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Generalized Bernstein Type Operators on Unbounded Interval and Some Approximation Properties

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Abstract. In the present paper, we construct a new family of Bernstein type operators on infinite interval by using exponential function a^x. We study some approximation results for these new operators on the interval $[0, \infty)$.

1. Introduction

In approximation theory, the most important basic result was given by Karl Weierstrass. In 1912, S.N. Bernstein [1] introduced the sequence of operators to give a constructive proof of the Weierstrass approximation theorem. In 1950 a new generalization of Bernstein's polynomials to the infinite interval was given by O. Szász [8]. The uniform convergence of a sequence of linear positive operators to continuous functions was introduced by Bohman [3] and Korovkin [7]. The Bernstein operators are defined as follows:

$$B_n(f;x) = \sum_{r=0}^{n} \binom{n}{r} x^r (1-x)^{n-r} f\left(\frac{r}{n}\right), \quad x \in [0,1] \quad \text{and} \quad n \in \mathbb{N}. \tag{1}$$

For detailed study we can see [6].
O. Szász [8] introduced the following operators on the infinite interval as follows:

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad f \in C[0,\infty), \quad n \geq 1. \tag{2}$$

In this paper we construct a new family of Bernstein type operators on an infinite interval and obtain some important approximation results for these operators. In next section, a new family of operators is constructed by using the exponential function a^x for the interval $[0, \infty)$. We use the symbol $\log a$ for $\log_e a$ throughout the paper.
Stirling's Formula:

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n. \tag{3}$$

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Definition 1.1. *By $f(n) = o(g(n))$, we mean* $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

2. Construction of operators and some auxiliary results

We define the following operators:

$$S_u^*(f;x) = a^{-ux}\sum_{v=0}^{\infty}\frac{(ux\log a)^v}{v!}f\left(\frac{v}{u\log a}\right), \quad f\in C[0,\infty), \quad u>0 \quad \text{and} \quad a>1. \tag{4}$$

For $a = e$ or $u = n/\log a$, then it reduces to (2).

Definition 2.1. *A set of continuous functions $S_u^*(f;x)$ is said to be convergent uniformly to the value M at $x = \alpha$ as $u \to \infty$ if $S_{u_n}^*(f;x_n) \to M$, whenever $x_n \to \alpha$ and $u_n \to \infty$ as $n \to \infty$.*

Lemma 2.2. *For $u > 0$ and $\lambda > 0$, we have*

$$\sum_{|v-u\log a|\geq\lambda}\frac{(u\log a)^v}{v!} < \lambda^{-2}u\log a(a^u), \quad \textit{for} \quad a>1. \tag{5}$$

Proof. We have the following identity which is easily verified:

$$\sum_{v=0}^{\infty}(v-u\log a)^2\frac{(u\log a)^v}{v!} = u\log a(a^u), \tag{6}$$

and then it follows that

$$\lambda^2\sum_{|v-u\log a|\geq\lambda}\frac{(u\log a)^v}{v!} > \sum_{v=0}^{\infty}(v-u\log a)^2\frac{(u\log a)^v}{v!} = u\log a(a^u).$$

The proof is completed. □

Lemma 2.3. *For $u \geq 0$, the following inequality holds:*

$$\sum_{v=0}^{\infty}|v-u\log a|\frac{(u\log a)^v}{v!} \leq \sqrt{u\log a}(a^u). \tag{7}$$

Proof. By using Schwarz's inequality and (6), we have

$$\begin{aligned}\left(\sum_{v=0}^{\infty}|v-u\log a|\frac{(u\log a)^v}{v!}\right)^2 &\leq \left\{\sum_{v=0}^{\infty}(v-u\log a)^2\frac{(u\log a)^v}{v!}\right\}\left(\sum_{v=0}^{\infty}\frac{(u\log a)^v}{v!}\right)\\ &= u\log a(a^u)(a^u)\\ &= u\log a(a^{2u}).\end{aligned}$$

The proof is completed. □

Note that

$$\sum_{v=0}^{\infty}(v-u\log a)\frac{(u\log a)^v}{v!} = 0. \tag{8}$$

Therefore, for a positive integer u, we have

$$\begin{aligned}
\sum_{v=0}^{\infty}|v-u\log a|\frac{(u\log a)^v}{v!} &= \sum_{v\leq u\log a}(u\log a-v)\frac{(u\log a)^v}{v!}+\sum_{v\geq u\log a}(v-u\log a)\frac{(u\log a)^v}{v!}\\
&= 2\sum_{v\leq u\log a}(u\log a-v)\frac{(u\log a)^v}{v!}\\
&= 2u\log a\sum_{v\leq u\log a}\frac{(u\log a)^v}{v!}-2\sum_{v\leq u\log a}\frac{v(u\log a)^v}{v!}\\
&= 2u\log a\sum_{v\leq u\log a}\frac{(u\log a)^v}{v!}-2\sum_{v\leq u\log a}\frac{(u\log a)^v}{(v-1)!}\\
&= 2u\log a\sum_{v\leq u\log a}\frac{(u\log a)^v}{v!}-2\sum_{v+1\leq u\log a}\frac{(u\log a)^{v+1}}{v!}\\
&= 2u\log a\sum_{v\leq u\log a}\frac{(u\log a)^v}{v!}-2u\log a\sum_{v\leq u\log a-1}\frac{(u\log a)^v}{v!}\\
&= 2u\log a\frac{(u\log a)^{u\log a}}{u\log a!}. \qquad (9)
\end{aligned}$$

Now by using Stirling formula (3), we have

$$\begin{aligned}
\sum_{v=0}^{\infty}|v-u\log a|\frac{(u\log a)^v}{v!} &= \frac{2\sqrt{u\log a}}{\sqrt{2\pi}}e^{u\log a}\\
&= \frac{2\sqrt{u\log a}}{\sqrt{2\pi}}a^u.
\end{aligned}$$

Thus, except for a constant factor, the estimate (7) is the sharpest possible.

3. Main results

In this section, we will study some convergence results, pointwise as well as uniformly convergence and also obtain a Voronovskaja type theorem.

Theorem 3.1. *Suppose that $f(x)$ is bounded in every finite interval, if $f(x) = o(x^k)$ for some $k > 0$ and if $f(x)$ is continuous at a point $\alpha > 0$, then $S_u^*(f;x)$ converges uniformly to $f(x)$ at $x = \alpha$.*

Proof.

$$\begin{aligned}
a^{ux}\{S_u^*(f;x)-f(x)\} &= \sum_{v=0}^{\infty}\{f(v/u\log a)-f(x)\}\frac{(ux\log a)^v}{v!}\\
&= \sum_{|v/u\log a-x|\leq\delta}+\sum_{|v/u\log a-x|\geq\delta}\\
&= L_1+L_2, \quad (say).
\end{aligned}$$

Let

$$\max|f(x)-f(\beta)| = m^*(\delta,\beta) = m^*(\delta), \quad \text{for} \quad |x-\beta|\leq\delta,$$

then $m^*(\delta) \to 0$ as $\delta \to 0$. Now

$$f(v/u\log a) - f(x) = f(v/u\log a) - f(\alpha) + f(\alpha) - f(x),$$

and

$$\frac{v}{u\log a} - \beta = \frac{v}{u\log a} - x + x - \beta. \tag{10}$$

In the sum L_1, $|\frac{v}{u\log a} - x| \leqq \delta$, hence from (10)

$$\left|\frac{v}{u\log a} - \beta\right| \leqq 2\delta$$

and

$$|f(v/u\log a) - f(x)| \leq m^*(2\delta) + m^*(\delta) \leqq 2m^*(2\delta).$$

Hence

$$|L_1| < 2m^*(2\delta)\sum_{v=0}^{\infty}\frac{(ux\log a)^v}{v!} = 2m^*(2\delta)a^{ux}.$$

Next write

$$L_2 = \sum_{v<u\log a(x-\delta)} + \sum_{v>u\log a(x+\delta)} = L_3 + L_4, \quad (say).$$

Then

$$L_3 < \sum_{v<u\log a(x-\delta)}\frac{(ux\log a)^v}{v!}|f(v/u\log a) - f(x)|.$$

Let

$$\sup|f(x)| = V(\delta), \quad \text{for} \quad x \leqq \delta.$$

Then

$$L_3 < 2V(\alpha+\delta)\sum_{ux\log a - v > u\delta\log a}\frac{(ux\log a)^v}{v!}.$$

By using Lemma 2.2 with $\lambda = u\delta\log a$, we have

$$L_3 < 2V(\alpha+\delta)\frac{ux\log a(a^{ux})}{(u\log a)^2\delta^2} = 2V(\alpha+\delta)\frac{xa^{ux}}{(u\log a)\delta^2}.$$

Now, assuming $u\log a(x+\delta) > k$,

$$\begin{aligned}
L_4 &= o\left(\sum_{v>u\log a(x+\delta)} \frac{(ux\log a)^v}{v!}\left(\frac{v}{u\log a}\right)^k\right)\\
&= o\left(\sum_{v>u\log a(x+\delta)} \frac{x^k(ux\log a)^{v-k}}{(v-k)!}\right)\\
&= o\left(\sum_{\mu>u\log a(x+\delta)-k} \frac{x^k(ux\log a)^{\mu}}{\mu!}\right).
\end{aligned}$$

Again by using Lemma 2.2, with $\lambda = u\delta\log a - k > 0$, we get

$$L_4 = o\left(x^k \frac{ux\log a(a^{ux})}{(u\delta\log a - k)^2}\right) = o\left(\frac{u\log a(a^{ux})}{(u\delta\log a-k)^2}\right), \quad \text{as} \quad u \to \infty.$$

Finally, we have

$$S_u^*(f;x) - f(x) = o\left\{m(2\delta) + \frac{1}{u\delta^2\log a} + \frac{u\log a}{(u\delta\log a - k)^2}\right\}.$$

Now for a fixed δ, letting $u \to \infty$, we have

$$\limsup |S_u^*(f;x) - f(x)| \leqq o(m(2\delta)), \quad u \to \infty, \quad \text{for} \quad |x-\alpha| \leqq \delta,$$

which gives the desired result. □

Theorem 3.2. *If $f(x)$ satisfies the Lipschitz-type condition*

$$|f(x_1) - f(x_2)| < M\frac{|x_2-x_1|^\rho}{(x_1+x_2)^{\frac{\rho}{2}}}, \quad 0 < x_1 < x_2 < \infty, \tag{11}$$

where M, ρ are constants, $0 < \rho \leqq 1$, then

$$|S_u^*(f;x) - f(x)| \leqq M\,(u\log a)^{-\frac{\rho}{2}},$$

converges uniformly for $0 < x < \infty$, as $u \to \infty$.

Proof. First of all, we consider the case $\rho = 1$.

$$\begin{aligned}
|S_u^*(f;x) - f(x)| &\leqq a^{-ux}\sum_{v=0}^{\infty}\frac{(ux\log a)^v}{v!}|f(v/u\log a) - f(x)|\\
&< a^{-ux}M\sum_{v=0}^{\infty}\frac{(ux\log a)^v}{v!}\frac{|v/u\log a - x|}{(v/u\log a + x)^{\frac{1}{2}}}\\
&< \frac{M}{(u\log a)^{\frac{1}{2}}}a^{-ux}\sum_{v=0}^{\infty}\frac{(ux\log a)^v}{v!}\frac{|v-ux\log a|}{(v+ux\log a)^{\frac{1}{2}}}\\
&< \frac{M}{u\log a\sqrt{x}}a^{-ux}\sum_{v=0}^{\infty}\frac{(ux\log a)^v}{v!}|v-ux\log a|\\
&< \frac{M}{u\log a\sqrt{x}}a^{-ux}\sqrt{ux\log a}(a^{ux}) \leqq \frac{M}{\sqrt{u\log a}},
\end{aligned}$$

by Lemma 2.3. Thus, the proof is completed for $\rho = 1$.

Now from Hölder's inequality, for $0 < \rho < 1$, we have

$$\begin{aligned} a^{ux}|S_u^*(f;x) - f(x)| &= \sum_{v=0}^{\infty} \frac{(ux\log a)^{v(1-\rho)}}{(v!)^{1-\rho}} \frac{(ux\log a)^{v\rho}}{(v!)^{\rho}} |f(v/u\log a) - f(x)| \\ &\leq \left(\sum_{v=0}^{\infty} \frac{(ux\log a)^{v}}{(v!)}\right)^{1-\rho} \left\{\sum_{v=0}^{\infty} \frac{(ux\log a)^{v}}{(v!)} \left|f(v/u\log a) - f(x)\right|^{\frac{1}{\rho}}\right\}^{\rho}. \end{aligned}$$

Now by (11), we have

$$\begin{aligned} a^{ux}|S_u^*(f;x) - f(x)| &< a^{(1-\rho)ux} M \left(\sum_{v=0}^{\infty} \frac{(ux\log a)^{v}}{v!} \frac{|v/u\log a - x|}{(v/u\log a + x)^{\frac{1}{2}}}\right)^{\rho} \\ &< Ma^{(1-\rho)ux} \left(\frac{1}{u\log a\sqrt{x}} \sqrt{ux\log a}(a^{ux})\right)^{\rho} \\ &= M(u\log a)^{\frac{-\rho}{2}} a^{ux}. \end{aligned}$$

Thus, the proof is completed. □

Example 3.3. *Let*

$$f(x) = \begin{cases} c - x, & \text{for } 0 \leq x \leq c \\ 0, & \text{for } x \geq c, \end{cases}$$

where c is a positive constant and the condition (11) *is fulfilled. Furthermore*

$$\begin{aligned} S_u^*(f;x) - f(c) = S_u^*(f;x) &= a^{-uc} \sum_{v \leq uc\log a} \frac{(uc\log a)^{v}}{v!} (c - v/u\log a) \\ &= \frac{1}{u\log a} a^{-uc} \sum_{v \leq uc\log a} (uc\log a - v) \frac{(uc\log a)^{v}}{v!}. \end{aligned}$$

Let $[uc] = r$ *and multiply by* $(u\log a)^{\frac{1}{2}}$ *on both sides. Then*

$$(u\log a)^{\frac{1}{2}} S_u^*(f;c) > (u\log a)^{\frac{-1}{2}} a^{-r} \sum_{v=0}^{r\log a} (r\log a - v) \frac{(r\log a)^{v}}{v!}.$$

By using (9), *we have*

$$\sum_{v=0}^{r\log a} (r\log a - v) \frac{(r\log a)^{v}}{v!} = \frac{(r\log a)^{r\log a + 1}}{(r\log a)!} \sim \left(\frac{r\log a}{2\pi}\right)^{\frac{1}{2}} a^{r},$$

since $a^x = e^{x\log a}$. *Thus,*

$$\liminf_{u\to\infty} (u\log a)^{\frac{1}{2}} S_u^*(f;c) > 0,$$

which proves that the order of estimate in Theorem 3.2 is sharpest possible for $\rho = 1$.

Theorem 3.4. *If* $f(x)$ *is continuous in* $(0, \infty)$, *then* $S_u^*(f;x) \to f(x)$ *uniformly in the interval* $(0, \infty)$.

Proof. Suppose $f(x)$ is continuous in $(0, \infty)$. Let

$$x = \frac{1}{\log a} \log\left(\frac{1}{t}\right), \quad 0 \leq t \leq 1,$$

$$f(x) = f\left(\frac{1}{\log a} \log \frac{1}{t}\right) = \psi(t)$$

is continuous in $0 \leq t \leq 1$.
Now for a given $\epsilon > 0$, we can find a polynomial

$$\sum_{r=0}^{n} b_r t^r = P_n(t),$$

so that

$$|\psi(t) - P_n(t)| < \epsilon.$$

It follows that

$$|f(x) - P_n(a^{-x})| < \epsilon, \quad x \in (0, \infty).$$

Now by (4) for $P_n(a^{-x})$, we have

$$\begin{aligned}
S_u^*(P_n; x) &= a^{-ux} \sum_{v=0}^{\infty} \frac{(ux \log a)^v}{v!} \sum_{r=0}^{n} b_r a^{\frac{-rv}{u \log a}} \\
&= a^{-ux} \sum_{r=0}^{n} b_r \sum_{v=0}^{\infty} \frac{(ux \log a (a^{-r/u \log a}))^v}{v!} \\
&= a^{-ux} \sum_{r=0}^{n} b_r a^{ux(a^{-r/u \log a})} \\
&= \sum_{r=0}^{n} b_r a^{-ux(1 - a^{-r/u \log a})}.
\end{aligned}$$

Hence, $S_u^*(P_n; x) \to P_n(a^{-x})$ uniformly in the interval $(0, \infty)$, as $u \to \infty$.
Furthermore

$$f(x) = P_n(a^{-x}) + \epsilon_n(x), \quad |\epsilon_n(x)| < \epsilon,$$

and

$$S_u^*(f; x) = S_u^*(f - P_n; x) + S_u^*(P_n; x).$$

Here,

$$|S_u^*(f - P_n; x)| < \epsilon,$$

so that

$$\begin{aligned}
|S_u^*(f; x) - f(x)| &< \epsilon + |S_u^*(P_n; x) - f(x)| \\
&< \epsilon + |S_u^*(P_n(a^{-x}); x) - P_n(a^{-x})| + |P_n(a^{-x}) - f(x)|.
\end{aligned}$$

Thus, the proof is completed. □

Theorem 3.5. *If $f(x)$ is r-times differentiable, $f^{(r)}(x) = o(x^k)$ as $x \to \infty$, for some $k > 0$, and if $f^{(r)}(x)$ is continuous at a point α, then $S_u^{*(r)}(f;x)$ converges uniformly to $f^{(r)}(x)$ at $x = \alpha$.*

Let $1/u = h$. We use the following notations

$$\begin{aligned} \Delta f(vh) &= f(\overline{v+1}h) - f(vh). \\ \Delta^2 f(vh) &= \Delta\Delta f(vh) = f(\overline{v+2}h) - 2f(\overline{v+1}h) + f(vh). \\ &\cdots \end{aligned} \tag{12}$$

$$\cdots \tag{13}$$

$$\Delta^r f(vh) = \Delta\Delta^{r-1} f(vh) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(\overline{v+k}h), \quad r \geq 0.$$

$$S_{1/h}(f;x) = Q_h(f;x) = a^{-x/h} \sum_{v=0}^{\infty} \frac{1}{v!} \left(\frac{x \log a}{h}\right)^v f\left(vh/\log a\right).$$

Lemma 3.6. *We have*

$$a^{x/h} \frac{d^r}{dx^r} Q_h(f;x) = \sum_{v=0}^{\infty} \Delta^r f(vh/\log a) \frac{1}{v!} \left(\frac{x \log a}{h}\right)^v \left(\frac{\log a}{h}\right)^r.$$

Differentiation gives

$$\begin{aligned} \frac{d}{dx} Q_h(f;x) &= a^{-x/h} \sum_{v=0}^{\infty} \frac{1}{(v-1)!} \left(\frac{x \log a}{h}\right)^{v-1} \frac{\log a}{h} f(vh/\log a) \\ &- \frac{\log a}{h} a^{-x/h} \sum_{v=0}^{\infty} \frac{1}{v!} \left(\frac{x \log a}{h}\right)^v f(vh/\log a) \\ &= \frac{\log a}{h} a^{-x/h} \sum_{v=0}^{\infty} \frac{1}{v!} \left(\frac{x \log a}{h}\right)^v \Delta f(vh/\log a), \end{aligned}$$

and the result follows by the induction.
It is known that

$$\left(\frac{\log a}{h}\right)^r \Delta^r f(vh) = f^{(r)}(\mu),$$

where

$$vh < \mu < (v+r)h.$$

$$\begin{aligned} D_r Q_h(f;x) - f^{(r)}(x) &= a^{-x/h} \sum_{v=0}^{\infty} \left\{ \frac{\left(\frac{\log a}{h}\right)^r \Delta^r f(vh) - f^{(r)}(x)}{v!} \right\} \left(\frac{x \log a}{h}\right)^v \\ &= a^{-x/h} \left\{ \sum_{|vh/\log a - x| \leq \delta} + \sum_{|vh/\log a - x| > \delta} \right\}, \end{aligned}$$

where $|x - \alpha| < \delta$. Now using the same technique as in the proof of Theorem 3.1, we obtain the above theorem.

Theorem 3.7. *Let $f(x)$ be bounded in every finite interval and differentiable at a point $\alpha > 0$. If $f(x) = o(x^k)$ for some $k > 0$, then*

$$\lim_{u\to\infty} (u \log a)^{1/2} \{S_u^*(f(\alpha); x) - f(\alpha)\} = 0.$$

Proof. Let

$$\max \left| \frac{f(\alpha+h) - f(\alpha)}{h} - f'(\alpha) \right| = \eta(\delta, \alpha) = \eta(\delta).$$

Then $\eta(\delta) \to 0$ as $\delta \to 0$. We may write

$$f(\alpha + h) - f(\alpha) = hf'(\alpha) + h\epsilon(\alpha, h),$$

where

$$|\epsilon(\alpha, h)| \leq \eta(\delta) \quad \text{for} \quad |h| \leq \delta.$$

Thus

$$S_u^*(f(\alpha); x) - f(\alpha) = a^{-u\alpha} \sum_{v=0}^{\infty} \frac{(u\alpha \log a)^v}{v!} \left\{ \left(\frac{v}{u \log a} - \alpha \right) f'(\alpha) + \left(\frac{v}{u \log a} - \alpha \right) \epsilon_v(u) \right\},$$

where

$$|\epsilon_v(u)| \leqq \eta(\delta) \quad \text{for} \quad \left| \frac{v}{u \log a} - \alpha \right| \leqq \delta.$$

Now by using formula (8), we have

$$\begin{aligned} S_u^*(f(\alpha); x) - f(\alpha) &= \frac{1}{u \log a} a^{-u\alpha} \sum_{v=0}^{\infty} \frac{(u\alpha \log a)^v}{v!} (v - \alpha u \log a) \epsilon_v(u) \\ &= \frac{1}{u \log a} a^{-u\alpha} \left\{ \sum_{|v - \alpha u \log a| \leq \delta u \log a} + \sum_{|v - \alpha u \log a| > \delta u \log a} \right\}. \end{aligned}$$

On the same technique as in the proof of Theorem 3.1, and using Lemma 2.3, we can get the desired result. □

The following result is generalize to higher derivatives. We restrict here that $f''(\alpha)$ exists and the same result proved for Bernstein polynomials (see [2], [9]).

Theorem 3.8. *Let $f(x)$ be bounded in every finite interval and twice differentiable at a point $\alpha > 0$. If for some $k > 0$, $f(x) = o(x^k)$, then*

$$\lim_{u\to\infty} u \log a \{S_u^*(f(\alpha); x) - f(\alpha)\} = \frac{1}{2} \alpha f''(\alpha).$$

Proof. We restrict here to the case that $f''(\alpha)$ exists. Thus

$$f(\alpha + h) - f(\alpha) = hf'(\alpha) + \frac{1}{2} h^2 \{f''(\alpha) + \epsilon(\alpha, h)\},$$

where

$$|\epsilon(\alpha, h)| \leq \mu(\delta) \quad \text{for} \quad |h| \leq \delta, \quad \text{and} \quad \mu(\delta) \to 0, \quad \delta \to 0.$$

Now

$$\begin{aligned} S_u^*(f(\alpha);x) - f(\alpha) &= a^{-u\alpha}\sum_{v=0}^{\infty}\frac{(u\alpha\log a)^v}{v!}\left\{\left(\frac{v}{u\log a}-\alpha\right)f'(\alpha)+\frac{1}{2}\left(\frac{v}{u\log a}-\alpha\right)^2 f''(\alpha)\right\} \\ &+ a^{-u\alpha}\sum_{v=0}^{\infty}\frac{(u\alpha\log a)^v}{v!}\frac{1}{2}\left(\frac{v}{u\log a}-\alpha\right)^2\epsilon_v(u), \end{aligned}$$

where

$$|\epsilon_v(u)| \leqq \mu(\delta) \quad \text{for} \quad \left|\frac{v}{u\log a}-\alpha\right| \le \delta. \tag{14}$$

From (6) and (8), we get

$$S_u^*(f(\alpha);x) - f(\alpha) = \frac{\alpha}{2u\log a}f''(\alpha) + \frac{a^{-u\alpha}}{2(u\log a)^2}\sum_{v=0}^{\infty}\frac{(u\alpha\log a)^v}{v!}(v-\alpha u\log a)^2\epsilon_v(u),$$

or

$$u\log a\{S_u^*(f(\alpha);x) - f(\alpha)\} = \frac{\alpha}{2}f''(\alpha) + \frac{a^{-u\alpha}}{2u\log a}\sum_{v=0}^{\infty}\frac{(u\alpha\log a)^v}{v!}(v-\alpha u\log a)^2\epsilon_v(u).$$

Write

$$\begin{aligned} \sum_{v=0}^{\infty}\frac{(u\alpha\log a)^v}{v!}(v-\alpha u\log a)^2\epsilon_v(u) &= \sum_{|v-\alpha u\log a|\le\delta u\log a} + \sum_{|v-\alpha u\log a|>\delta u\log a} \\ &= T_1 + T_2, \quad (say). \end{aligned}$$

Then from (14) and (6), we have

$$|T_1| < \mu(\delta)\alpha u\log a(a^{u\alpha}). \tag{15}$$

Hence

$$\frac{a^{-u\alpha}}{2u\log a}|T_1| < \frac{1}{2}\mu(\delta)\alpha.$$

Next write

$$T_2 = \sum_{v<u\log a(\alpha-\delta)} + \sum_{v>u\log a(\alpha+\delta)} = T_3 + T_4, \quad (say)$$

and also note that

$$\begin{aligned} \frac{1}{2}\left(\frac{v}{u\log a}-\alpha\right)^2\epsilon_v(u) &= f\left(\frac{v}{u\log a}\right) - f(\alpha) - \left(\frac{v}{u\log a}-\alpha\right)f'(\alpha) \\ &- \frac{1}{2}\left(\frac{v}{u\log a}-\alpha\right)^2 f''(\alpha). \end{aligned}$$

Let

$$\sup|f(x)| = M(\alpha), \quad x \leqq \alpha.$$

Then

$$|T_3| < \left\{2M(\alpha) + \alpha|f'(\alpha)| + \frac{1}{2}\alpha^2|f''(\alpha)|\right\}\sum_{v<u\log a(\alpha-\delta)}(u\log a)^2\alpha^2\frac{(u\alpha\log a)^v}{v!}.$$

Now by using the formula (see e.g. [4], p.200)

$$\sum_{|v-u|>\delta u} e^{-u}\frac{(u)^v}{v!} = o\left(\exp\left(\frac{-1}{3}\delta^2 u\right)\right), \quad u \to \infty,$$

it follows that

$$\sum_{u\alpha\log a - v > \delta u \log a} e^{-u\alpha\log a}\frac{(u\alpha\log a)^v}{v!} = o\left(\exp\left(\frac{-1}{3}\frac{\delta^2}{\alpha}u\log a\right)\right).$$

Hence

$$\begin{aligned}\frac{a^{-u\alpha}}{u\log a}T_3 &= o\left\{(u\log a)\exp\left(\frac{-1}{3}\frac{\delta^2}{\alpha}u\log a\right)\right\}\\ &= o\left\{u\log a\left(a^{-\frac{1}{3}\frac{\delta^2}{\alpha}u}\right)\right\}.\end{aligned}$$

In view of $f(x) = o(x^k)$, for $v > u\alpha\log a$ and $k \geq 2$, we have

$$(v - u\alpha\log a)^2\epsilon_v(u) = o\left((u\log a)^2\frac{v^k}{(u\log a)^k}\right).$$

Therefore,

$$\begin{aligned}T_4 &= o\left(\sum_{v-u\alpha\log a>\delta u\log a}\frac{(u\alpha\log a)^v}{v!}\frac{v^k}{(u\log a)^k}\right)\\ &= o\left(\sum_{v-u\alpha\log a>\delta u\log a}(u\log a)^2\alpha^k\frac{(u\alpha\log a)^{v-k}}{(v-k)!}\right)\\ &= o\left(\sum_{v-u\alpha\log a>\delta u\log a-k}(u\log a)^2\frac{(u\alpha\log a)^v}{v!}\right)\\ &= o\left\{(u\log a)^2\exp\left(u\alpha\log a - \frac{\delta^2}{3\alpha}u\log a\right)\right\}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{a^{-u\alpha}}{u\log a}T_4 &= o\left\{(u\log a)\exp\left(-\frac{1}{3\alpha}\delta^2 u\log a\right)\right\}\\ &= o\left\{u\log a\left(a^{-\frac{1}{3\alpha}\delta^2 u}\right)\right\}.\end{aligned} \tag{16}$$

From (15) and (16), we finally get

$$\limsup\left|u\log a\{S_u^*(f(\alpha);x) - f(\alpha)\} - \frac{\alpha}{2}f''(\alpha)\right| \leqq \delta.$$

But δ is arbitrarily small, hence the proof is completed. □

Remark 3.9. *From a well-known property of the Beta function*

$$\binom{n}{v}\int_0^1 t^v(1-t)^{n-v}dt = \frac{1}{n+1}, \quad v = 0,1,2,\cdots n;$$

we have

$$\int_0^1 B_n(t)dt = \frac{1}{n+1}\sum_{v=0}^{n} f(v/n).$$

So that, for any Riemann integrable function

$$\int_0^1 B_n(t)dt \to \int_0^1 f(t)dt.$$

Similarly,

$$\begin{aligned}\int_0^\infty S_u^*(f;x)dx &= \sum_{v=0}^{\infty}\frac{(u\log a)^v}{v!} f(v/u\log a)\int_0^\infty e^{-ux\log a}x^v dx\\ &= \frac{1}{u\log a}\sum_{v=0}^{\infty} f(v/u\log a),\end{aligned}$$

the interchange of integration and summation is legitimate if the series $\sum f(v/u\log a)$ *is convergent. Thus, the formula*

$$\int_0^\infty S_u^*(f;x)dx = \frac{1}{u\log a}\sum_{v=0}^{\infty} f(v/u\log a)$$

is valid if both sides exist.

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