $n=1$, $\alpha=2$, we obtain*

$$\left|\frac{H^{-1}\left(\frac{a}{2},b\right)}{3}-\frac{1}{b-a}\left[A(a,b)-abL^{-1}(a,b)\right]\right|\leq\frac{b-a}{3}\left(\frac{2\sqrt{3}}{9}\right)^{1-\frac{1}{q}}\left[\frac{5}{24a^{2q}}+\left(\frac{2\sqrt{3}}{9}-\frac{5}{24}\right)\frac{1}{b^{2q}}\right]^{\frac{1}{q}}.$$

Proof. 1. $\left|f'\right|$ is convex function on $[a,b]$ and the coefficients are

$$K_3(1,0)\;=\;\int_0^1\left|B(1,1)-2B_t(1,1)\right|dt=\int_0^1\left|1-2t\right|dt=\frac{1}{2},$$

2. $\left|f'\right|$ is convex function on $[a,b]$ and the coefficients are

$$K_3(2,1) = \int_0^1 |B(2,1) - 2B_t(2,1)|\,dt = \int_0^1 \left|\frac{1}{2} - \frac{3t^2}{2}\right| dt = \frac{2\sqrt{3}}{9}.$$

Thus, the example is completed. □

Theorem 2.13. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b). If $\left|f'\right|^q$ is convex on $[a,b]$ for $q > 1$, then the following left conformable fractional integral inequality holds:*

$$\left|\frac{(n+1)\,f(a) + (\alpha - n)\,f(b)}{\alpha + 1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha - n)}\; I_{\alpha}^{a} f(b)\right| \tag{18}$$

$$\leq \frac{b-a}{(\alpha+1)\,B(n+1, \alpha-n)} K_4^{\frac{1}{p}}(\alpha, n) \left(\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}$$

where

$$K_4(\alpha, n) = \int_0^1 |(\alpha - n)\,B(n+1, \alpha - n) - (\alpha+1)\,B_t(n+1, \alpha-n)|^p\,dt,$$

with $\frac{1}{p} + \frac{1}{q} = 1$, $n = 0, 1, 2...$ *and* $\alpha \in (n, n+1]$.

Proof. Using Lemma 2.3, Hölder inequality and the convexity of $\left|f'\right|^q$, we have

$$\left|\frac{(n+1)\,f(a) + (\alpha - n)\,f(b)}{\alpha + 1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha - n)}\; I_{\alpha}^{a} f(b)\right|$$

$$\leq \frac{b-a}{(\alpha+1)\,B(n+1, \alpha-n)} \int_0^1 \left|\begin{array}{c}(\alpha-n)\,B(n+1,\alpha-n)\\ -(\alpha+1)\,B_t(n+1,\alpha-n)\end{array}\right| \left|f'(ta + (1-t)\,b)\right| dt$$

$$\leq \frac{b-a}{(\alpha+1)\,B(n+1, \alpha-n)} \left(\int_0^1 \left|\begin{array}{c}(\alpha-n)\,B(n+1,\alpha-n)\\ -(\alpha+1)\,B_t(n+1,\alpha-n)\end{array}\right|^p dt\right)^{\frac{1}{p}}$$
$$\left(\int_0^1 \left|f'(ta + (1-t)\,b)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{(\alpha+1)\,B(n+1, \alpha-n)} \left(\int_0^1 \left|\begin{array}{c}(\alpha-n)\,B(n+1,\alpha-n)\\ -(\alpha+1)\,B_t(n+1,\alpha-n)\end{array}\right|^p dt\right)^{\frac{1}{p}}$$
$$\left(\int_0^1 \left[t\left|f'(a)\right|^q + (1-t)\left|f'(b)\right|^q\right] dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{(\alpha+1)\,B(n+1, \alpha-n)} \left(\int_0^1 \left|\begin{array}{c}(\alpha-n)\,B(n+1,\alpha-n)\\ -(\alpha+1)\,B_t(n+1,\alpha-n)\end{array}\right|^p dt\right)^{\frac{1}{p}} \left(\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Remark 2.14. *In Theorem 2.13,*

1. *If one takes $\alpha = n + 1$, one has the inequality proved in [6, Theorem 6].*

2. *If one takes $\alpha = n+1$, after that if one takes $\alpha = 1$, one has the inequality proved in [2, Theorem 2.3].*

Theorem 2.15. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b). If $\left|f'\right|$ is convex on $[a,b]$, then the following left conformable fractional integral inequality holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}\ I_{\alpha}^{a}f(b)\ -f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)\right| \\
&\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\left|f'(a)\right|K_5(\alpha,n)+\left|f'(b)\right|K_6(\alpha,n)\right]
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
K_5(\alpha,n) &= \left(\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right| t dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left|B_t(n+1,\alpha-n)-B(n+1,\alpha-n)\right| t dt \end{array}\right), \\
K_6(\alpha,n) &= \left(\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right| (1-t) dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left|B_t(n+1,\alpha-n)-B(n+1,\alpha-n)\right| (1-t) dt \end{array}\right),
\end{aligned}$$

with $n = 0, 1, 2...$ and $\alpha \in (n, n+1]$.

Proof. Using Lemma 2.5 and the convexity of $\left|f'\right|$, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}\ I_{\alpha}^{a}f(b)\ -f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)\right| \\
&\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right| \left|f'(ta+(1-t)b)\right| dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array}\right| \left|f'(ta+(1-t)b)\right| dt \end{array}\right] \\
&\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right| \left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right] dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array}\right| \left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right] dt \end{array}\right] \\
&\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c} \left|f'(a)\right| \int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right| t dt \\ +\left|f'(b)\right| \int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right| (1-t) dt \\ +\left|f'(a)\right| \int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array}\right| t dt \\ +\left|f'(b)\right| \int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array}\right| (1-t) dt \end{array}\right] \\
&\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c} \left|f'(a)\right|\left(\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right| t dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left|B_t(n+1,\alpha-n)-B(n+1,\alpha-n)\right| t dt \end{array}\right) \\ +\left|f'(b)\right|\left(\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right| (1-t) dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array}\right| (1-t) dt \end{array}\right) \end{array}\right].
\end{aligned}$$

This completes the proof. □

Remark 2.16. *In Theorem 2.15,*

1. *If one takes $\alpha = n+1$, one has the inequality proved in [6, Theorem 7],*

2. *If one takes $\alpha = n+1$, after that if one takes $\alpha = 1$, one has the inequality proved in [5, Theorem 2.2].*

Example 2.17. *Under the condition of Theorem 2.15 with $f : [a,b] \subset (0,\infty) \to \mathbb{R}$,$f(x) = \frac{1}{x}$,*

1. *If we take $n = 0$, $\alpha = 1$, we obtain*

$$\left|L^{-1}(a,b) - A^{-1}(a,b)\right| \leq \frac{b-a}{4} H^{-1}\left(a^2, b^2\right).$$

2. *If we take $n = 1$, $\alpha = 2$, we obtain*

$$\left|\frac{1}{b-a}\left[A(a,b) - abL^{-1}(a,b)\right] - \frac{3A^{-1}(2a,b)}{2}\right| \leq \frac{b-a}{324}\left[\frac{41}{a^2} + \frac{23}{b^2}\right].$$

Proof. 1. $\left|f'\right|$ is convex function on $[a,b]$ and the coefficients are

$$\begin{aligned} K_5(1,0) &= \int_0^{1/2} |B_t(1,1)|\, tdt + \int_{1/2}^1 |B_t(1,1) - B(1,1)|\, tdt = \int_0^{1/2} t^2 dt + \int_{1/2}^1 |t-1|\, tdt = \frac{1}{8}, \\ K_6(1,0) &= \int_0^{1/2} |B_t(1,1)|\,(1-t)dt + \int_{1/2}^1 |B_t(1,1) - B(1,1)|\,(1-t)dt \\ &= \int_0^{1/2} t(1-t)dt + \int_{1/2}^1 |t-1|\,(1-t)dt = \frac{1}{8}. \end{aligned}$$

2. $\left|f'\right|$ is convex function on $[a,b]$ and the coefficients are

$$\begin{aligned} K_5(1,0) &= \int_0^{2/3} |B_t(2,1)|\, tdt + \int_{2/3}^1 |B_t(2,1) - B(2,1)|\, tdt = \int_0^{2/3} \frac{t^3}{2} dt + \int_{2/3}^1 \left|\frac{t^2-1}{2}\right| tdt = \frac{41}{648}, \\ K_6(1,0) &= \int_0^{2/3} |B_t(2,1)|\,(1-t)dt + \int_{2/3}^1 |B_t(2,1) - B(2,1)|\,(1-t)dt \\ &= \int_0^{2/3} \frac{t^2}{2}(1-t)dt + \int_{2/3}^1 \left|\frac{t^2-1}{2}\right|(1-t)dt = \frac{23}{648}. \end{aligned}$$

Thus, the example is completed. □

Theorem 2.18. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b). If $\left|f'\right|^q$ is convex on $[a,b]$ for $q \geq 1$, then the following left conformable fractional integral inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha \Gamma(\alpha-n)} \; I_\alpha^a f(b) \; - f\left(\frac{(n+1)a + (\alpha-n)b}{\alpha+1}\right)\right| \tag{20}$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)} \left[\begin{array}{c} K_7^{1-\frac{1}{q}}(\alpha,n) \begin{pmatrix} \left|f'(a)\right|^q K_8(\alpha,n) \\ +\left|f'(b)\right|^q K_9(\alpha,n) \end{pmatrix}^{\frac{1}{q}} \\ +K_{10}^{1-\frac{1}{q}}(\alpha,n) \begin{pmatrix} \left|f'(a)\right|^q K_{11}(\alpha,n) \\ +\left|f'(b)\right|^q K_{12}(\alpha,n) \end{pmatrix}^{\frac{1}{q}} \end{array}\right]$$

where

$$\begin{aligned}
K_7(\alpha,n) &= \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,dt,\\
K_8(\alpha,n) &= \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,tdt,\\
K_9(\alpha,n) &= \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,(1-t)\,dt,\\
K_{10}(\alpha,n) &= \int_{\frac{n+1}{\alpha+1}}^1 |B_t(n+1,\alpha-n)-B(n+1,\alpha-n)|\,dt,\\
K_{11}(\alpha,n) &= \int_{\frac{n+1}{\alpha+1}}^1 |B_t(n+1,\alpha-n)-B(n+1,\alpha-n)|\,tdt,\\
K_{12}(\alpha,n) &= \int_{\frac{n+1}{\alpha+1}}^1 |B_t(n+1,\alpha-n)-B(n+1,\alpha-n)|\,(1-t)\,dt,
\end{aligned}$$

with $n=0,1,2...$ *and* $\alpha\in(n,n+1]$.

Proof. Using Lemma 2.5, power mean inequality and the convexity of $\left|f'\right|^q$, we have

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\,\Gamma(\alpha-n)}\ I_\alpha^a f(b)\ -f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)\right|$$

$$\le \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c}\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\left|f'(ta+(1-t)b)\right|dt\\ +\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{c}B_t(n+1,\alpha-n)\\-B(n+1,\alpha-n)\end{array}\right|\left|f'(ta+(1-t)b)\right|dt\end{array}\right]$$

$$\le \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c}\left(\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}}\\ \times\left(\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{c}B_t(n+1,\alpha-n)\\-B(n+1,\alpha-n)\end{array}\right|dt\right)^{1-\frac{1}{q}}\\ \times\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{c}B_t(n+1,\alpha-n)\\-B(n+1,\alpha-n)\end{array}\right|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\le \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c}\left(\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}}\\ \times\left(\begin{array}{c}\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\\ \left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\end{array}\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{n+1}{\alpha+1}}^1 |B_t(n+1,\alpha-n)-B(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}}\\ \times\left(\begin{array}{c}\int_{\frac{n+1}{\alpha+1}}^1 |B_t(n+1,\alpha-n)-B(n+1,\alpha-n)|\\ \left[t\left|f'(a)\right|^q+\left|f'(b)\right|^q(1-t)\right]dt\end{array}\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)} \left[\begin{array}{c} \left(\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\, dt \right)^{1-\frac{1}{q}} \\ \times \left(\begin{array}{c} |f'(a)|^q \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\, t dt \\ + |f'(b)|^q \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| (1-t)\, dt \end{array} \right)^{\frac{1}{q}} \\ + \left(\int_{\frac{n+1}{\alpha+1}}^1 |B_t(n+1,\alpha-n) - B(n+1,\alpha-n)|\, dt \right)^{1-\frac{1}{q}} \\ \times \left(\begin{array}{c} |f'(a)|^q \int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array} \right| t dt \\ + |f'(b)|^q \int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array} \right| (1-t)\, dt \end{array} \right)^{\frac{1}{q}} \end{array} \right].$$

This completes the proof. □

Remark 2.19. *In Theorem 2.18,*

1. *If one takes $\alpha = n+1$, one has the inequality proved in [6, Theorem 8],*
2. *If one takes $\alpha = n+1$, after that if one takes $\alpha = 1$, one has the inequality proved in [6, Remark 9].*

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