Filomat 33:9 (2019), 2867–2873
https://doi.org/10.2298/FIL1909867A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Two Scale Defect Measure and Linear Equations with Oscillating Coefficients

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Abstract. Microlocal measure μ is associated to a two-scale convergent sequence u_n over $\mathbb{R}^d$ with the limit $u \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$, $\mathbb{T}^d$ is a torus, to analyze possible strong limit. μ is an operator valued measure absolutely continuous with respect to the product of scalar microlocal defect measure and a measure on the d-dimensional torus. The result is applied to the first order linear PDE with the oscillating coefficients.

1. Introduction

Microlocal defect measures, also called H-measures, are introduced independently in [7] and [16] as ones associated to weakly convergent sequences $u_n \rightharpoonup u$ in L^2_{loc}. They are used for the analysis of the strong convergence ($u_n \to u$) of a weakly convergent sequence of solutions to some types of PDEs. Our aim is to analyze an associated microlocal defect measure, called *two scale microlocal defect measure*, related to a two scale weakly convergent sequence $u_n \overset{2}{\rightharpoonup} u$, where $u_n \in L^2(\mathbb{R}^d)$, $u \in L^2(\mathbb{R}^d \times Y)$, $Y = \mathbb{T}^d$ is a torus, and compare it with an associated measure for a weakly convergent sequence $A_n(u_n)(x,y) = v_n(x,y) \rightharpoonup u(x,y)$, $(x,y) \in R^d \times Y$, where A_n is a certain family of operators introduced and discussed in [1] and [11] (see Sections 1 and 2 for the definitions). Actually we will show that the projections of these measures on $b_l(y) = e^{2l\pi i y}$, $y \in Y$, $l \in \mathbb{Z}^d$ coincide. Then, we apply obtained results to a sequence of weak solutions to a linear partial differential equation with oscillating coefficients.

The generalization of microlocal defect measures to $L^p(\mathbb{R}^d)$, $p \in (1,\infty)$, and Sobolev spaces, called microlocal defect distributions (or H-distributions), [3, 5], are also used to determine whether a weakly convergent sequence of solutions to certain classes of equations converges strongly. Using microlocal defect measures the authors of [2] obtained L^1_{loc}-precompactness of solutions to diffusion-dispersion approximation for a scalar conservation law. Applications of these objects for the problems of homogenization can be found e.g. in [1], [4], [6], [7], [9], [10] and [16]. In [12], H-measures are applied to family of entropy solutions of a first order quasilinear equation and in [14] to an ultraparabolic equation. The list of applications of these objects is far from being complete.

The paper is organized as follows. In Section 1 we recall the weak and two scale convergence as well as the basic notions of pseudo differential operators needed in the sequel. In Section 2 we present results for a

2010 *Mathematics Subject Classification*. Primary 28B10 ; Secondary 40A30, 35F05, 47G30
Keywords. Microlocal defect measure, two-scale convergence, PDE with oscillating coefficients
Received: 10 June 2019; Accepted: 07 July 2019
Communicated by Dragan S. Djordjević
Research partially supported by Ministry of Education, Science and Technological Development, Republic of Serbia, project no. 174024.
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two scale microlocal defect measure associated to a two scale convergent sequence, compare it with the one associated to the corresponding weakly convergent sequence $v_n = A_n u_n$ and prove that obtained measure is absolutely continuous with respect to the product of the scalar microlocal defect measure over $\mathbb{R}^d$ and a measure over $\mathbb{T}^d$. In Section 3 we apply results from Section 2 for a two scale convergent sequence of solutions to a linear first order PDE with oscillating coefficients. The sequence of solutions gives a solution of a corresponding integro-differential equation.

1.1. Notation

We use notation $\mathcal{F}u(\xi) = \hat{u}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} u(x)\,dx$, to denote the Fourier transform of a function $u \in L^2(\mathbb{R}^d)$ ($\mathcal{F}^{-1}$ is defined with the exponent $2\pi i x\xi$). The space $C^\infty(Y) = \mathcal{E}(Y) = \mathcal{D}(Y) = \mathcal{S}(Y)$, (cf. [15] as well as [13] for the calculus over a torus), is separable and Fréchet with the usual topology. The set of linear combinations of functions $b_l(y) = e^{2\pi i l y}$, $l \in \mathbb{Z}^d$, is dense in $C^\infty(Y)$ (also in $C(Y)$). The analysis of the two scale limit (1) is the same if instead of test functions $\varphi(x)b(y)$ we take test function $\phi(x,y) \in C_0^\infty(\mathbb{R}^d, C_\sharp^\infty(Y)) \equiv \mathcal{D}(\mathbb{R}^d, C_\sharp^\infty(Y))$. $C_\sharp^\infty(Y)$ denotes the space of smooth Y-periodic functions on $\mathbb{R}^d$, i.e. for every $y \in Y$ and every $k \in \mathbb{Z}^d$, $b(y+k) = b(y)$.

Smooth functions $a(x,\xi)$, $(x,\xi) \in T^*\mathbb{R}^d$, belonging to the Hörmander class $S^{1,0}(T^*\mathbb{R}^d)$ (cf. [8]), are symbols of pseudo-differential operators $A : L^2 \to L^2$ with the standard notation $A = \mathrm{Op}(a)$, $Au(x) := \displaystyle\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} a(x,\xi)\hat{u}(\xi)\,d\xi$, $u \in L^2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$. The kernel of A is $Ker(A)(x,z) = \displaystyle\int_{\mathbb{R}^d} e^{2\pi i\xi\cdot(x-z)} a(x,\xi)d\xi$.

Recall ([11]) that a sequence $u_n(x) \in L^2(\mathbb{R}^d)$ *two scale converges* to $u(x,y) \in L^2(\mathbb{R}^d \times Y)$, $u_n(x) \xrightharpoonup{2} u(x,y)$, if for all test functions $\varphi(x) \in C_0^\infty(\mathbb{R}^d)$, $b(y) \in C_\sharp^\infty(Y)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} u_n(x)\varphi(x)b(nx)\,dx = \int_{\mathbb{R}^d\times Y} u(x,y)\varphi(x)b(y)\,dxdy. \tag{1}$$

Note that every sequence bounded in $L^2(\mathbb{R}^d)$ has a two scale convergent subsequence. We refer to [1, 11] and references therein for the properties of two scale convergent sequences.

Two scale convergence $u_n \xrightharpoonup{2} u$ implies the weak convergence $u_n \rightharpoonup \int_Y u(x,y)dy$ in $L^2(\mathbb{R}^d)$. Since the weak limit of an oscillating function is not always easily identifiable, it is more appropriate to assume that the sequence of weak solutions to PDE with oscillating coefficients two-scale converges, cf. [6, 9]. This will be considered in the last part of the paper.

1.2. Preliminaries for microlocal defect measures

In the sequel, for a Hilbert space H we use $\langle\cdot,\cdot\rangle$ to denote the scalar product (consequently for the scalar product in L^2). Sometimes we use the notation for the action of a linear functional f on a test function φ, $f(\varphi) = \langle f, \bar{\varphi}\rangle$. Let $\mathcal{K}(H)$, resp. $\mathcal{L}^1(H)$, be the space of compact, resp. trace class, operators on H. By $\Psi^0_{\mathrm{comp}}(\mathbb{R}^d, \mathcal{K}(H))$ is denoted the space of zero order pseudo-differential operators with compactly supported kernel in the cotangent bundle $T^*(\mathbb{R}^d) = \mathbb{R}^d_x \times \mathbb{R}^d_\xi$ as well as with the vector valued symbol $a(x,\xi)$ being a compact operator on H ($a(x,\xi) \in \mathcal{K}(H)$). Recall that $\mathcal{L}^1(H)) = (\mathcal{K}(H)))'$.

Gerard analyzed in [7] a weakly convergent sequence $u_n \in L^2_{\mathrm{loc}}(\mathbb{R}^d, H)$, $u_n \rightharpoonup u$, $n \to \infty$, and showed that, up to a subsequence, there exists an operator valued microlocal defect measure $\mu \in \mathcal{M}_+(\mathbb{R}^d, \mathcal{L}^1(H))$, such that for every pseudo-differential operator $A \in \Psi^0_{\mathrm{comp}}(\mathbb{R}^d, \mathcal{K}(H))$,

$$\lim_{n\to\infty} \langle A(u_n - u)\,, u_n - u\rangle = \int_{T^*\mathbb{R}^d} \mathrm{tr}\{a(x,\xi)\mu(dxd\xi)\}. \tag{2}$$

The operator valued measure μ is positive on the Borel sets of $T^*\mathbb{R}^d$ with values in $\mathcal{L}^1(H)$, that is $\mu \in \mathcal{M}_+(T^*\mathbb{R}^d, \mathcal{L}^1(H))$.

In the case $H = \mathbf{C}$, we use notation $\tilde{\mu}$ for the scalar measure on the right hand side of (2). Also we will consider global $L^2(\mathbb{R}^d)$ convergence in order to compare it with the new notion two scale convergence.

A sequence $u_n - u \rightharpoonup 0$ in $L^2(\mathbb{R}^d, H)$ is called *pure* if there exists measure $\mu \in \mathcal{M}_+(T^*\mathbb{R}^d, \mathcal{L}^1(H))$ such that for any operator $A = \mathrm{Op}(a) \in \Psi^0_{\mathrm{comp}}(\mathbb{R}^d, \mathcal{K}(H))$ the limit (2) exists without passing to a subsequence. Then we call measure μ *microlocal defect measure associate to the sequence* $u_n - u$.

The existence of a subsequence of $u_n - u$ for which there exists the limit in (2) for every $\mathrm{Op}(a)$ implies that this subsequence of $u_n - u$ is pure.

2. Existence of two scale measure and relation to scalar miclolocal defect measure

Let (X, M) be a measurable space with σ-algebra M of subsets of X, and H be a Hilbert space. We consider a positive operator valued measure μ that maps elements of M into bounded positive self-adjoint operators on H such that $\mu(X) = Id_H$, the identity operator of H. Recall [7] that for every $h, k \in H$, one can define a complex measure $\mu_{h,k}$ on M,

$$\mu_{h,k}(E) = \langle \mu(E)h, k\rangle, \qquad E \in M,$$

that becomes positive measure when $h = k$, or probability measure when $h = k$ and $|h| = 1$. Denote

$$\mu_h := \mu_{h,h}, \quad h \in H.$$

Moreover, there exists unique bounded linear operator T defined on bounded M-measurable functions $\varphi = \varphi(x)$, $x \in X$, via

$$\langle T(\varphi)h, h\rangle = \int_X \varphi(x)\, d\mu_h.$$

Thus, a positive operator valued measure μ can be seen as a mapping from the space of bounded M-measurable function to the space of positive self-adjoint operators on H.

Denote by m_Y the Haar measure over the torus $Y = \mathbb{T}^d$. Recall [11] that $A_n : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d \times Y)$ is a sequence of unitary operators defined by

$$L^2(\mathbb{R}^d) \ni f \mapsto A_n(f)(x, y) := \sum_{k\in\mathbb{Z}^d} f_n^k(x) e^{2\pi i k y}, \quad (x, y) \in \mathbb{R}^d \times Y,$$

where f_n^k is given by its Fourier transform

$$\hat{f}_n^k(\xi) = \hat{f}(\xi + nk)\chi_n(\xi), \quad \xi \in \mathbb{R}^d,$$

with χ_n denoting the characteristic function of the hyper-interval $[-\frac{n}{2}, \frac{n}{2}]^d$. It is proved in [11] that the two scale convergence $u_n(x) \overset{2}{\rightharpoonup} u(x, y)$ is equivalent to the weak convergence of the sequence $v_n(x, y) := (A_n u_n)(x, y)$ in $L^2(\mathbb{R}^d \times Y)$. Put

$$(A_n u_n)(x, y) = \sum_{k\in\mathbb{Z}^d} v_n^k(x) e^{2\pi i k y}.$$

where, by the use of inverse Fourier transform, one has

$$v_n^k(\xi) = \left(u_n(x) e^{-2\pi i k n x}\right) * \delta_n(x). \tag{3}$$

Note that $\delta_n(x) := \mathcal{F}^{-1}(\chi_n)(x) = \Pi_{i=1}^d \frac{\sin \pi n x_i}{\pi x_i}$ is a delta sequence.

The weak convergence $v_n \rightharpoonup u$ in $L^2(\mathbb{R}^d \times Y)$,

$$\int_{\mathbb{R}^d\times Y} v_n(x, y)\varphi(x)\overline{b(y)} dx dy \longrightarrow \int_{\mathbb{R}^d} \varphi(x) \langle u(x, \cdot), b(\cdot)\rangle_{L^2(Y)}\, dx,$$

$\varphi \in C_0^\infty(\mathbb{R}^d)$, $b \in C^\infty(Y)$, clearly shows the boundedness of v_n in $L^2(\mathbb{R}^d, L^2(Y))$.

Now we can define two scale defect measure starting with $v_n = A_n(u_n)$, where $u_n(x) \xrightharpoonup{2} u(x,y)$. The first one is determined by $v_n(x,y) - u(x,y)$. It is denoted as $\mu \in \mathcal{M}_+(T^*\mathbb{R}^d, \mathcal{L}^1(L^2(Y))$ and it is called *two scale microlocal defect measure associated to the sequence u_n*. Note that if v_n is pure, then u_n is also pure sequence. We will assume this in Theorem 2.1.

With $b = 1$ in (1) we obtain the weak convergence

$$u_n \rightharpoonup u_0 := \int_Y u(x,y)\,dy, \text{ in } L^2(\mathbb{R}^d) \tag{4}$$

and the second microlocal defect measure $\tilde{\mu} \in \mathcal{M}_+(T^*\mathbb{R}^d)$ prescribed to $u_n - u_0 \rightharpoonup 0$, in $L^2(\mathbb{R}^d)$. By ([7, 16]), for every $\Phi \in \Psi^0_{\text{comp}}(\mathbb{R}^d)$ with the symbol $\phi(x,\xi)$,

$$\lim_{n\to\infty} \langle \Phi(u_n - u_0)\,,\, u_n - u_0\rangle = \int_{T^*\mathbb{R}^d} \phi(x,\xi)\,\tilde{\mu}(dxd\xi).$$

We collect the previous observations in the next theorem keeping the same notation.

Theorem 2.1. *With the assumptions and notation given above we have:*
Let $l \in \mathbb{Z}^d$ and denote the associated measure by $\mu^l = \langle \mu b_l, b_l\rangle$. Then,

$$\mu^l = \tilde{\mu}^l,$$

where $\tilde{\mu}_l \in \mathcal{M}_+(T^*\mathbb{R}^d)$ is a scalar measure associate to

$$u_n(x)b_l(nx) - u^l(x) \rightharpoonup 0 \text{ with } u^l(x) := \langle u(x,y), \overline{b_l(y)}\rangle.$$

In particular, the sequence $u_n - u_0 \rightharpoonup 0$, where u_0 is given by (4) *is pure with the associated (scalar) microlocal defect measure $\tilde{\mu} = \tilde{\mu}_0 = (\mu, 1)$.*

Proof. Since $\langle v_n, b_l\rangle_{L^2(Y)}$ weakly converges to $\langle u, b_l\rangle = u^l$ in $L^2(\mathbb{R}^d)$ for every $l \in \mathbb{Z}^d$, the sequence $\langle v_n - u, b_l\rangle$ is pure with the associated defect measure $\langle \mu b_l, b_l\rangle$.

Now, we will prove that $\mu^l = \tilde{\mu}^l$ a.e. on $T^*\mathbb{R}^d$. Let $\phi \in C_0^\infty(T^*\mathbb{R}^d)$, $\Phi = \mathrm{Op}(\phi) = \phi(x,D)$ and

$$u_n^l(x) = u_n(x)b_{-l}(nx),\ v_n^l = u_n^l * \delta_n.$$

By (3), there exist constants c_1, c_2 and compact set $K \subset \mathbb{R}^d$ such that

$$\begin{aligned}
&\langle \Phi\left(u_n^l(x) - u_n^l * \delta_n(x)\right), u_n^l(x) - u_n^l * \delta_n(x)\rangle \\
&\quad\le \|\Phi\left(u_n^l(x) - u_n^l * \delta_n(x)\right)\|^2\, \|u_n^l(x) - u_n^l * \delta_n(x)\|^2 \\
&\quad\le c_1 \int_K \Big| \int_{\mathbb{R}^d_\xi} e^{2\pi i x\cdot\xi}\phi(x,\xi)\left(\hat{u}_n^l(\xi) - \hat{u}_n^l(\xi)\chi_n(\xi)\right) d\xi\Big|^2 dx \\
&\quad\le c_1 \int_{\mathbb{R}^d_\xi}\int_K \left|\phi(x,\xi)\right|^2 \left|\hat{u}_n^l(\xi) - \hat{u}_n^l(\xi)\chi_n(\xi)\right|^2 dxd\xi \\
&\quad\le c_2 \int_{\mathbb{R}^d_\xi} \left|\hat{u}_n^l(\xi) - \hat{u}_n^l(\xi)\chi_n(\xi)\right|^2 d\xi \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Moreover,

$$\begin{aligned}
(\tilde{\mu}_l, \phi(x,\xi)) &:= \int_{T^*\mathbb{R}^d} \phi(x,\xi) d\tilde{\mu}_l(x,\xi) = \langle \Phi(u_n^l - u^l), u_n^l - u^l\rangle \\
&\le \langle \Phi(u_n^l - v_n^l), u_n^l - v_n^l\rangle + \langle \Phi(u_n^l - v_n^l), v_n^l - u^l\rangle \\
&\quad + \langle \Phi(v_n^l - u^l), u_n^l - v_n^l\rangle + \langle \Phi(v_n^l - u^l), v_n^l - u^l\rangle.
\end{aligned} \tag{5}$$

We have to estimate the first summand on the right hand side of (5):

$$\left|\langle\Phi(u_n^l - v_n^l), u_n^l - v_n^l\rangle\right| \le c_1 \int_K \int_{|\xi|>\frac{n}{2}} |\phi(x,\xi)|^2 |\hat{u}_n^l(\xi)|^2 \, d\xi dx \to 0.$$

A similar estimate holds for the second and third summand in (5), since both v_n^l and u_n^l weakly converge to u^{-l}, which provides the boundedness of the sequences $v_n^l - u^{-l}$ and $u_n^l - u^{-l}$, i.e.

$$\langle\Phi(u_n^l - v_n^l), v_n^l - u^{-l}\rangle \le c_3 \int_K \left|\Phi\left(u_n^l(x) - u_n^l * \delta_n(x)\right)\right|^2 \to 0$$
$$\langle\Phi(v_n^l - u^{-l}), u_n^l - v_n^l\rangle \le c_4 \int_K \int_{\mathbb{R}^d_\xi} |\phi(x,\xi)(1-\chi_n(\xi)|^2 |\hat{u}_n^l(\xi)|^2 \, d\xi dx \to 0.$$

We see that the forth summand in (5) converges to (μ_l, ϕ) which gives us that $\mu_l = \tilde{\mu}_l$ a.e. on $T^*\mathbb{R}^d$ that is

$$(\tilde{\mu}_l, \phi) = \lim_{n\to\infty} \langle\Phi(v_n^l - u^l), v_n^l - u^l\rangle = (\mu_l, \phi).$$

The last assertion is clear. □

Now we can prove that two scale defect measure can be represented via scalar miclolocal defect measure and Haar measure m_Y (d-dimensional Lebesgue measure) on Y.

Theorem 2.2. *Two scale defect measure μ is absolutely continuous with respect to $\tilde{\mu} \otimes m_Y$. (Recall, $\tilde{\mu} = \tilde{\mu}^0$.)*

Proof. The proof is the same as the one of Proposition A.1 in [7]. Since $\mu_{l,k} = \langle \mu b_l, b_k\rangle$, $l,k \in \mathbb{Z}$, are absolutely continuous ($\mu_{l,l} \ge 0$) with respect to the trace measure $\lambda = \mathrm{tr}\mu = \sum_{l\in\mathbb{Z}^d} \mu_{ll}$, there exists $f = (f_{k,l})_{l,k\in\mathbb{Z}^d}$ so that $\mu = \lambda f$, $\mu_{l,k} = f_{l,k}\lambda$, $l,k \in \mathbb{Z}^d$, with $\overline{f_{l,k}} = f_{k,l}$.

By Theorem 2.1, $\mu_{ll} = \mu_l = \tilde{\mu}_l$. So $\tilde{\mu}(\varphi) = 0$ implies $\lambda(\varphi) = 0$. Thus, μ is absolutely continuous with respect to $\tilde{\mu} \otimes m_Y$. □

From Theorem 2.2 we have the following representation.

Corollary 2.3. *There exists $f \in L^1(T^*\mathbb{R}^d \times Y)$ such that*

$$\mu(x,y,\xi) = f(x,y,\xi)\tilde{\mu}(x,\xi)m_y(y).$$

3. Applications

Let $u_n(x), x \in \mathbb{R}^d$, be a sequence of solutions to equation

$$\sum_{i=1}^{m} \partial_{x_i}\Big(a_i(nx)u_n(x)\Big) = \Pi_n(x), \tag{6}$$

where coefficients $a_i \in C^\infty_\sharp(Y), i = 1, ..., m$, and $\Pi_n(x)$ is a sequence in $H^{-1}(\mathbb{R}^d)$ converging strongly to $\Pi(x)$ in $H^{-1}(\mathbb{R}^d)$. We assume that

$$u_n(x) \overset{2}{\rightharpoonup} u(x,y).$$

Our aim is to find some sufficient conditions which will imply the strong convergence $u_n \to u$ in $L^2(\mathbb{R}^d)$ so that one has the strong convergence in $H^{-1}(\mathbb{R}^d)$ of the left and the right hand sides of (6) to the left and right hand sides of

$$\sum_{i=1}^{m} \partial_{x_i} \int_Y \Big(a_i(y)u(x,y)\Big)dy = \Pi(x). \tag{7}$$

Define operator $\mathcal{P}$

$$\mathcal{P}v(x,y) := \sum_{i=1}^{d} \partial_{x_i} \left(a_i(y)\, v(x,y)\right),\ v \in L^2(\mathbb{R}^d \times Y).$$

Assume that the symbol p of $\mathcal{P}$ satisfies the following non-degenerecy condition (cf. [7]):

$$m_Y\{y \,|\, p(x,y,\xi) \text{ is not injective}\} = 0, \quad (x,\xi) \in T^*\mathbb{R}^d.$$

Non-degeneracy condition implies that for every $b \in L^2(Y)$,

$$\langle p(x,y,\xi)\,,\, b(y)\rangle_{L^2(Y)} = 0 \quad \Longrightarrow \quad b = 0 \text{ a.e. in } y \in Y.$$

Let $l \in \mathbb{Z}^d$ and

$$\mathcal{P}_{l,n}u_n(x) := \sum_{i=1}^{m} \partial_{x_i}\Big(a_i(nx)b_l(nx)\,u_n(x)\Big) = \Pi_n^l(x), \text{ with } \Pi_n^0 = \Pi_n.$$

Assume that

$$\Pi_n^l(x) \text{ strongly converges in } H^{-1}(\mathbb{R}^d), \text{ for every } l \in \mathbb{Z}^d. \tag{8}$$

Lemma 3.1. *For every $l \in \mathbb{Z}^d$, $\Pi_n^l * \delta_n(x)$ strongly converges in $H^{-1}(\mathbb{R}^d)$.*

Proof. This follows from the fact that for any B bounded in $H^1(\mathbb{R}^d)$ the set $B_\delta = \{\psi * \delta_n \,:\, \psi \in B,\, n \in \mathbb{N}\}$ is also bounded in $H^1(\mathbb{R}^d)$ since $\|\psi * \delta_n\|_{L^2} \le \|\psi\|_{L^2}$. □

Proposition 3.2. *Let $u_n \overset{2}{\rightharpoonup} u$ in $L^2(\mathbb{R}^d \times Y)$, Π_n^l satisfy* (8) *and $u_0(x) = \displaystyle\int_Y u(x,y)dy$. Then, $u_n - u_0 \to 0$, in $L^2(\mathbb{R}^d)$ and* (7) *holds.*

Proof. We know from [11] that $v_n(x,y) \rightharpoonup u$ in $L^2(\mathbb{R}^d \times Y)$, where

$$v_n(x,y) = (A_n u_n)(x,y) = \sum_{k \in \mathbb{Z}^d} b_k(y)\Big(u_n(x)b_{-k}(nx) * \delta_n(x)\Big).$$

Thus we can associate a two scale microlocal defect measure μ to $v_n - u$. If $v_n - u$ is not a pure sequence, we continue with a pure subsequence and associated two scale microlocal defect measure μ.

We will show that for every $l \in \mathbb{Z}^d$,

$$\Big\langle \mathcal{P}v_n(x,y),\, b_l(y)\Big\rangle_{L^2(Y)}$$

strongly converges in $H^{-1}(\mathbb{R}^d)$. Decompose coefficients $a_i(y) = \displaystyle\sum_{l \in \mathbb{Z}^d} a_{il}b_l(y), \quad i = 1, ..., d$. The non-degeneracy condition implies that

$$\sum_{i=1}^{d} \sum_{l \in \mathbb{Z}^d} a_{il}b_l(y)\xi_i \neq 0.$$

Then, with $v_n^l = e^{2\pi i l y}(u_n b_{-ln}) * \delta_n$ and $k + l = s$,

$$\begin{aligned}\Big\langle \mathcal{P}v_n(x,y), b_l(y)\Big\rangle &= \Big\langle \sum_{i=1}^{d}\sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} \partial_{x_i}\Big(a_{ik}v_n^l(x)\Big)e^{2\pi i(k+l)y}\,,\, b_s\Big\rangle \\ &= \sum_{i=1}^{d}\sum_{k\in\mathbb{Z}^d}\partial_{x_i}\Big(a_{ik}v_n^{s-k}(x)\Big) \\ &= \sum_{i=1}^{d}\partial_{x_i}\Big(a_i(nx)u_n(x)b_{-l}(nx) * \delta_n(x)\Big).\end{aligned} \tag{9}$$

Since the lineal of the set $\{b_l\,:\, l\in\mathbb{Z}^d\}$ is dense in $L^2(Y)$ and $\Pi_n^l(x) * \delta_n(x)$ strongly converges in $H^{-1}(\mathbb{R}^d)$, we have that for every $l\in\mathbb{Z}^d$,

$$\Big\langle \mathcal{P}v_n(x,y),\, b_l(y)\Big\rangle_{L^2(Y)}$$

strongly converges in $H^{-1}(\mathbb{R}^d)$.

So we can apply Corollary 2.2 from [7], to conclude that

$$p\mu = 0.$$

From (9) and the previous theorem, we see that $\mu \in \mathcal{M}_+(T^*\mathbb{R}^d, \mathcal{L}^1(L^2(Y)))$ equals zero, provided by the non-degeneracy condition. From Corollary 2.3, we conclude that the scalar microlocal defect measure $\tilde{\mu} \in \mathcal{M}_+(T^*\mathbb{R}^d)$ associated to the sequence $u_n - u_0$ satisfies $\tilde{\mu} = 0$, i.e. for every $\Phi \in \Psi^0_{\mathrm{comp}}(\mathbb{R}^d)$ with a principal symbol $\phi(x,\xi)$,

$$\lim_{n\to\infty}\langle \Phi(u_n-u_0)\,,\, u_n-u_0\rangle = \int_{T^*\mathbb{R}^d}\phi(x,\xi)\,\tilde{\mu}(dxd\xi) = 0.$$

Thus, $u_n - u_0$ strongly converges to zero in $L^2(\mathbb{R}^d)$ and (7) holds.

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