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f-Biharmonic Integral Submanifolds in Generalized Sasakian Space Forms

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Abstract. We study f-biharmonic integral submanifolds and integral C-parallel submanifolds in generalized Sasakian space forms. As an application, we find the f-biharmonicity conditions for the integral and integral C-parallel submanifolds in Sasakian, λ-Sasakian, Kenmotsu and cosymplectic space forms. Finally, we give also some examples of f-biharmonic integral submanifolds in Sasakian space forms.

1. Introduction

Let (M,g) and (N,h) be two Riemannian manifolds. If a map $\varphi : (M,g) \to (N,h)$ is a critical point of the *energy functional* and *bienergy functional*

$$E(\varphi) = \frac{1}{2}\int_{\Omega} \left\|d\varphi\right\|^2 dv_g,$$

$$E_2(\varphi) = \frac{1}{2}\int_{\Omega} \left\|\tau(\varphi)\right\|^2 dv_g,$$

where Ω is a compact domain of M, then it is called a *harmonic map* and a *biharmonic map*, respectively. The Euler-Lagrange equation of harmonic maps is given by

$$\tau(\varphi) = tr(\nabla d\varphi) = 0,$$

where $\tau(\varphi)$ is the *tension field* of φ [9]. In [17], Jiang obtained the Euler-Lagrange equation of biharmonic maps, where

$$\tau_2(\varphi) = tr(\nabla^N \nabla^N - \nabla^N_{\nabla})\tau(\varphi) - tr(R^N(d\varphi, \tau(\varphi))d\varphi) = 0 \qquad (1)$$

is the *bitension field* of φ. An f-*biharmonic map* with function $f : M \overset{C^\infty}{\to} \mathbb{R}$ is a critical point of the f-*bienergy functional*

$$E_{2,f}(\varphi) = \frac{1}{2}\int_{\Omega} f\left\|\tau(\varphi)\right\|^2 dv_g,$$

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where Ω is a compact domain of M [21]. The f-biharmonic map equation is given by

$$\tau_{2,f}(\varphi) = f\tau_2(\varphi) + \Delta f \tau(\varphi) + 2\nabla^N_{\mathrm{grad} f}\tau(\varphi) = 0, \tag{2}$$

where $\tau_{2,f}(\varphi)$ is the *f-bitension field* of φ [21]. If the f-biharmonic map is neither harmonic nor biharmonic then we call it by *proper f-biharmonic* [21].

Biharmonic and f-biharmonic submanifolds have become popular in recent years (see [6], [13], [16], [20], [26], [27], [28]). In [3], Baikoussis and Blair gave a classification of 3-dimensional flat integral C-parallel submanifolds in the unit sphere $S^7(1)$ with the standard Sasakian structure. In [10], Fetcu and Oniciuc studied integral C-parallel submanifolds in 7-dimensional Sasakian space form. In [12], the same authors studied biharmonic integral C-parallel submanifolds in 7-dimensional Sasakian space forms and classified such submanifolds in this space. In [30], Roth and Upadhyay studied biharmonic submanifolds in both generalized complex and Sasakian space forms. In [29], Ou considered f-biharmonic maps and f-biharmonic submanifolds. In [1], Alegre, Blair and Carriazo defined the notion of a generalized Sasakian space form. In [2], Alegre and Carriazo studied submanifolds of generalized Sasakian space forms. For some recent study of generalized Sasakian space forms see [7], [8], [15], [23], [24], [25]. Motivated by these studies, in this paper, we find the necessary and sufficient conditions for integral and integral C-parallel submanifolds in generalized Sasakian space forms to be f-biharmonic. We also obtain the f-biharmonicity conditions for the integral and integral C-parallel submanifolds in Sasakian, λ-Sasakian, Kenmotsu and cosymplectic space forms. Finally, we give some examples of f-biharmonic integral submanifolds in Sasakian space forms.

2. Preliminaries

Let $M^{2n+1} = (M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold with almost contact metric structure (φ, ξ, η, g). If a contact metric manifold is normal, then the manifold is called a *Sasakian manifold* [5]. An almost contact metric manifold M^{2n+1} is called a *Kenmotsu manifold* [19] if

$$(\nabla_X \varphi) Y = g(\varphi X, Y)\xi - \eta(Y)X$$

where ∇ is the Levi-Civita connection. An almost contact metric manifold M^{2n+1} is called a *cosymplectic manifold* if $\nabla\varphi = 0$, which implies that $\nabla\xi = 0$ [22]. An almost contact metric manifold M^{2n+1} is called a *λ-Sasakian manifold* if

$$(\nabla_X \varphi) Y = \lambda \left[g(\varphi X, Y)\xi - \eta(Y)X \right],$$

(see [18]). If $\lambda = 1$, a λ-Sasakian manifold turns into a Sasakian manifold.

The sectional curvature of a φ-section is called a *φ-sectional curvature*. When the φ-sectional curvature is constant, the manifold is called a *space form* (*Sasakian,Kenmotsu, cosymplectic, λ-Sasakian*) (see [5], [19], [22], [18]). The manifold $M^{2n+1} = (M, \varphi, \xi, \eta, g)$ is called a *generalized Sasakian space form* if its curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z &= f_1 \{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2 \{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\} \\ &+ f_3 \{\eta(X)\eta(Z)Y - \eta(X)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \tag{3}$$

for certain differentiable functions f_1, f_2 and f_3 on M^{2n+1} [1]. If M is a Sasakian space form then $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ [5], if M is a Kenmotsu space form $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$ [19], if M is a cosymplectic space form $f_1 = f_2 = f_3 = \frac{c}{4}$ [22] and if M is a λ-Sasakian space form then $f_1 = \frac{c+3\lambda}{4}$, $f_2 = f_3 = \frac{c-\lambda}{4}$ [18].

A submanifold M^m of a Sasakian manifold N^{2n+1} is called an *integral submanifold* if $\eta(X) = 0$ for any vector field X tangent to M [5]. An integral submanifold M^m of a Sasakian manifold N^{2n+1} is said to be *integral C-parallel* [3] if $\nabla^{\perp}B$ is parallel to the characteristic vector field , where B is the second fundamental form of M and $\nabla^{\perp}B$ is given by

$$\nabla^{\perp}B(X,Y,Z) = \nabla^{\perp}_X B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

for any vector fields X, Y, Z tangent to M, $\nabla^{\perp}$ and ∇ being the normal connection and the Levi-Civita connection on M, respectively.

3. f-Biharmonic integral submanifolds in generalized Sasakian space forms

By B, A, H, $\nabla^{\perp}$ and $\Delta^{\perp}$, we will denote the second fundamental form of a integral submanifold M^n in a generalized Sasakian space form N^{2n+1}, the shape operator and the mean curvature vector field, the connection and the Laplacian in normal bundle, respectively. We have the following theorem:

Theorem 3.1. *Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a generalized Sasakian space form with constant φ-sectional curvature c and M^n an integral submanifold of N^{2n+1}. The integral submanifold $i : M^n \to N^{2n+1}$ is f-biharmonic if and only if*

$$\Delta^{\perp}H + traceB(., A_H.) - \frac{\Delta f}{f}H - (f_1 n + 3f_2)H - 2\nabla^{\perp}_{grad \ln f}H = 0 \tag{4}$$

and

$$\frac{n}{2}grad(\| H \|^2) + 2traceA_{\nabla^{\perp}_{(.)}H}(.) + 2A_H grad \ln f = 0. \tag{5}$$

Proof. Let $\{e_i\}$, $1 \le i \le n$ be a local geodesic orthonormal frame at $p \in M$ and $\tau(i) = nH$. Using the equation (1), bitension field of i is

$$\tau_2(i) = -n\left\{\frac{n}{2}\mathrm{grad}(\| H \|^2) + 2traceA_{\nabla^{\perp}_{(.)}H}(.) + traceB(., A_H.) \right.$$
$$\left. +\Delta^{\perp}H + \sum_{i=1}^{n} R^N(e_i, H)e_i \right\}. \tag{6}$$

Since $\{e_i\}_{i=1}^n$is a local orthonormal frame on M, $\left\{e_i, \varphi e_j, \xi\right\}_{i,j=1}^n$ is a local orthonormal frame on N. From the equation (3) and $H \in span\{\varphi e_i : i = 1, .., n\}$, after a straightforward computation, we have

$$R^N(e_i, H)e_i = -f_1 g(e_i, e_i)H + 3f_2 g(e_i, \varphi H)\varphi e_i.$$

Hence,

$$\sum_{i=1}^{n} R^N(e_i, H)e_i = -\sum_{i=1}^{n} f_1 g(e_i, e_i)H + 3\sum_{i=1}^{n} f_2 g(e_i, \varphi H)\varphi e_i$$
$$= -(f_1 n + 3f_2)H. \tag{7}$$

Using the Weingarten formula, we find

$$\nabla^N_{\mathrm{grad} f}\tau(i) = \nabla^N_{\mathrm{grad} f} nH = n(-A_H(\mathrm{grad} f) + \nabla^{\perp}_{\mathrm{grad} f}H). \tag{8}$$

In addition, substituting equations (6), (7) and (8) into equation (2), we obtain

$$-fn\left\{\frac{n}{2}\mathrm{grad}(\| H \|^2) + 2traceA_{\nabla^{\perp}_{(.)}H}(.) + traceB(., A_H.) + \Delta^{\perp}H - (f_1 n + 3f_2)H\right\}$$
$$+n\Delta f H + 2n(-A_H(\mathrm{grad} f) + \nabla^{\perp}_{\mathrm{grad} f}H) = 0. \tag{9}$$

Finally, taking the tangent and normal parts the equation (9), we obtain the desired result. □

Corollary 3.2. *Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a generalized Sasakian space form with constant φ-sectional curvature c. Then there does not exist a proper f-biharmonic integral submanifold M^n such that $\frac{\Delta f}{f} + (f_1 n + 3f_2) < 0$ with constant mean curvature $\| H \|$ in N^{2n+1}.*

Proof. Let M^n be an f-biharmonic integral submanifold with constant mean curvature $\| H \|$ in N^{2n+1}. Then taking the scalar product of the first equation of Theorem 3.1 with H, we obtain

$$\begin{aligned}
g(\Delta^{\perp} H, H) &= -g(traceB(.,A_H.), H) + \frac{\Delta f}{f} g(H,H) \\
&+ (f_1 n + 3f_2) g(H,H) + 2g(\nabla^{\perp}_{\mathrm{grad}\ln f} H, H) \\
&= -\sum_{i=1}^{n} g(B(e_i, A_H e_i), H) + \frac{\Delta f}{f} \| H \|^2 + (f_1 n + 3f_2) \| H \|^2 + 2g(\nabla^{\perp}_{\mathrm{grad}\ln f} H, H) \\
&= -\sum_{i=1}^{n} g(A_H e_i, A_H e_i) + \frac{\Delta f}{f} \| H \|^2 + (f_1 n + 3f_2) \| H \|^2 + 2g(\nabla^{\perp}_{\mathrm{grad}\ln f} H, H) \\
&= - \| A_H \|^2 + \left(\frac{\Delta f}{f} + (f_1 n + 3f_2) \right) \| H \|^2 + 2g(\nabla^{\perp}_{\mathrm{grad}\ln f} H, H). \qquad (10)
\end{aligned}$$

Using the equation $g(H,H) = \| H \|^2 =$ constant, we find

$$2g(\nabla^{\perp}_{\mathrm{grad}\ln f} H, H) = 0. \qquad (11)$$

Then, putting equation (11) into equation (10), we have

$$g(\Delta^{\perp} H, H) = - \| A_H \|^2 + \left(\frac{\Delta f}{f} + (f_1 n + 3f_2) \right) \| H \|^2 . \qquad (12)$$

Thus, from the Weitzenböck formula,

$$\frac{1}{2} \Delta \| H \|^2 = g(\Delta^{\perp} H, H) - \| \nabla^{\perp} H \|^2 \qquad (13)$$

and since M^n is an f-biharmonic integral submanifold with constant mean curvature, the equation (13) is reduced to

$$g(\Delta^{\perp} H, H) = \| \nabla^{\perp} H \|^2 . \qquad (14)$$

In view of equation (14) into equation (12), we obtain

$$\| \nabla^{\perp} H \|^2 + \| A_H \|^2 = \left(\frac{\Delta f}{f} + (f_1 n + 3f_2) \right) \| H \|^2 . \qquad (15)$$

Since we assume $\frac{\Delta f}{f} + (f_1 n + 3f_2) < 0$, from the equation (15), we get $\| H \|^2 = 0$, so M^n is minimal. This completes the proof. □

Corollary 3.3. *Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a generalized Sasakian space form with constant φ-sectional curvature c. Then there does not exist a proper f-biharmonic compact integral submanifold M^n in N^{2n+1} such that $\frac{\Delta f}{f} + (f_1 n + 3f_2) \leq 0$.*

Proof. Let M^n be an f-biharmonic compact integral submanifold. Then using the same method in the proof of Corollary 3.2, from equation (12), $\frac{\Delta f}{f} + (f_1 n + 3f_2) \leq 0$ and Weitzenböck formula, we obtain

$$\| \nabla^{\perp} H \|^2 + \| A_H \|^2 \leq \left(\frac{\Delta f}{f} + (f_1 n + 3f_2) \right) \| H \|^2 .$$

Hence, we obtain the result. □

For integral C-parallel submanifold, we obtain the following propositions:

Proposition 3.4. *Let M^n be an integral C-parallel submanifold of N^{2n+1}. Then, we have*

$$A_H grad \ln f = 0.$$

Proof. By the use of Proposition 3.40 in [12], we have $\| H \|$ is constant and $\nabla^{\perp} H$ is parallel to ξ. Thus, we have $A_{\nabla^{\perp}_X H} = 0$ for any vector field X tangent to M, since $A_{\xi} = 0$. Hence from tangent part of Theorem 3.1, we have

$$A_H \mathrm{grad} \ln f = 0.$$

This completes the proof. □

Proposition 3.5. *A non-minimal integral C-parallel submanifold M^n with constant mean curvature $\| H \|$ in N^{2n+1} is proper f-biharmonic if and only if*

$$\frac{\Delta f}{f} + f_1 n + 3f_2 - 1 > 0$$

and

$$traceB(., A_H .) - 2\nabla^{\perp}_{grad \ln f} H = \left(\frac{\Delta f}{f} + f_1 n + 3f_2 - 1 \right) H.$$

Proof. It is known that $\Delta^{\perp} H = H$ [12]. Thus, from normal part of Theorem 3.1 and the above proposition, we obtain

$$traceB(., A_H .) - 2\nabla^{\perp}_{\mathrm{grad} \ln f} H = \left(\frac{\Delta f}{f} + f_1 n + 3f_2 - 1 \right) H.$$

Then taking the scalar product of the above equation with H, we find

$$\| A_H \|^2 = \left(\frac{\Delta f}{f} + f_1 n + 3f_2 - 1 \right) \| H \|^2 .$$

Hence, it follows that

$$\frac{\Delta f}{f} + f_1 n + 3f_2 - 1 > 0.$$

□

4. Applications

In this section, we apply Theorem 3.1, Corollary 3.2, Corollary 3.3 and Proposition 3.5 to Sasakian, Kenmotsu, cosymplectic and λ-Sasakian space forms. Firstly, we investigate above results for Sasakian space form and then, using Theorem 3.1, we have following theorem:

Theorem 4.1. *Let M^n be an integral submanifold of a Sasakian space form $(N^{2n+1}, \varphi, \xi, \eta, g)$ with constant φ-sectional curvature c. The integral submanifold $i : M^n \to N^{2n+1}$ is f-biharmonic if and only if*

$$\Delta^{\perp} H + traceB(., A_H.) - \frac{\Delta f}{f} H - \left(\frac{(n+3)c + 3n - 3}{4}\right) H - 2\nabla^{\perp}_{grad \ln f} H = 0$$

and

$$\frac{n}{2} grad(\| H \|^2) + 2traceA_{\nabla^{\perp}_{(.)} H}(.) + 2A_H grad \ln f = 0.$$

Proof. Using the equations (4), (5) and $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, we obtain the result. □

From Corollary 3.2 and Corollary 3.3, we have the following corollaries:

Corollary 4.2. *There does not exist a proper f-biharmonic integral submanifold M^n with constant mean curvature $\| H \|$ such that $\frac{\Delta f}{f} + \left(\frac{(n+3)c+3n-3}{4}\right) < 0$ in a Sasakian space form $(N^{2n+1}, \varphi, \xi, \eta, g)$.*

Corollary 4.3. *There does not exist a proper f-biharmonic compact integral submanifold M^n such that $\frac{\Delta f}{f} + \left(\frac{(n+3)c+3n-3}{4}\right) \leq 0$ in a Sasakian space form $(N^{2n+1}, \varphi, \xi, \eta, g)$.*

By the use of Proposition 3.5, we find the following proposition:

Proposition 4.4. *A non-minimal integral C-parallel submanifold M^n with constant mean curvature $\| H \|$ in a Sasakian space form $(N^{2n+1}, \varphi, \xi, \eta, g)$ is proper f-biharmonic if and only if*

$$\frac{\Delta f}{f} + \left(\frac{(n+3)c + 3n - 7}{4}\right) > 0$$

and

$$traceB(., A_H.) - 2\nabla^{\perp}_{grad \ln f} H = \left(\frac{\Delta f}{f} + \left(\frac{(n+3)c + 3n - 7}{4}\right)\right) H.$$

Now, we analyze f-biharmonic integral and integral C-parallel submanifolds in Kenmotsu space forms. Then we have the following theorem:

Theorem 4.5. *Let M^n be an integral submanifold of a Kenmotsu space form $(N^{2n+1}, \varphi, \xi, \eta, g)$. The integral submanifold $i : M^n \to N^{2n+1}$ is f-biharmonic if and only if*

$$\Delta^{\perp} H + traceB(., A_H.) - \frac{\Delta f}{f} H - \left(\frac{(n+3)c - 3n + 3}{4}\right) H - 2\nabla^{\perp}_{grad \ln f} H = 0$$

and

$$\frac{n}{2} grad(\| H \|^2) + 2traceA_{\nabla^{\perp}_{(.)} H}(.) + 2A_H grad \ln f = 0.$$

Proof. Putting $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$ into the equations (4) and (5), we get the result. □

By utilizing, Corollary 3.2 and Corollary 3.3, we find the following corollaries:

Corollary 4.6. *There does not exist a proper f-biharmonic integral submanifold M^n with constant mean curvature $\| H \|$ such that* $\frac{\Delta f}{f} + \left(\frac{(n+3)c-3n+3}{4}\right) < 0$ *in a Kenmotsu space form* $(N^{2n+1}, \varphi, \xi, \eta, g)$.

Corollary 4.7. *There does not exist a proper f-biharmonic compact integral submanifold M^n such that* $\frac{\Delta f}{f} + \left(\frac{(n+3)c-3n+3}{4}\right) \leq 0$ *in a Kenmotsu space form* $(N^{2n+1}, \varphi, \xi, \eta, g)$.

Using Proposition 3.5, we obtain the following proposition:

Proposition 4.8. *A non-minimal integral C-parallel submanifold M^n with constant mean curvature $\| H \|$ in a Kenmotsu space form $(N^{2n+1}, \varphi, \xi, \eta, g)$ is proper f-biharmonic if and only if*

$$\frac{\Delta f}{f} + \left(\frac{(n+3)c+3n-1}{4}\right) > 0$$

and

$$traceB(., A_H.) - 2\nabla^{\perp}_{grad \ln f} H = \left(\frac{\Delta f}{f} + \left(\frac{(n+3)c+3n-1}{4}\right)\right) H.$$

Now, we consider cosymplectic space forms. Then we obtain the following theorem:

Theorem 4.9. *Let M^n be an integral submanifold of a cosymplectic space form $(N^{2n+1}, \varphi, \xi, \eta, g)$. The integral submanifold $i : M^n \to N^{2n+1}$ is f-biharmonic if and only if*

$$\Delta^{\perp} H + traceB(., A_H.) - \frac{\Delta f}{f} H - \frac{(n+3)c}{4} H - 2\nabla^{\perp}_{grad \ln f} H = 0$$

and

$$\frac{n}{2} grad(\| H \|^2) + 2traceA_{\nabla^{\perp}_{(.)} H}(.) + 2A_H grad \ln f = 0.$$

Proof. In view of equation $f_1 = f_2 = f_3 = \frac{c}{4}$ into the equations (4) and (5), we have the desired result. □

So, we have the following corollaries for an integral submanifold of cosymplectic space forms.

Corollary 4.10. *There does not exist a proper f-biharmonic integral submanifold M^n with constant mean curvature $\| H \|$ such that* $\frac{\Delta f}{f} + \frac{(n+3)c}{4} < 0$ *in a cosymplectic space form* $(N^{2n+1}, \varphi, \xi, \eta, g)$.

Corollary 4.11. *There does not exist a proper f-biharmonic compact integral submanifold M^n such that* $\frac{\Delta f}{f} + \frac{(n+3)c}{4} \leq 0$ *in a cosymplectic space form* $(N^{2n+1}, \varphi, \xi, \eta, g)$.

By Proposition 3.5, we obtain the following proposition for an integral C-parallel submanifold of cosymplectic space forms.

Proposition 4.12. *A non-minimal integral C-parallel submanifold M^n with constant mean curvature $\| H \|$ in a cosymplectic space form $(N^{2n+1}, \varphi, \xi, \eta, g)$ is proper f-biharmonic if and only if*

$$\frac{\Delta f}{f} + \left(\frac{(n+3)c-4}{4}\right) > 0$$

and

$$traceB(., A_H.) - 2\nabla^{\perp}_{grad \ln f} H = \left(\frac{\Delta f}{f} + \left(\frac{(n+3)c-4}{4}\right)\right) H.$$

Finally, we study an integral submanifold and an integral C-parallel submanifold of λ-Sasakian space forms. Thus we find the following results:

Theorem 4.13. *Let M^n be an integral submanifold of a λ-Sasakian space form $(N^{2n+1}, \varphi, \xi, \eta, g)$. The integral submanifold $i : M^n \to N^{2n+1}$ is f-biharmonic if and only if*

$$\Delta^{\perp} H + traceB(., A_H.) - \frac{\Delta f}{f} H - \left(\frac{(n+3)c + 3\lambda(n-1)}{4}\right) H - 2\nabla^{\perp}_{grad \ln f} H = 0$$

and

$$\frac{n}{2} grad(\| H \|^2) + 2traceA_{\nabla^{\perp}_{(.)} H}(.) + 2A_H grad \ln f = 0.$$

Proof. Substituting $f_1 = \frac{c+3\lambda}{4}$, $f_2 = f_3 = \frac{c-\lambda}{4}$ into the equations (4) and (5), we obtain the result. □

Corollary 4.14. *There does not exist a proper f-biharmonic integral submanifold M^n with constant mean curvature $\| H \|$ such that $\frac{\Delta f}{f} + \left(\frac{(n+3)c+3\lambda(n-1)}{4}\right) < 0$ in a λ-Sasakian space form $(N^{2n+1}, \varphi, \xi, \eta, g)$.*

Corollary 4.15. *There does not exist a proper f-biharmonic compact integral submanifold M^n such that $\frac{\Delta f}{f} + \left(\frac{(n+3)c+3\lambda(n-1)}{4}\right) \leq 0$ in a λ-Sasakian space form $(N^{2n+1}, \varphi, \xi, \eta, g)$.*

Proposition 4.16. *A non-minimal integral C-parallel submanifold M^n with constant mean curvature $\| H \|$ in a λ-Sasakian space form $(N^{2n+1}, \varphi, \xi, \eta, g)$ is proper f-biharmonic if and only if*

$$\frac{\Delta f}{f} + \left(\frac{(n+3)c + 3\lambda(n-1) - 4}{4}\right) > 0$$

and

$$traceB(., A_H.) - 2\nabla^{\perp}_{grad \ln f} H = \left(\frac{\Delta f}{f} + \left(\frac{(n+3)c + 3\lambda(n-1) - 4}{4}\right)\right) H.$$

5. Examples of f-Biharmonic Integral Submanifolds

In the present section, we give some examples of f-biharmonic integral submanifolds of Sasakian space forms. To obtain examples of f-biharmonic integral submanifolds of Sasakian space forms., similar to the proof of Theorem 4.1, Remark 4.2 and Theorem 4.3 in [11], we state the following Theorem 5.1, Remark 5.2 and Theorem 5.3:

Theorem 5.1. *Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian space form with constant φ-sectional curvature c and $i : M \to N$ an r-dimensional integral submanifold of N, $1 \leq r \leq n$. Consider*

$$F : \widetilde{M} = I \times M \to N \quad , \quad F(t, p) = \phi_t(p) = \phi_p(t),$$

where $I = \mathbb{S}^1$ or $I = \mathbb{R}$ and $\left\{\phi_t\right\}_{t \in I}$ is the flow of the vector field ξ. Then $F : \left(\widetilde{M}, \widetilde{g} = dt^2 + i^* g\right) \to N$ is a Riemannian immersion [11]. *Then $\widetilde{M}$ is proper f-biharmonic if and only if M is a proper f-biharmonic submanifold of N, where $f : M \to \mathbb{R}$ is a differentiable function.*

Proof. From [11], we have

$$\tau(F)_{(t,p)} = \left(d\phi_t\right)_p \tau(i) \tag{16}$$

and

$$\tau_2(F)_{(t,p)} = \left(d\phi_t\right)_p \tau_2(i). \tag{17}$$

Let $\sigma \in C(F^{-1}(TN))$ be a section in $F^{-1}(TN)$ defined by $\sigma_{(t,p)} = \left(d\phi_t\right)_p (Z_p)$, where Z is a vector field along M. Then we have

$$\left(\nabla^F_X \sigma\right)_{(t,p)} = \left(d\phi_t\right)_p \left(\nabla^N_X Z\right) \quad , \quad \forall X \in C(TM), \tag{18}$$

where ∇^F is the pull-back connection determined by the Levi-Civita connection on N (see [11]). Using the equations (16) and (18), we calculate

$$\nabla^F_{grad f}\tau(F) = \nabla^F_{grad f}\left(\left(d\phi_t\right)_p \tau(i)\right)$$
$$= \left(d\phi_t\right)_p \nabla^i_{grad f}\tau(i). \tag{19}$$

In view of the equations (16), (17) and (19) into the equation (2), we get

$$\tau_{2,f}(F)_{(t,p)} = \left(d\phi_t\right)_p \tau_{2,f}(i).$$

This completes the proof. □

By the use of f-biharmonicity of F and Fubini Theorem, we have

Remark 5.2. *Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a compact strictly regular Sasakian manifold and $G : M \to N$ be an arbitrary smooth map from a compact Riemannian manifold M. If F is f-biharmonic, then G is f-biharmonic, where*

$$F : \widetilde{M} = \mathbb{S}^1 \times M \to N \quad , \quad F(t,p) = \phi_t(G(p)).$$

Using the above remark, we can state the following theorem:

Theorem 5.3. *Let $N^{2n+1}(c)$ be a Sasakian space form with constant φ-sectional curvature c and $\widetilde{M}^2$ a surface of $N^{2n+1}(c)$ invariant under the flow-action of the characteristic vector field ξ. Then $\widetilde{M}$ is proper f-biharmonic if and only if, locally, it is given by $F(t,s) = \phi_t(\gamma(s))$, where γ is a proper f-biharmonic Legendre curve.*

Let us consider $M = \mathbb{R}^7$ with the standard coordinate functions $(x_1, x_2, x_3, y_1, y_2, y_3, z)$, the contact structure $\eta = \frac{1}{2}(dz - \sum_{i=1}^{3} y_i dx_i)$, the characteristic vector field $\xi = 2\frac{\partial}{\partial z}$ and the tensor field φ are given by

$$\varphi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{bmatrix}.$$

The associated Riemannian metric is $g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{3}((dx_i)^2 + (dy_i)^2)$. Then $(M, \varphi, \xi, \eta, g)$ is a Sasakian space form with constant φ-sectional curvature $c = -3$ and it is denoted by $\mathbb{R}^7(-3)$. The vector fields

$$X_i = 2\frac{\partial}{\partial y_i},\ X_{3+i} = \varphi X_i = 2(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z}), 1 \leq i \leq 3,\ \xi = 2\frac{\partial}{\partial z} \tag{20}$$

form a g-orthonormal basis and the Levi-Civita connection is

$$\nabla_{X_i}X_j = \nabla_{X_{3+i}}X_{3+j} = 0, \nabla_{X_i}X_{3+j} = \delta_{ij}\xi, \nabla_{X_{3+i}}X_j = -\delta_{ij}\xi,$$
$$\nabla_{X_i}\xi = \nabla_\xi X_i = -X_{3+i}, \nabla_{X_{3+i}}\xi = \nabla_\xi X_{3+i} = X_i,$$

(see [5]).

Now we give the following four examples of proper f-biharmonic Legendre curves in $\mathbb{R}^7(-3)$.

Example 5.4. ([14]) *Let us take* $\gamma(t) = (2\sinh^{-1}(t), \sqrt{1+t^2}, \sqrt{3}\sqrt{1+t^2}, 0, 0, 0, 1)$ *in* $\mathbb{R}^7(-3)$. *Then* γ *is a proper* f*-biharmonic Legendre curve with osculating order* $r = 2$, $\kappa_1 = \frac{1}{1+t^2}$, $f = c_1(1+t^2)^{3/2}$, *where* $c_1 > 0$ *is a constant.*

Example 5.5. ([14]) *Let* $\gamma(t) = (a_1, a_2, a_3, \sqrt{2}t, 2\sinh^{-1}(\frac{t}{\sqrt{2}}), \sqrt{2}\sqrt{2+t^2}, a_4)$ *be a curve in* $\mathbb{R}^7(-3)$, *where* $a_i \in \mathbb{R}$, $1 \leq i \leq 4$. *Then* γ *is proper* f*-biharmonic Legendre curve with osculating order* $r = 3$, $\kappa_1 = \kappa_2 = \frac{1}{2+t^2}$, $f = c_1(2+t^2)^{3/2}$, *where* $c_1 > 0$ *is a constant.*

Example 5.6. *Let us take* $\gamma(t) = (\sqrt{2}\ln\left(\sqrt{2t^2+1} + \sqrt{2}t\right), \sqrt{2t^2+1}, \sqrt{2t^2+1}, 0, 0, 0, 1)$ *in* $\mathbb{R}^7(-3)$. *Then* γ *is proper* f*-biharmonic Legendre curve with osculating order* $r = 2$, $\kappa_1 = \frac{\sqrt{2}}{2t^2+1}$, $f = 2^{-3/4}c_1(1+2t^2)^{3/2}$, *where* $c_1 > 0$ *is a constant.*

Example 5.7. *Let* $\gamma(t) = (a_1, a_2, a_3, \sinh^{-1}(2t), -\frac{\sqrt{4t^2+1}}{\sqrt{2}}, \frac{\sqrt{4t^2+1}}{\sqrt{2}}, a_4)$ *be a curve in* $\mathbb{R}^7(-3)$, *where* $a_i \in \mathbb{R}$, $1 \leq i \leq 4$. *Then* γ *is proper* f*-biharmonic Legendre curve with osculating order* $r = 2$, $\kappa_1 = \frac{2}{1+4t^2}$, $f = 2^{-3/2}c_1(1+4t^2)^{3/2}$, *where* $c_1 > 0$ *is a constant.*

Using Example 5.4, Example 5.5, Example 5.6, Example 5.7 and Theorem 5.3, we can give the following example of proper f-biharmonic surfaces:

Example 5.8. *Let* $\widetilde{M}^2$ *be a surface of* $\mathbb{R}^7(-3)$ *endowed with its canonical Sasakian structure which is invariant under the flow-action of the characteristic vector field* ξ. *If* γ *is a Legendre curve given in Example 5.4, Example 5.5, Example 5.6 or Example 5.7 and locally,* $\widetilde{M}^2$ *is given by* $F(t,s) = \phi_t(\gamma(s))$, *then* $\widetilde{M}^2$ *is proper* f*-biharmonic.*

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