Generalized Jordan Triple \((\sigma, \tau)\)-Higher Derivation on Triangular Algebras

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Abstract. Let \(\mathcal{R}\) be a commutative ring with unity, \(\mathfrak{A} = \text{Tri}(\mathcal{A}, \mathcal{B}, \mathcal{M})\) be a triangular algebra consisting of unital algebras \(\mathcal{A}, \mathcal{B}\) and \((\mathcal{A}, \mathcal{B})\)-bimodule \(\mathcal{M}\) which is faithful as a left \(\mathcal{A}\)-module and also as a right \(\mathcal{B}\)-module. Let \(\sigma\) and \(\tau\) be two automorphisms of \(\mathfrak{A}\). A family \(\Delta = \{\delta_0\}_{k \in \mathbb{N}}\) of \(\mathcal{R}\)-linear mappings \(\delta_0 : \mathfrak{A} \to \mathfrak{A}\) is said to be a generalized Jordan triple \((\sigma, \tau)\)-higher derivation on \(\mathfrak{A}\) if there exists a Jordan triple \((\sigma, \tau)\)-higher derivation \(\mathcal{D} = \{d_k\}_{k \in \mathbb{N}}\) on \(\mathfrak{A}\) such that \(\delta_0 = I_{\mathfrak{A}}\), the identity map of \(\mathfrak{A}\) and \(\delta_k(XYX) = \sum_{(i,j,k) = n} \delta_i(\sigma^{x^i}(X)d_j(\sigma^{x^{i+1}}(Y))d_k(\tau^{n-x}(X)))\) holds for all \(X, Y \in \mathfrak{A}\) and each \(n \in \mathbb{N}\). In this article, we study generalized Jordan triple \((\sigma, \tau)\)-higher derivation on \(\mathfrak{A}\) and prove that every generalized Jordan triple \((\sigma, \tau)\)-higher derivation on \(\mathfrak{A}\) is a generalized \((\sigma, \tau)\)-higher derivation on \(\mathfrak{A}\).

1. Introduction

Let \(\mathcal{R}\) be a commutative ring with unity and \(\mathcal{A}\) be an unital algebra over \(\mathcal{R}\) and \(Z(\mathcal{A})\) be the center of \(\mathcal{A}\). Recall that an \(\mathcal{R}\)-linear map \(d : \mathcal{A} \to \mathcal{A}\) is called a derivation (resp. Jordan derivation) on \(\mathcal{A}\) if \(d(xy) = d(x)y + xd(y)\) (resp. \(d(x^2) = d(x)x + xd(x)\)) holds for all \(x, y \in \mathcal{A}\). An \(\mathcal{R}\)-linear map \(d : \mathcal{A} \to \mathcal{A}\) is said to be a Jordan triple derivation on \(\mathcal{A}\) if \(d(xyx) = d(x)yx + xyd(x)\) holds for all \(x, y \in \mathcal{A}\). An \(\mathcal{R}\)-linear map \(d : \mathcal{A} \to \mathcal{A}\) is called a generalized derivation (resp. generalized Jordan derivation) on \(\mathcal{A}\) if there exists a derivation (resp. Jordan derivation) \(d\) on \(\mathcal{A}\) such that \(d(xy) = d(x)y + xd(y)\) (resp. \(d(x^2) = d(x)x + xd(x)\)) holds for all \(x, y \in \mathcal{A}\). Clearly, every generalized derivation on \(\mathcal{A}\) is a generalized Jordan derivation on \(\mathcal{A}\) but the converse need not be true in general. Zhu and Xiong [16] proved that every generalized Jordan derivation from a 2-torsion free semiprime ring with identity into itself is a generalized derivation.

The concept of derivation was extended in various directions to different rings and algebras. Let \(\sigma, \tau\) be two endomorphisms on a ring \(\mathcal{R}\). An additive map \(d : \mathcal{R} \to \mathcal{R}\) is called a generalized \((\sigma, \tau)\)-derivation (resp. generalized \(\sigma\)-derivation) on \(\mathcal{R}\) if there exists a \((\sigma, \tau)\)-derivation (resp. \(\sigma\)-derivation) \(d\) such that \(d(xy) = d(x)y + \sigma(x)d(y)\) (resp. \(d(x^2) = d(x)x + \sigma(x)d(x))\) holds for all \(x, y \in \mathcal{R}\) and \(d\) is said to be a generalized Jordan \((\sigma, \tau)\)-derivation on \(\mathcal{R}\) if there exists a Jordan \((\sigma, \tau)\)-derivation \(d\) such that \(d(xy) = d(x)y + \sigma(x)d(y)\) holds for all \(x, y \in \mathcal{R}\). It is easy to observe that a generalized \((I_{\mathcal{R}}, I_{\mathcal{R}})\)-derivation, generalized Jordan \((I_{\mathcal{R}}, I_{\mathcal{R}})\)-derivation and generalized Jordan triple \((I_{\mathcal{R}}, I_{\mathcal{R}})\)-derivation is...
simply a generalized derivation, generalized Jordan derivation and generalized Jordan triple derivation on \(\mathcal{R}\) respectively.

Let \(\mathbb{N}\) be the set of all nonnegative integers. Following Hasse and Schmidt [12], a family \(\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}\) of additive mappings \(d_n : \mathcal{R} \to \mathcal{R}\) such that \(d_0 = \text{id}_\mathcal{R}\), the identity map on \(\mathcal{R}\), is said to be a higher derivation (resp. Jordan higher derivation) on \(\mathcal{R}\) if \(d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)\) (resp. \(d_n(x^2) = \sum_{i+j=n} d_i(x)d_j(x)\)) holds for all \(x, y \in \mathcal{R}\) and for each \(n \in \mathbb{N}\). Furthermore, motivated by the concept of generalized derivation Cortes and Haetinger [8] introduced the notion of generalized higher derivation. A family \(\Delta = \{\delta_n\}_{n \in \mathbb{N}}\) of additive mappings \(\delta_n : \mathcal{R} \to \mathcal{R}\) such that \(\delta_0 = \text{id}_\mathcal{R}\), the identity map on \(\mathcal{R}\), is said to be a generalized higher derivation (resp. generalized Jordan higher derivation) on \(\mathcal{R}\) if there exists a higher derivation (resp. Jordan higher derivation) \(\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}\) on \(\mathcal{R}\) such that \(\delta_n(xy) = \sum_{i+j=n} \delta_i(x)d_j(y)\) (resp. \(\delta_n(x^2) = \sum_{i+j=n} \delta_i(x)d_j(x)\)) holds for all \(x, y \in \mathcal{R}\) and for each \(n \in \mathbb{N}\). Motivated by the notion of \((\sigma, \tau)\)-derivation the first author together with Khan and Haetinger [2] introduced the concept of \((\sigma, \tau)\)-derivation as follows: A family of \(\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}\) of additive mappings \(d_n : \mathcal{R} \to \mathcal{R}\) is said to be a \((\sigma, \tau)\)-higher derivation (resp. Jordan \((\sigma, \tau)\)-higher derivation) on \(\mathcal{R}\) if \(d_0 = \text{id}_\mathcal{R}\), the identity map on \(\mathcal{R}\), is said to be a generalized \((\sigma, \tau)\)-higher derivation (resp. generalized Jordan \((\sigma, \tau)\)-higher derivation) on \(\mathcal{R}\) if there exists a higher derivation (resp. Jordan higher derivation) \(\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}\) on \(\mathcal{R}\) such that \(\delta_n(xy) = \sum_{i+j=n} \delta_i(x)d_j(y)\) (resp. \(\delta_n(x^2) = \sum_{i+j=n} \delta_i(x)d_j(x)\)) holds for all \(x, y \in \mathcal{R}\) and for each \(n \in \mathbb{N}\). Also, they obtained that under certain assumptions, if \(\mathcal{R}\) is a prime ring of characteristic different from \(2\), then every generalized Jordan \((\sigma, \tau)\)-higher derivation on \(\mathcal{R}\) is a generalized \((\sigma, \tau)\)-higher derivation on \(\mathcal{R}\), where \(\sigma, \tau\) are commuting endomorphisms of \(\mathcal{R}\).

The \(\mathcal{R}\)-algebra \(\mathfrak{N} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mid a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}\) under the usual matrix operations is called a triangular algebra, where \(\mathcal{A} \) and \(\mathcal{B}\) are unital algebras over \(\mathcal{R}\) and \(\mathcal{M}\) is an \((\mathcal{A}, \mathcal{B})\)-bimodule. The notion of triangular algebra was first introduced by Chase [4] in 1960. Further, in the year 2000, Cheung [5] initiated the study of linear maps on triangular algebras. He described Lie derivations, commuting maps and automorphisms of triangular algebras [6, 7]. Recently, Han and Wei [11] studied generalized Jordan \((\sigma, \tau)\)-derivation on the triangular algebras \(\mathfrak{N}\) and obtained that if \(\mathfrak{N}\) is a triangular algebra consisting of unital algebra \(\mathcal{A} \), \(\mathcal{B}\) and \((\mathcal{A}, \mathcal{B})\)-bimodule \(\mathcal{M}\) which is faithful as a left \(\mathcal{A}\)-module and also faithful as a right \(\mathcal{B}\)-module, then the following statements are equivalent (i) \(\delta\) is a generalized Jordan \((\sigma, \tau)\)-derivation on \(\mathfrak{N}\), (ii) \(\delta\) is a generalized Jordan triple \((\sigma, \tau)\)-derivation on \(\mathfrak{N}\), (iii) \(\delta\) is a generalized \((\sigma, \tau)\)-derivation on \(\mathfrak{N}\). Motivated by [2, 11], our main purpose is to study generalized \((\sigma, \tau)\)-higher derivations on triangular algebras. In fact, we obtain the condition on a triangular algebra \(\mathfrak{N}\) under which every generalized Jordan triple \((\sigma, \tau)\)-higher derivation on \(\mathfrak{N}\) is a generalized \((\sigma, \tau)\)-higher derivation on \(\mathfrak{N}\).

In the last section of this article we shall give some applications of our results in a special case viz. nest algebra.

2. Preliminaries

Throughout, this paper we shall use the following notions: Let \(\mathcal{A}\) and \(\mathcal{B}\) be unital algebras over \(\mathcal{R}\) and let \(\mathcal{M}\) be \((\mathcal{A}, \mathcal{B})\)-bimodule which is faithful as a left \(\mathcal{A}\)-module, that is, for \(A \in \mathcal{A}\), \(AM = 0\) implies \(A = 0\) and also as a right \(\mathcal{B}\)-module, that is, for \(B \in \mathcal{B}\), \(MB = 0\) implies \(B = 0\). The triangular algebra \(\mathfrak{N} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})\) is 2-torsion free. The center of \(\mathfrak{N}\) is \(Z(\mathfrak{N}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid am = mb \text{ for all } m \in \mathcal{M} \right\}\). Define two natural projections \(\pi_\mathcal{A} : \mathfrak{N} \to \mathcal{A}\) and \(\pi_\mathcal{B} : \mathfrak{N} \to \mathcal{B}\) by \(\pi_\mathcal{A} \left( \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \right) = a\) and \(\pi_\mathcal{B} \left( \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \right) = b\). Moreover, \(\pi_\mathcal{A}(Z(\mathfrak{N})) \subseteq Z(\mathcal{A})\) and \(\pi_\mathcal{B}(Z(\mathfrak{N})) \subseteq Z(\mathcal{B})\) and there exists a unique algebraic isomorphism \(\xi : \pi_\mathcal{A}(Z(\mathfrak{N})) \to \pi_\mathcal{B}(Z(\mathfrak{N}))\) such that \(am = m\xi(a)\) for all \(a \in \pi_\mathcal{A}(Z(\mathfrak{N})), m \in \mathcal{M}\).
Let $1_{A}$ (resp. $1_{B}$) be the identity of the algebra $A$ (resp. $B$) and let $I$ be the identity of triangular algebra $A$, $e = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$, $f = I - e = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$ and $A_{11} = e A e$, $A_{12} = e A f$, $A_{22} = f A f$. Thus $A = e A e + e A f + f A f = A_{11} + A_{12} + A_{22}$, where $A_{11}$ is subalgebra of $A$ isomorphic to $A$, $A_{22}$ is subalgebra of $A$ isomorphic to $B$ and $A_{12}$ is $(A_{11}, A_{22})$-bimodule isomorphic to $M$. Also, $\pi_{A}(Z(A))$ and $\pi_{B}(Z(B))$ are isomorphic to $e Z(A) e$ and $f Z(B) f$ respectively. Then there is an algebra isomorphisms $\xi : e Z(A) e \rightarrow f Z(B) f$ such that $am = m \xi(a)$ for all $m \in e A f$.

Let $N$ be the set of all nonnegative integers, $\sigma, \tau$ be automorphisms of triangular algebra $A$ and $D = \{d_{n}\}_{n \in N}$ be the family of $R$-linear maps $d_{n} : A \rightarrow A$ such that $d_{0} = I_{A}$. Then $D$ is said to be a $(\sigma, \tau)$-higher derivation(resp. Jordan $(\sigma, \tau)$-higher derivation) on $A$ if $d_{n}(XY) = \sum_{i+j=n} d_{i}(\sigma^{n-i}(X))d_{j}(\tau^{n-j}(Y))$ (resp. $d_{n}(X^{2}) = \sum_{i+j=n} d_{i}(\sigma^{n-i}(X))d_{j}(\tau^{n-j}(X))$) for all $X, Y \in A$ and for each $n \in N$ and $D$ is said to be a Jordan triple $(\sigma, \tau)$-higher derivation on $A$ if $d_{n}(XYX) = \sum_{i+j+k=n} d_{i}(\sigma^{n-i}(X))d_{j}(\sigma^{k}(\tau(X)))d_{k}(\tau^{n-k}(X))$ for all $X, Y, Z \in A$ and for each $n \in N$. Obviously, every $(\sigma, \tau)$-higher derivation is a Jordan $(\sigma, \tau)$-higher derivation on $A$ and every Jordan $(\sigma, \tau)$-higher derivation is a Jordan triple $(\sigma, \tau)$-higher derivation on $A$ but the converse statements are not true in general. First two authors together with Parveen [3] proved the following result:

**Theorem 2.1.** Let $A = \text{Tri}(A, M, B)$ be a triangular algebra where $A$ and $B$ have only trivial idempotents, $\sigma, \tau$ be automorphisms of $A$ such that $\sigma \tau = \tau \sigma$ and let $D = \{d_{n}\}_{n \in N}$ be the family of $R$-linear maps $d_{n} : A \rightarrow A$ such that $d_{0} = I_{A}$. Then the following statements are equivalent:

(i) $D$ is a $(\sigma, \tau)$-higher derivation on $A$,
(ii) $D$ is a Jordan $(\sigma, \tau)$-higher derivation on $A$,
(iii) $D$ is a Jordan triple $(\sigma, \tau)$-higher derivation on $A$.

Motivated by the notion of generalized higher derivation on triangular algebra $A$, we introduce the notion of generalized $(\sigma, \tau)$-higher derivation on $A$.

Let $\Delta = \{\delta_{n}\}_{n \in N}$ be the family of $R$-linear maps $\delta_{n} : A \rightarrow A$ such that $\delta_{0} = I_{A}$, the identity map of $A$. Then $\Delta = \{\delta_{n}\}_{n \in N}$ is said to be a generalized $(\sigma, \tau)$-higher derivation on $A$ if there exists a $(\sigma, \tau)$-higher derivation $D = \{d_{n}\}_{n \in N}$ on $A$ and

\[
\delta_{n}(XY) = \sum_{i+j=n} \delta_{i}(\sigma^{n-i}(X))d_{j}(\tau^{n-j}(Y))
\]

for all $X, Y \in A$ and for each $n \in N$,

(ii) generalized Jordan $(\sigma, \tau)$-higher derivation on $A$ if there exists a Jordan $(\sigma, \tau)$-higher derivation $D = \{d_{n}\}_{n \in N}$ on $A$ and

\[
\delta_{n}(X^{2}) = \sum_{i+j=n} \delta_{i}(\sigma^{n-i}(X))d_{j}(\tau^{n-j}(X))
\]

for all $X \in A$ and for each $n \in N$,

(iii) generalized Jordan triple $(\sigma, \tau)$-higher derivation on $A$ if there exists a Jordan triple $(\sigma, \tau)$-higher derivation $D = \{d_{n}\}_{n \in N}$ on $A$ and

\[
\delta_{n}(XYX) = \sum_{i+j+k=n} \delta_{i}(\sigma^{n-i}(X))d_{j}(\sigma^{k}(\tau(X)))d_{k}(\tau^{n-k}(X))
\]

for all $X, Y \in A$ and for each $n \in N$. 
It can be easily seen that every generalized \((\sigma, \tau)-\)higher derivation is a generalized Jordan \((\sigma, \tau)-\)higher derivation on \(\mathcal{A}\) and every generalized Jordan \((\sigma, \tau)-\)higher derivation is a generalized Jordan triple \((\sigma, \tau)-\)higher derivation on \(\mathcal{A}\). But the converse need not be true in general. In fact, if \(\Delta = [\delta_n]_{n \in \mathbb{N}}\) is a generalized \((\sigma, \tau)-\)higher derivation associated with \((\sigma, \tau)-\)higher derivation \(\mathcal{D} = [\delta_n]_{n \in \mathbb{N}}\) on \(\mathcal{A}\), then \(\delta_n(XY) = \sum_{i+j=n} \delta_i(\sigma^{i-}(X))d_j(\tau^{j-}(Y))\) for all \(X, Y \in \mathcal{A}\). Replacing \(Y\) by \(X\), we obtain \(\delta_n(X^2) = \sum_{i+j=n} \delta_i(\sigma^{i-}(X))d_j(\tau^{j-}(X))\) for all \(X \in \mathcal{A}\) and for each \(n \in \mathbb{N}\). That is \(\Delta = [\delta_n]_{n \in \mathbb{N}}\) is a generalized Jordan \((\sigma, \tau)-\)higher derivation on \(\mathcal{A}\). Again, replacing \(Y\) by \(X\), we obtain \(\delta_n(XYX) = \sum_{i+j+k=n} \delta_i(\sigma^{i-}(X))d_j(\sigma^{j-}(\tau^{k-}(Y)))d_k(\tau^{k-}(X))\) for all \(X, Y \in \mathcal{A}\) and for each \(n \in \mathbb{N}\). That is \(\Delta = [\delta_n]_{n \in \mathbb{N}}\) is a generalized Jordan triple \((\sigma, \tau)-\)higher derivation on \(\mathcal{A}\).

In the present paper, our objective is to prove every generalized Jordan (triple) \((\sigma, \tau)-\)higher derivation is a generalized \((\sigma, \tau)-\)higher derivation on \(\mathcal{A}\). In fact, our results generalize \([3, \text{Theorem 3.7}, \text{Theorem 3.8}]\) and \([11, \text{Proposition 4.1}, \text{Theorem 4.3}]\).

3. Main Results

The main result of the present paper states as follows:

**Theorem 3.1.** Let \(\mathcal{A} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})\) be a triangular algebra consisting of \(\mathcal{A}, \mathcal{B}\) and \(\mathcal{M}\), where \(\mathcal{A}\) and \(\mathcal{B}\) have only trivial idempotents, \(\sigma, \tau\) be automorphisms of \(\mathcal{A}\) such that \(\sigma \tau = \tau \sigma\) and let \(\Delta = [\delta_n]_{n \in \mathbb{N}}\) be the family of \(\mathcal{R}\)-linear maps \(\delta_n : \mathcal{A} \rightarrow \mathcal{A}\) on \(\mathcal{A}\) such that \(\delta_0 = 1_\mathcal{A}\). Then the following statements are equivalent:

\begin{enumerate}
  \item \(\Delta\) is a generalized \((\sigma, \tau)-\)higher derivation on \(\mathcal{A}\),
  \item \(\Delta\) is a generalized Jordan \((\sigma, \tau)-\)higher derivation on \(\mathcal{A}\),
  \item \(\Delta\) is a generalized Jordan triple \((\sigma, \tau)-\)higher derivation on \(\mathcal{A}\).
\end{enumerate}

In order to prove our main results, we begin with the following sequence of lemmas:

**Lemma 3.2.** \([14, \text{Theorem 1}]\) Let \(\mathcal{A} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})\) be a triangular algebra consisting of \(\mathcal{A}, \mathcal{B}\) and \(\mathcal{M}\), where \(\mathcal{A}\) and \(\mathcal{B}\) have only trivial idempotents. Then an \(\mathcal{R}\)-linear map \(\sigma : \mathcal{A} \rightarrow \mathcal{A}\) is an automorphism of \(\mathcal{A}\) if and only if it has the form

\[\sigma \left( \begin{array}{cc} a & m \\ 0 & b \end{array} \right) = \left( \begin{array}{cc} \theta(a) & \theta(a)m' - m'\eta(b) + \nu(m) \\ 0 & \eta(b) \end{array} \right),\]

where \(\theta : \mathcal{A} \rightarrow \mathcal{A}\) and \(\eta : \mathcal{B} \rightarrow \mathcal{B}\) are automorphisms, \(m'\) is a fixed element in \(\mathcal{M}\) and \(\nu : \mathcal{M} \rightarrow \mathcal{M}\) is an \(\mathcal{R}\)-linear bijective mapping such that \(\nu(am) = \theta(a)\nu(m), \nu(mb) = \nu(m)\eta(b)\) for all \(a \in \mathcal{A}, b \in \mathcal{B}\) and \(m \in \mathcal{M}\).

Obviously, for any \(\left( \begin{array}{cc} a & m \\ 0 & b \end{array} \right) \in \mathcal{A}\), we have

\[\sigma(e) = \left( \begin{array}{cc} 1 & m \\ 0 & 0 \end{array} \right), \quad \sigma(f) = \left( \begin{array}{cc} 0 & -m \\ 0 & 1 \end{array} \right) = f_{-m}.\]  \hfill (1)

**Lemma 3.3.** \([3, \text{Lemma 3.3}]\) Let \(\mathcal{A} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})\) be a triangular algebra consisting of \(\mathcal{A}, \mathcal{B}\) and \(\mathcal{M}\), where \(\mathcal{A}\) and \(\mathcal{B}\) have only trivial idempotents and \(\sigma, \tau\) be automorphisms of \(\mathcal{A}\). Let \(\mathcal{D} = [\delta_n]_{n \in \mathbb{N}}\) be a Jordan \((\sigma, \tau)-\)higher derivation on \(\mathcal{A}\). Then for all \(m \in \mathcal{M}\) and for each fixed \(n \in \mathbb{N}\)

\begin{enumerate}
  \item \(\sigma^n(e) = e_m\) and \(\sigma^n(f) = f_{-m},\)
  \item \(\delta_n(l) = 0,\) where \(l\) is the identity element of \(\mathcal{A}\),
  \item \(\delta_n(e), \delta_n(f) \in \mathcal{M}\).
\end{enumerate}

**Lemma 3.4.** \([1, \text{Theorem 2.2}]\) Suppose that \(\mathcal{A} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})\) is a triangular algebra having algebras \(\mathcal{A}, \mathcal{B}\) with only trivial idempotents and an \((\mathcal{A}, \mathcal{B})\)-bimodule \(\mathcal{M}\). Let \(\sigma\) and \(\tau\) be two automorphisms of \(\mathcal{A}\) and a multiplicative map \(\delta : \mathcal{A} \rightarrow \mathcal{A}\) be a generalized Jordan \((\sigma, \tau)-\)derivation (not necessarily linear) on \(\mathcal{A}\) associated with a multiplicative Jordan \((\sigma, \tau)-\)derivation \(d\) on \(\mathcal{A}\). Then \(\delta\) is an additive generalized \((\sigma, \tau)-\)derivation on \(\mathcal{A}\).
Lemma 3.5. Let $\mathfrak{A} = \text{Tri}(\mathcal{A}, \mathcal{B}, \mathcal{M})$ be a triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, where $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents; $\sigma, \tau$ be endomorphisms of $\mathfrak{A}$ such that $\sigma \tau = \tau \sigma$ and $\Delta = \{d_n\}_{n \in \mathbb{N}}$ be a generalized Jordan $(\sigma, \tau)$-higher derivation associated with a Jordan $(\sigma, \tau)$-higher derivation $\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}$ on $\mathfrak{A}$. Then for all $X, Y, Z \in \mathfrak{A}$ and for each fixed $n \in \mathbb{N}$

(i) $\delta_n(XY + YX) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(Y)) + \delta_i(\sigma^{n-i}(Y))d_j(\tau^{n-j}(X))$,

(ii) $\delta_n(XXY) = \sum_{i+j+k=n} \delta_i(\sigma^{n-i}(X))d_j(\sigma^k \tau(Y))d_k(\tau^{n-k}(X))$,

(iii) $\delta_n(XYZ + ZYX) = \sum_{i+j+k=n} \delta_i(\sigma^{n-i}(X))d_j(\sigma^k \tau(Y))d_k(\tau^{n-k}(Z)) + \delta_i(\sigma^{n-i}(Z))d_j(\sigma^k \tau(Y))d_k(\tau^{n-k}(X))$.

Proof. (i) By our hypothesis for $X \in \mathfrak{A}, n \in \mathbb{N}$, we have

$$\delta_n(X^2) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(X)).$$

Now replace $X$ by $X + Y$ in the above relation to get

$$\delta_n((X + Y)^2) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(X + Y))d_j(\tau^{n-j}(X + Y))$$

$$= \sum_{i+j=n} \delta_i(\sigma^{n-i}(X) + \sigma^{n-i}(Y))d_j(\tau^{n-j}(X) + \tau^{n-j}(Y))$$

$$= \sum_{i+j=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(X)) + \delta_i(\sigma^{n-i}(Y))d_j(\tau^{n-j}(Y)) + \delta_i(\sigma^{n-i}(Y))d_j(\tau^{n-j}(X)) + \delta_i(\sigma^{n-i}(Y))d_j(\tau^{n-j}(Y)).$$

(2)

Also,

$$\delta_n((X + Y)^2) = \delta_n(X^2) + \delta_n(XY + YX) + \delta_n(Y^2)$$

$$= \sum_{i+j=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(X)) + \delta_n(XY + YX) + \sum_{i+j=n} \delta_i(\sigma^{n-i}(Y))d_j(\tau^{n-j}(Y))$$

(3)

On comparing (2) and (3), we obtain the required result.

(ii) Now, replacing $Y$ by $XY + YX$ in (i), we find that

$$\delta_n(X(XY + YX) + (XY + YX)X)$$

$$= \sum_{i+j=n} \{\delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(XY + YX)) + \delta_i(\sigma^{n-i}(XY + YX))d_j(\tau^{n-j}(X))\}$$

$$= \sum_{i+j+k=n} \delta_i(\sigma^{n-i}(X))d_j(\sigma^k \tau(X))d_k(\tau^{n-k}(Y)) + 2\sum_{i+j+k=n} \delta_i(\sigma^{n-i}(X))d_j(\sigma^k \tau(Y))d_k(\tau^{n-k}(X)) + \sum_{i+j+k=n} \delta_i(\sigma^{n-i}(Y))d_j(\sigma^k \tau(X))d_k(\tau^{n-k}(X)).$$

(4)
On the other hand,
\[ \delta_n(X(XX + YX) + (XY + X)X) = \sum_{i+j=n} \delta_i(a^{n-i}(X^2))d_j((\tau^{n-i}(Y)) + \delta_i(a^{n-i}(Y))d_j((\tau^{n-i}(X^2))) + 2\delta_n(XXY) \]
\[ = \sum_{r+s+j=n} \delta_r(a^{n-r}(X))d_s(\sigma^j\tau^r(X))d_j((\tau^{n-i}(Y))) + \sum_{i+j=n} \delta_i(a^{n-i}(Y))d_j(\tau^i(X))d_r((\tau^{n-j}(X))) + 2\delta_n(XXY). \]

Combining (4) and (5) and using Theorem 2.1, we get the required result.

(iii) Linearizing \( X \) in (ii), we have
\[ \delta_n(XYZ + ZYX) = \sum_{i+j+k=n} \delta_i(a^{n-i}(X))d_j(\sigma^k\tau^j(Y))d_k((\tau^{n-k}(Z))) + \delta_j(a^{n-j}(Z))d_i(\sigma^k\tau^j(Y))d_k((\tau^{n-k}(X))) \]
for all \( X, Y, Z \in \mathfrak{A} \) and for each fixed \( n \in \mathbb{N} \). 

Following the above notations we prove that:

**Lemma 3.6.** Let \( \mathfrak{A} = \text{Tri}(\mathcal{A}, \mathcal{B}, \mathcal{M}) \) be a triangular algebra consisting of \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{M} \), where \( \mathcal{A} \) and \( \mathcal{B} \) have only trivial idempotents and \( \sigma, \tau \) be automorphisms of \( \mathfrak{A} \). Let \( \Delta = \{\delta_n\}_{n \in \mathbb{N}} \) be a generalized Jordan \((\sigma, \tau)\)-higher derivation associated with a Jordan \((\sigma, \tau)\)-higher derivation \( \mathfrak{D} = \{d_n\}_{n \in \mathbb{N}} \) on \( \mathfrak{A} \). Then \( \delta_n(e) \in \mathcal{A} + \mathcal{M} \) and \( \delta_n(f) \in \mathcal{B} + \mathcal{M} \) for each fixed \( n \in \mathbb{N} \).

**Proof.** By Lemma 3.4, we have \( \delta(e) \in \mathcal{A} + \mathcal{M} \). Now suppose that \( \delta_r(e) \in \mathcal{A} + \mathcal{M} \) for all \( 1 < r < n \). Using method of induction and by the definition of generalized Jordan \((\sigma, \tau)\)-higher derivation, we have
\[ \delta_n(e) = \delta_n(e^2) \]
\[ = \sum_{i+j=n} \delta_i(a^{n-i}(e))d_j((\tau^{n-i}(e))) \]
\[ = \delta_n(e)(\delta_i(a^{n-i}(e))d_j((\tau^{n-i}(e))) + \delta_{n-i-1}(\sigma(e))d_i((\tau^{n-i-1}(e))) + \delta_{n-2}(\sigma^2(e))d_{i+1}(\tau^{n-2}(e))) \]
\[ + \cdots + \delta_1(\sigma^{n-1}(e))d_{n-i}(\tau^1(e)) + \sigma^n(e)d_n(e) \]
\[ = \delta_n(e)(\sigma^n(e)d_n(e) + \delta_n(e)). \]

Put \( \delta_n(e) = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \) where \( a \in \mathcal{A}, m \in \mathcal{M} \) and \( b \in \mathcal{B} \) and using Lemma 3.3 in the above expression, we obtain that \( b = 0 \). This implies that \( \delta_n(e) \in \mathcal{A} + \mathcal{M} \). Similarly, we can prove that \( \delta_n(f) \in \mathcal{B} + \mathcal{M} \).

**Lemma 3.7.** Let \( \mathfrak{A} = \text{Tri}(\mathcal{A}, \mathcal{B}, \mathcal{M}) \) be a triangular algebra consisting of \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{M} \), where \( \mathcal{A} \) and \( \mathcal{B} \) have only trivial idempotents and \( \sigma, \tau \) be automorphisms of \( \mathfrak{A} \) such that \( \sigma \tau = \tau \sigma \). If \( \Delta = \{\delta_n\}_{n \in \mathbb{N}} \) is a generalized Jordan \((\sigma, \tau)\)-higher derivation associated with a Jordan \((\sigma, \tau)\)-higher derivation \( \mathfrak{D} = \{d_n\}_{n \in \mathbb{N}} \) on \( \mathfrak{A} \), then for each \( n \in \mathbb{N} \)

(i) \( \delta_n(M) \subseteq \mathcal{M} \),
(ii) \( \delta_n(\mathcal{A}) \subseteq \mathcal{A} + \mathcal{M} \),
(iii) \( \delta_n(\mathcal{B}) \subseteq \mathcal{B} + \mathcal{M} \).

**Proof.** (i) In order to prove our lemma we follow induction method. From Lemma 3.4, we have
\[ \delta_1(\mathcal{M}) \subseteq \mathcal{M}, \delta_1(\mathcal{A}) \subseteq \mathcal{A} + \mathcal{M}, \delta_1(\mathcal{B}) \subseteq \mathcal{B} + \mathcal{M} \]
Assume that our lemma holds for component index \( i \), where \( 1 < i < n \), i.e.,
\[ \delta_i(\mathcal{M}) \subseteq \mathcal{M}, \delta_i(\mathcal{A}) \subseteq \mathcal{A} + \mathcal{M}, \delta_i(\mathcal{B}) \subseteq \mathcal{B} + \mathcal{M} \]

(6)
Now
\[
\delta_n(m) = \sum_{i+j=n} \delta_i(a^{n-i}(m))d_i((\tau^{n-i}(a))) + \sum_{i+j=n} \delta_i(a^{n-i}(e))d_i((\tau^{n-i}(m)))
\]
\[
= \delta_n(m)\tau^n(a) + \delta_{n-1}(\sigma(m))d_1((\tau^{n-1}(e))) + \delta_{n-2}(\sigma^2(m))d_2((\tau^{n-2}(e)))
+ \cdots + \delta_1(a^{n-1}(m))d_{n-1}(\tau^1(e)) + \sigma^m(m)d_n(e) + \delta_n(e)\tau^n(m)
\]
\[
+ \delta_{n-1}(\sigma(e))d_1((\tau^{n-1}(m))) + \delta_{n-2}(\sigma^2(e))d_2((\tau^{n-2}(m)))
+ \cdots + \delta_1(a^{n-1}(e))d_{n-1}(\tau^1(m)) + \sigma^m(e)d_n(m).
\]

Put \(\delta_n(m) = \begin{pmatrix} a_1 & m_1 \\ 0 & b_1 \end{pmatrix}\) where \(a_1 \in \mathcal{A}, m_1 \in \mathcal{M} \) and \(b_1 \in \mathcal{B}\). Now using Lemmas 3.3 and 3.6 in the above expression, we obtain that \(b_1 = 0\). On the other way using \(m = mf + fm\), we have \(a_1 = 0\). This implies that \(\delta_n(\mathcal{M}) \subseteq \mathcal{M}\).

(ii) For any \(a \in \mathcal{A}\), we have
\[
0 = \delta_n(af + fa) = \sum_{i+j=n} \delta_i(a^{n-i}(f))d_i((\tau^{n-i}(a))) + \sum_{i+j=n} \delta_i(a^{n-i}(f))d_i((\tau^{n-i}(a)))
\]
\[
= \delta_n(a)\tau^n(f) + \delta_{n-1}(\sigma(a))d_1((\tau^{n-1}(f))) + \delta_{n-2}(\sigma^2(a))d_2((\tau^{n-2}(f)))
+ \cdots + \delta_1(a^{n-1}(a))d_{n-1}(\tau^1(f)) + \sigma^m(a)d_n(f) + \delta_n(f)\tau^n(a)
\]
\[
+ \delta_{n-1}(\sigma(f))d_1((\tau^{n-1}(a))) + \delta_{n-2}(\sigma^2(f))d_2((\tau^{n-2}(a)))
+ \cdots + \delta_1(a^{n-1}(f))d_{n-1}(\tau^1(a)) + \sigma^m(f)d_n(a).
\]

Put \(\delta_n(a) = \begin{pmatrix} a_2 & m_2 \\ 0 & b_2 \end{pmatrix}\) where \(a_2 \in \mathcal{A}, m_2 \in \mathcal{M} \) and \(b_2 \in \mathcal{B}\) and using Lemmas 3.3 and 3.6 in the above expression, we obtain that \(b_2 = 0\). This implies that \(\delta_n(\mathcal{A}) \subseteq \mathcal{A} + \mathcal{M}\).

(iii) Similar to (ii). \(\square\)

**Lemma 3.8.** Let \(\mathcal{A} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})\) be a triangular algebra consisting of \(\mathcal{A}, \mathcal{B}\) and \(\mathcal{M}\), where \(\mathcal{A}\) and \(\mathcal{B}\) have only trivial idempotents and \(\sigma, \tau\) be automorphisms of \(\mathcal{A}\) such that \(\sigma \tau = \tau \sigma\). If \(\Delta = \{\delta_n\}_{n \in \mathbb{N}}\) is a generalized Jordan \((\sigma, \tau)\)-higher derivation associated with a Jordan \((\sigma, \tau)\)-higher derivation \(\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}\) on \(\mathcal{A}\), then for all \(a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B}\) and for each \(n \in \mathbb{N}\)

(i) \(\delta_n(am) = \sum_{i+j=n} \delta_i(a^{n-i}(a))d_i((\tau^{n-i}(m)))\),

(ii) \(\delta_n(mb) = \sum_{i+j=n} \delta_i(a^{n-i}(m))d_i((\tau^{n-i}(b)))\).

**Proof.** (i) For any \(a \in \mathcal{A}\) and \(m \in \mathcal{M}\) using Lemma 3.7, we have
\[
\delta_n(am) = \sum_{i+j=n} \delta_i(a^{n-i}(m))d_i((\tau^{n-i}(m))) + \sum_{i+j=n} \delta_i(a^{n-i}(m))d_i((\tau^{n-i}(m)))
\]
\[
= \delta_n(a)\tau^n(m) + \delta_{n-1}(\sigma(m))d_1((\tau^{n-1}(m))) + \delta_{n-2}(\sigma^2(m))d_2((\tau^{n-2}(m)))
+ \cdots + \delta_1(a^{n-1}(m))d_{n-1}(\tau^1(m)) + \sigma^m(m)d_n(m) + \delta_n(m)\tau^n(a)
\]
\[
+ \delta_{n-1}(\sigma(m))d_1((\tau^{n-1}(m))) + \delta_{n-2}(\sigma^2(m))d_2((\tau^{n-2}(m)))
+ \cdots + \delta_1(a^{n-1}(m))d_{n-1}(\tau^1(m)) + \sigma^m(m)d_n(m).
\]

(ii) Similar to (i). \(\square\)
Lemma 3.9. Let $\mathcal{A} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, where $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and $\sigma, \tau$ be automorphisms of $\mathcal{A}$ such that $\sigma\tau = \tau\sigma$. If $\Delta = [\delta_n]_{n\in\mathbb{N}}$ is a generalized Jordan $(\sigma, \tau)$-higher derivation associated with a Jordan $(\sigma, \tau)$-higher derivation $\Delta' = [d_n]_{n\in\mathbb{N}}$ on $\mathcal{A}$, then for all $a_1, a_2 \in \mathcal{A}, b_1, b_2 \in \mathcal{B}$ and for each $n \in \mathbb{N}$

(i) $\delta_n(a_1a_2) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(a_2))$,

(ii) $\delta_n(b_1b_2) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(b_1))d_j(\tau^{n-j}(b_2))$.

Proof. From Lemma 3.8 for any $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$, we obtain that

$$\delta_n(a_1a_2m) = \sigma^n(a_1a_2)d_n(m) + \delta_n(a_1a_2)\tau^n(m) + \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(m)).$$

On the other hand,

$$\delta_n(a_1a_2m) = \sigma^n(a_1)d_n(a_2m) + \delta_n(a_1)\tau^n(a_2m) + \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(a_2m))
= \sigma^n(a_1a_2)d_n(m) + \sigma^n(a_1)d_n(a_2)\tau^n(m) + \delta_n(a_1)\tau^n(a_2m) + \sigma^n(a_1) \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_2))d_j(\tau^{n-j}(m)) + \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(a_2m)).$$

From last two expressions, it follows that

$$[\delta_n(a_1a_2) - \delta_n(\sigma^n(a_1))d_n(a_2)] = \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(a_2)))\tau^n(m).$$

(7)

Since $\mathcal{M}$ is faithful left $\mathcal{A}$-module, using (7) we find that

$$\delta_n(a_1a_2) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(a_2))$$

for all $a_1, a_2 \in \mathcal{A}$.

(ii) Similar to (i). $\square$

Theorem 3.10. Let $\mathcal{A} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, where $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and $\sigma, \tau$ be automorphisms of $\mathcal{A}$ such that $\sigma\tau = \tau\sigma$. If $\Delta = [\delta_n]_{n\in\mathbb{N}}$ is a generalized Jordan $(\sigma, \tau)$-higher derivation associated with a Jordan $(\sigma, \tau)$-higher derivation $\Delta' = [d_n]_{n\in\mathbb{N}}$ on $\mathcal{A}$, then $\Delta$ is a generalized $(\sigma, \tau)$-higher derivations on $\mathcal{A}$.

Proof. For any $X, Y \in \mathcal{A}$. Suppose that $X = a_1 + m_1 + b_1$ and $Y = a_2 + m_2 + b_2$, where $a_1, a_2 \in \mathcal{A}, m_1, m_2 \in \mathcal{M}$ and $b_1, b_2 \in \mathcal{B}$. Using Lemmas 3.8 and 3.9, we have

$$\delta_n(XY) = \delta_n((a_1, m_1 + b_1)(a_2, m_2 + b_2))
= \delta_n(a_1a_2 + a_1m_2 + m_1b_2 + b_1b_2)
= \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(a_2)) + \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(m_2))
+ \sum_{i+j=n} \delta_i(\sigma^{n-i}(m_1))d_j(\tau^{n-j}(b_2)) + \sum_{i+j=n} \delta_i(\sigma^{n-i}(b_1))d_j(\tau^{n-j}(b_2)).$$

(8)
On the other hand, using Lemmas 3.3, 3.5 and 3.6, we arrive at

\[ \sum_{i+j=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(Y)) \]

\[ = \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1 + m_1 + b_1))d_j(\tau^{n-j}(a_2 + m_2 + b_2)) \]

\[ = \sum_{i+j=n} [\delta_i(\sigma^{n-i}(a_1)) + \delta_i(\sigma^{n-i}(m_1)) + \delta_i(\sigma^{n-i}(b_1))] \]

\[ [d_j(\tau^{n-j}(a_2)) + d_j(\tau^{n-j}(m_2)) + d_j(\tau^{n-j}(b_2))] \]

\[ = \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(a_2)) + \sum_{i+j=n} \delta_i(\sigma^{n-i}(m_1))d_j(\tau^{n-j}(m_2)) \]

\[ + \sum_{i+j=n} \delta_i(\sigma^{n-i}(b_1))d_j(\tau^{n-j}(b_2)). \]

(9)

Since from Theorem 2.1 every Jordan \((\sigma, \tau)\)-higher derivation on \(\mathfrak{H}\) is a \((\sigma, \tau)\)-higher derivation on \(\mathfrak{H}\). So that (8) and (9) implies that \(\Delta\) is a generalized \((\sigma, \tau)\)-higher derivation with associated \((\sigma, \tau)\)-higher derivation on \(\mathfrak{H}\). \(\square\)

Now we are in position to prove our main result:

Proof. [Proof of Theorem 3.1] (i) \(\iff\) (ii) It is obvious by Theorem 3.10.

(ii) \(\iff\) (iii) It can be easily seen that every generalized Jordan \((\sigma, \tau)\)-higher derivation on \(\mathfrak{H}\) is a generalized Jordan triple \((\sigma, \tau)\)-higher derivation on \(\mathfrak{H}\) by Lemma 3.5(ii). Conversely, by the definition of generalized Jordan triple \((\sigma, \tau)\)-higher derivation on \(\mathfrak{H}\), we have

\[ \delta_n(XYX) = \sum_{i+j+k=n} \delta_i(\sigma^{n-i}(X))d_j(\sigma^k\tau(Y))d_k(\tau^{n-k}(X)) \]

for all \(X, Y \in \mathfrak{H}\). Replace \(Y\) by \(I\), the identity map of \(\mathfrak{H}\), in the above expression, we arrive at

\[ \delta_n(XIX) = \sum_{i+j+k=n} \delta_i(\sigma^{n-i}(X))d_j(\sigma^k\tau(I))d_k(\tau^{n-k}(X)) \]

\[ = \sum_{i+j+k=n} \delta_i(\sigma^{n-i}(X))d_j(\sigma^k\tau(I))d_k(\tau^{n-k}(X)) \]

\[ + \sum_{i+j+k=n} \delta_i(\sigma^{n-i}(X))(\sigma^k\tau(I))d_k(\tau^{n-k}(X)). \]

This implies that

\[ \delta_n(X^2) = \sum_{i+j+k=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(X)) \]

for all \(X \in \mathfrak{H}\). Since from Theorem 2.1 every Jordan triple \((\sigma, \tau)\)-higher derivation on \(\mathfrak{H}\) is a Jordan \((\sigma, \tau)\)-higher derivation on \(\mathfrak{H}\). Therefore, \(\Delta\) is a generalized Jordan \((\sigma, \tau)\)-higher derivation with associated Jordan \((\sigma, \tau)\)-higher derivation on \(\mathfrak{H}\). \(\square\)

In particular, for \(n = 1\), we find that the following result due to Han and Wei [11].

Corollary 3.11. [11, Theorem 4.3] Let \(\mathfrak{H} = \text{Tri}(\mathfrak{A}, \mathfrak{M}, \mathfrak{B})\) be a triangular algebra consisting of \(\mathfrak{A}, \mathfrak{B}\) and \(\mathfrak{M}\), where \(\mathfrak{A}\) and \(\mathfrak{B}\) have only trivial idempotents, \(\sigma, \tau\) be automorphisms of \(\mathfrak{A}\) such that \(\sigma\tau = \tau\sigma\) and let \(\delta\) be an \(\mathfrak{R}\)-linear map on \(\mathfrak{H}\). Then the following statements are equivalent:

(i) \(\delta\) is a generalized \((\sigma, \tau)\)-derivation on \(\mathfrak{H}\),

(ii) \(\delta\) is a generalized Jordan \((\sigma, \tau)\)-derivation on \(\mathfrak{H}\),

(iii) \(\delta\) is a generalized Jordan triple \((\sigma, \tau)\)-derivation on \(\mathfrak{H}\).
4. Applications

As an immediate consequence we will apply Theorem 3.1 to a classical example of triangular algebra viz. nest algebras. By Theorem 3.1, we have the following results:

If we choose the identity map in place of $\sigma$ and $\tau$ in Theorem 3.1, we obtain the following corollary. Note that similar result still holds if the condition that $A$ and $B$ have only trivial idempotents is deleted.

**Corollary 4.1.** [15, Theorem 4.7] Let $A = (A, M, B)$ be a triangular algebra and $\Delta = \{\delta_n\}_{n \in \mathbb{N}}$ be a family of $R$-linear maps $\delta_n : A \to A$ such that $\delta_0 = 1_A$. Then the following statements are equivalent:

(i) $\Delta$ is a generalized higher derivation on $A$,
(ii) $\Delta$ is a generalized Jordan higher derivation on $A$,
(iii) $\Delta$ is a generalized Jordan triple higher derivation on $A$.

**Corollary 4.2.** [15, Corollary 4.8] For any one of the following two cases:

(a) Assume, $N$ is a nest on a Banach space $X$, $Alg(N)$ is the nest algebra associated with $N$ and $\Delta = \{\delta_n\}_{n \in \mathbb{N}}$ is a sequence of linear mappings of $Alg(N)$. Suppose that there exists a non-trivial element in $N$ which is complemented in $X$.

(b) Let $N$ be a nest on a complex Hilbert space $H$, $Alg(N)$ be the nest algebra associated with $N$ and $\Delta = \{\delta_n\}_{n \in \mathbb{N}}$ be a sequence of linear mappings of $Alg(N)$.

Then the following statements are equivalent:

(i) $\Delta$ is a generalized higher derivation on $Alg(N)$,
(ii) $\Delta$ is a generalized Jordan higher derivation on $Alg(N)$,
(iii) $\Delta$ is a generalized Jordan triple higher derivation on $Alg(N)$.

**References**


