



Another Class of Warped Product Skew CR-Submanifolds of Kenmotsu Manifolds

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Abstract. Recently, Naghi et al. [32] studied warped product skew CR-submanifold of the form $M_1 \times_f M_\perp$ of order 1 of a Kenmotsu manifold \bar{M} such that $M_1 = M_T \times M_\theta$, where M_T , M_\perp and M_θ are invariant, anti-invariant and proper slant submanifolds of \bar{M} . The present paper deals with the study of warped product submanifolds by interchanging the two factors M_T and M_\perp , i.e, the warped products of the form $M_2 \times_f M_T$ such that $M_2 = M_\perp \times M_\theta$. The existence of such warped product is ensured by an example and then we characterize such warped product submanifold. A lower bound of the squared norm of second fundamental form is derived with sharp relation, whose equality case is also considered.

1. Introduction

In 1986, Bejancu [4] introduced the notion of CR-Submanifolds. This family of submanifolds was generalized by Chen [9] as slant submanifolds. Then a more generalization is given as semi-slant submanifolds by Papaghiuc [33]. Next, Cabrerizo et al. [7] defined and studied bi-slant submanifolds and simultaneously gave the notion of pseudo-slant submanifolds. The contact version of slant, semi slant and pseudo-slant submanifolds are studied in [28], [7] and [24], respectively. As a generalization of all these class of submanifolds, Ronsse [34] introduced the notion of skew CR-submanifolds of Kaehler manifolds.

The notion of warped product was introduced by Bishop and O'Neill in [6] to construct the examples of manifolds with negative curvature. The study of warped product submanifolds was initiated by Chen ([10], [11]). Then several authors studied warped product submanifolds. For detailed study of warped product submanifolds, we may refer to ([12], [19]-[22], [31]). In this connection it may be mentioned that warped product submanifolds of Kenmotsu manifold are studied in ([1]-[3], [25]-[27], [30], [37]-[41]).

Warped product skew CR-submanifolds of Kaehler manifold was studied by Sahin [35] and in [13] Haider et al. studied this class of submanifolds in cosymplectic ambient. Recently Naghi et al. [32] studied warped product skew CR-submanifolds of the form $M_1 \times_f M_\perp$ of order 1 of a Kenmotsu manifold \bar{M} such that $M_1 = M_T \times M_\theta$, where M_T , M_\perp and M_θ are invariant, anti-invariant and proper slant submanifolds of \bar{M} . In this paper we have concentrated on another class of warped product skew CR-submanifolds of Kenmotsu manifolds of the form $M_2 \times_f M_T$, where $M_2 = M_\perp \times M_\theta$. The present paper is organized as follows:

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in section 2, some preliminaries are given, section 3 is dedicated to the study of skew CR-submanifold of Kenmotsu manifold, in section 4, we provide an example of warped product skew CR-submanifolds of the form $M_2 \times_f M_T$ and some basic results of such type of submanifolds are obtained, a characterization of skew CR-warped product of the form $M_2 \times_f M_T$ is obtained in section 5. In section 6, we have established two inequalities on a warped product skew CR-submanifold $M = M_2 \times_f M_T$ of a Kenmotsu manifold \bar{M} .

2. Preliminaries

In [36] Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c . He proved that they could be divided into three classes: (i) homogeneous normal contact Riemannian manifolds with $c > 0$, (ii) global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature if $c = 0$ and (iii) a warped product space $\mathbb{R} \times_f \mathbb{C}^n$ if $c < 0$.

Kenmotsu [23] characterized the differential geometric properties of the manifolds of class (iii) which are nowadays called Kenmotsu manifolds and later studied by several authors ([16]-[18]) etc.

An odd dimensional smooth manifold \bar{M}^{2m+1} is said to be an almost contact metric manifold [5] if it admits a $(1, 1)$ tensor field ϕ , a vector field ξ , an 1-form η and a Riemannian metric g which satisfy

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \phi^2 X = -X + \eta(X)\xi, \tag{1}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{3}$$

for all vector fields X, Y on \bar{M} .

An almost contact metric manifold $\bar{M}^{2m+1}(\phi, \xi, \eta, g)$ is said to be Kenmotsu manifold if the following conditions hold [23]:

$$\bar{\nabla}_X \xi = X - \eta(X)\xi, \tag{4}$$

$$(\bar{\nabla}_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{5}$$

where $\bar{\nabla}$ denotes the Riemannian connection of g .

Let M be an n -dimensional submanifold of a Kenmotsu manifold \bar{M} . Throughout the paper we assume that the submanifold M of \bar{M} is tangent to the structure vector field ξ . Let ∇ and ∇^\perp be the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M respectively. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{6}$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N \tag{7}$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into \bar{M} and they are related by $g(h(X, Y), N) = g(A_N X, Y)$ for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where g is the Riemannian metric on \bar{M} as well as on M .

The mean curvature H of M is given by $H = \frac{1}{n} \text{trace } h$. A submanifold M of a Kenmotsu manifold \bar{M} is said to be totally umbilical if $h(X, Y) = g(X, Y)H$ for any $X, Y \in \Gamma(TM)$. If $h(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$, then M is totally geodesic and if $H = 0$ then M is minimal in \bar{M} .

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent bundle TM and $\{e_{n+1}, \dots, e_{2m+1}\}$ be that of the normal bundle $T^\perp M$. Set

$$h_{ij}^r = g(h(e_i, e_j), e_r) \text{ and } \|h\|^2 = g(h(e_i, e_j), h(e_i, e_j)), \tag{8}$$

for $i, j \in \{1, \dots, n\}$ and $r \in \{n+1, \dots, 2m+1\}$. For a differentiable function f on M , the gradient ∇f is defined by

$$g(\nabla f, X) = Xf \quad (9)$$

for any $X \in \Gamma(TM)$. As a consequence, we get

$$\|\nabla f\|^2 = \sum_{i=1}^n (e_i(f))^2. \quad (10)$$

For any $X \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, we can write

$$(a) \phi X = PX + QX, \quad (b) \phi N = bN + cN \quad (11)$$

where PX , bN are the tangential components and QX , cN are the normal components.

A submanifold M of an almost contact metric manifold \bar{M} is said to be invariant if $\phi(T_p M) \subseteq T_p M$ and anti-invariant if $\phi(T_p M) \subseteq T_p^\perp M$ for every $p \in M$.

A submanifold M of an almost contact metric manifold \bar{M} is said to be slant if for each non-zero vector $X \in T_p M$, the angle θ between ϕX and $T_p M$ is a constant, i.e. it does not depend on the choice of $p \in M$. Invariant and anti-invariant submanifolds are particular cases of slant submanifolds with slant angles $\theta = 0$ and $\frac{\pi}{2}$ respectively.

Theorem 2.1. [8] Let M be a submanifold of an almost contact metric manifold \bar{M} such that $\xi \in \Gamma(TM)$. Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$P^2 = \lambda(-I + \eta \otimes \xi), \quad (12)$$

furthermore if θ is slant angle then $\lambda = \cos^2 \theta$.

If M is a slant submanifold of an almost contact metric manifold \bar{M} , the following relation holds [38]:

$$bQX = \sin^2 \theta \{-X + \eta(X)\xi\}, \quad cQX = -QPX. \quad (13)$$

Definition 2.2. [6] Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds with Riemannian metric g_1 and g_2 respectively and f be a positive smooth function on N_1 . The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + f^2 g_2. \quad (14)$$

A warped product manifold $N_1 \times_f N_2$ is said to be trivial if the warping function f is constant. For a warped product manifold $M = N_1 \times_f N_2$, we have [6]

$$\nabla_U X = \nabla_X U = (X \ln f)U \quad (15)$$

for any $X, Y \in \Gamma(TN_1)$ and $U \in \Gamma(TN_2)$.

We now recall the following:

Theorem 2.3. (Hiepko's Theorem, see [15]). Let \mathcal{D}_1 and \mathcal{D}_2 be two orthogonal distribution on a Riemannian manifold M . Suppose that \mathcal{D}_1 and \mathcal{D}_2 both are involutive such that \mathcal{D}_1 is a totally geodesic foliation and \mathcal{D}_2 is a spherical foliation. Then M is locally isometric to a non-trivial warped product $M_1 \times_f M_2$, where M_1 and M_2 are integral manifolds of \mathcal{D}_1 and \mathcal{D}_2 , respectively.

3. Skew CR-submanifolds of Kenmotsu manifolds

Let M be a submanifold of a Kenmotsu manifold \bar{M} . First from [34], we recall the definition of skew CR-submanifolds. Throughout the paper we consider the structure vector field ξ is tangent to the submanifold otherwise the submanifold is C -totally real [14].

For any X and Y in T_pM , we have $g(PX, Y) = -g(X, PY)$. Hence it follows that P^2 is symmetric operator on the tangent space TM , for all $p \in M$. Therefore the eigen values are real and it is diagonalizable. Moreover its eigen values are bounded by -1 and 0 . For each $p \in M$, we may set

$$\mathcal{D}_p^\lambda = \ker\{P^2 + \lambda^2(p)I\}_p,$$

where I is the identity transformation and $\lambda(p) \in [0, 1]$ such that $\lambda^2(p)$ is an eigen value of P_p^2 . We note that $\mathcal{D}_p^1 = \ker Q$ and $\mathcal{D}_p^0 = \ker P$. \mathcal{D}_p^1 is the maximal ϕ -invariant subspace of T_pM and \mathcal{D}_p^0 is the maximal ϕ -anti-invariant subspace of T_pM . From now on, we denote the distributions \mathcal{D}^1 and \mathcal{D}^0 by $\mathcal{D}^T \oplus \langle \xi \rangle$ and \mathcal{D}^\perp , respectively. Since P_p^2 is symmetric and diagonalizable, if $-\lambda_1^2(p), \dots, -\lambda_k^2(p)$ are the eigenvalues of P^2 at $p \in M$, then T_pM can be decomposed as direct sum of mutually orthogonal eigen spaces, i.e.

$$T_pM = \mathcal{D}_p^{\lambda_1} \oplus \mathcal{D}_p^{\lambda_2} \cdots \oplus \mathcal{D}_p^{\lambda_k}.$$

Each $\mathcal{D}_p^{\lambda_i}$, $1 \leq i \leq k$ defined on M with values in $(0, 1)$ such that

(i) Each $-\lambda_i^2(p)$, $1 \leq i \leq k$ is a distinct eigen value of P^2 with

$$T_pM = \mathcal{D}_p^T \oplus \mathcal{D}_p^\perp \oplus \mathcal{D}_p^{\lambda_1} \oplus \mathcal{D}_p^{\lambda_2} \cdots \oplus \mathcal{D}_p^{\lambda_k} \oplus \langle \xi \rangle_p$$

for any $p \in M$.

(ii) The dimensions of \mathcal{D}_p^T , \mathcal{D}_p^\perp and $\mathcal{D}_p^{\lambda_i}$, $1 \leq i \leq k$ are independent on $p \in M$.

Moreover, if each λ_i is constant on M , then M is called a skew CR-submanifold. Thus, we observe that CR-submanifolds are a particular class of skew CR-submanifolds with $k = 0$, $\mathcal{D}^T \neq \{0\}$ and $\mathcal{D}^\perp \neq \{0\}$. And slant submanifolds are also a particular class of skew CR-submanifold with $k = 1$, $\mathcal{D}^T = \{0\}$, $\mathcal{D}^\perp = \{0\}$ and λ_1 is constant. Moreover, if $\mathcal{D}^\perp = \{0\}$, $\mathcal{D}^T \neq \{0\}$ and $k = 1$, then M is semi-slant submanifold. Furthermore, if $\mathcal{D}^T = \{0\}$, $\mathcal{D}^\perp \neq \{0\}$ and $k = 1$, then M is a pseudo-slant (or hemi-slant) submanifold.

A submanifold M of \bar{M} is said to be proper skew CR-submanifold of order 1 if M is a skew CR-submanifold with $k = 1$ and λ_1 is constant. In that case, the tangent bundle of M is decomposed as

$$TM = \mathcal{D}^T \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \langle \xi \rangle.$$

The normal bundle $T^\perp M$ of a skew CR-submanifold M is decomposed as

$$T^\perp M = \phi\mathcal{D}^\perp \oplus Q\mathcal{D}^\theta \oplus \nu,$$

where ν is a ϕ -invariant normal subbundle of $T^\perp M$.

Now for the sake of further study we give the following useful results.

Lemma 3.1. *Let M be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that $\xi \in \Gamma(\mathcal{D}^\perp \oplus \mathcal{D}^\theta)$, then we have*

$$g(\nabla_X Y, Z) = g(A_{\phi Z} X, \phi Y) - \eta(Z)g(X, Y), \quad (16)$$

$$g(\nabla_X Y, U) = \csc^2 \theta [g(A_{QU} X, \phi Y) - g(A_{QPU} X, Y)] - \eta(U)g(X, Y) \quad (17)$$

for every $X, Y \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$ and $U \in \Gamma(\mathcal{D}^\theta)$.

Proof. For any $X, Y \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\phi \bar{\nabla}_X Y, \phi Z) + \eta(Z)g(\bar{\nabla}_X Y, \xi) \\ &= g(\bar{\nabla}_X \phi Y, \phi Z) - g((\bar{\nabla}_X \phi)Y, \phi Z) - \eta(Z)g(Y, \bar{\nabla}_X \xi). \end{aligned}$$

Using (2.4), (2.5) and (2.6) in the above equation, we get (16). Also, for $X, Y \in \Gamma(\mathcal{D}^T)$, $U \in \Gamma(\mathcal{D}^\theta)$, we have

$$\begin{aligned} g(\nabla_X Y, U) &= g(\phi \bar{\nabla}_X Y, \phi U) + \eta(U)g(\bar{\nabla}_X Y, \xi) \\ &= g(\bar{\nabla}_X \phi Y, \phi U) - g((\bar{\nabla}_X \phi)Y, \phi U) - \eta(U)g(Y, \bar{\nabla}_X \xi) \\ &= g(\bar{\nabla}_X \phi Y, PU) + g(\bar{\nabla}_X \phi Y, QU) - \eta(U)g(X, Y) \\ &= -g(\phi Y, \bar{\nabla}_X PU) - g(\phi Y, \bar{\nabla}_X QU) - \eta(U)g(X, Y) \\ &= g(\bar{\nabla}_X \phi PU, Y) - g((\bar{\nabla}_X \phi)PU, Y) - g(\bar{\nabla}_X QU, \phi Y) - \eta(U)g(X, Y) \\ &= g(\bar{\nabla}_X P^2 U, Y) + g(\bar{\nabla}_X QPU, Y) - g(\bar{\nabla}_X QU, \phi Y) - \eta(U)g(X, Y). \end{aligned}$$

By virtue of (4), (7) and (12) the above equation yields

$$\begin{aligned} g(\nabla_X Y, U) &= -\cos^2 \theta g(\bar{\nabla}_X U, Y) + \cos^2 \theta \eta(U)g(X, Y) \\ &\quad - g(A_{QPU}X, Y) + g(A_{QU}X, \phi Y) - \eta(U)g(X, Y), \end{aligned}$$

Thus we get

$$\sin^2 \theta g(\nabla_X Y, U) = g(A_{QU}X, \phi Y) - g(A_{QPU}X, Y) - \sin^2 \theta \eta(U)g(X, Y).$$

From which the relation (17) follows. \square

Corollary 3.2. *Let M be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that $\xi \in \Gamma(\mathcal{D}^\perp \oplus \mathcal{D}^\theta)$, then we have*

$$g([X, Y], Z) = g(A_{\phi Z}X, \phi Y) - g(A_{\phi Z}Y, \phi X) \tag{18}$$

$$g([X, Y], U) = \csc^2 \theta \{g(A_{QU}X, \phi Y) - g(A_{QU}Y, \phi X)\} \tag{19}$$

for every $X, Y \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$ and $U \in \Gamma(\mathcal{D}^\theta)$.

Lemma 3.3. *Let M be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that $\xi \in \Gamma(\mathcal{D}^\perp \oplus \mathcal{D}^\theta)$, then we have*

$$g(\nabla_Z W, X) = -g(A_{\phi W} \phi X, Z), \tag{20}$$

$$g(\nabla_Z U, X) = \csc^2 \theta \{g(A_{QPU}X, Z) - g(A_{QU} \phi X, Z)\}, \tag{21}$$

$$g(\nabla_U Z, X) = -g(A_{\phi Z}X, U), \tag{22}$$

$$g(\nabla_U V, X) = \csc^2 \theta \{g(A_{QPV}X, U) - g(A_{QV} \phi X, U)\}, \tag{23}$$

$$g(\nabla_X Z, U) = \sec^2 \theta \{g(A_{QPU}Z, X) - g(A_{\phi Z}PU, X)\} \tag{24}$$

for $X \in \Gamma(\mathcal{D}^T)$, $Z, W \in \Gamma(\mathcal{D}^\perp)$ and $U, V \in \Gamma(\mathcal{D}^\theta)$.

Proof. For every $X \in \Gamma(\mathcal{D}^T)$ and $Z, W \in \Gamma(\mathcal{D}^\perp)$, we have

$$\begin{aligned} g(\nabla_Z W, X) &= g(\phi \bar{\nabla}_Z W, \phi X), \\ &= g(\bar{\nabla}_Z \phi W, \phi X) - g((\bar{\nabla}_Z \phi)W, \phi X). \end{aligned}$$

Using (5), (7) and orthogonality of vector fields in the above equation, we get (20). Also, for $X \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$ and $U \in \Gamma(\mathcal{D}^\theta)$, we have

$$\begin{aligned} g(\nabla_Z U, X) &= g(\phi \bar{\nabla}_Z U, \phi X), \\ &= g(\bar{\nabla}_Z \phi U, \phi X) - g((\bar{\nabla}_Z \phi)U, \phi X), \\ &= g(\bar{\nabla}_Z P U, \phi X) + g(\bar{\nabla}_Z Q U, \phi X), \\ &= -g(\bar{\nabla}_Z P^2 U, X) - g(\bar{\nabla}_Z Q P U, X) + g(\bar{\nabla}_Z Q U, \phi X). \end{aligned}$$

Using (7), (12) and the symmetry of shape operator in the above equation, we obtain

$$g(\nabla_Z U, X) = \cos^2 \theta g(\bar{\nabla}_Z U, X) + g(A_{QP} U, Z) - g(A_{QU} \phi X, Z),$$

from which the relation (21) follows.

Again, for $X \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$ and $U \in \Gamma(\mathcal{D}^\theta)$, we have

$$g(\nabla_U Z, X) = g(\phi \bar{\nabla}_U Z, \phi X) = g(\bar{\nabla}_U \phi Z, \phi X) - g((\bar{\nabla}_U \phi)Z, X).$$

By virtue of (5) and (7), the above equation yields

$$g(\nabla_U Z, X) = -g(A_{\phi Z} U, \phi X) - g(\phi U, X)\eta(Z),$$

from which the relation (22) follows. Again we have

$$\begin{aligned} g(\nabla_U V, X) &= g(\phi \bar{\nabla}_U V, \phi X), \\ &= g(\bar{\nabla}_U \phi V, \phi X) - g((\bar{\nabla}_U \phi)V, \phi X), \\ &= g(\bar{\nabla}_U P V, \phi X) + g(\bar{\nabla}_U Q V, \phi X), \\ &= -g(\bar{\nabla}_U P^2 V, X) - g(\bar{\nabla}_U Q P V, X) + g(\bar{\nabla}_U Q V, \phi X). \end{aligned}$$

Using (7), (12) and the symmetry of shape operator in the above equation, we get

$$g(\nabla_U V, X) = \cos^2 \theta g(\bar{\nabla}_U V, X) + g(A_{QP} V, U) - g(A_{QV} \phi X, U),$$

from which we get (23).

For every $X \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$ and $U \in \Gamma(\mathcal{D}^\theta)$, we have

$$\begin{aligned} g(\nabla_X Z, U) &= g(\phi \bar{\nabla}_X Z, \phi U) + \eta(U)g(\bar{\nabla}_X Z, \xi), \\ &= g(\bar{\nabla}_X \phi Z, \phi U) - g((\bar{\nabla}_X \phi)Z, \phi U) - \eta(U)g(Z, \bar{\nabla}_X \xi), \\ &= g(\bar{\nabla}_X \phi Z, P U) + g(\bar{\nabla}_X \phi Z, Q U) + \eta(Z)g(\phi X, \phi U) - \eta(U)g(X, Z), \\ &= g(\bar{\nabla}_X \phi Z, P U) - g(\bar{\nabla}_X Q U, \phi Z), \\ &= g(\bar{\nabla}_X \phi Z, P U) + g(\bar{\nabla}_X b Q U, Z) + g(\bar{\nabla}_X c Q U, Z). \end{aligned}$$

In view of (7), (13) and the symmetry of shape operator, the above equation reduces to

$$g(\nabla_X Z, U) = -g(A_{\phi Z} P U, X) - \sin^2 \theta g(\bar{\nabla}_X U, Z) + g(A_{QP} U, Z, X),$$

from which the relation (24) follows. \square

4. Warped product skew CR-submanifolds of the form $M_2 \times_f M_T$

Let $M = M_2 \times_f M_T$ be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that ξ is tangent to $M_2 = M_\perp \times M_\theta$, where M_T, M_θ and M_\perp are invariant, proper slant and anti-invariant submanifold of \bar{M} , respectively. Let the dimensions of these submanifolds are $\dim M_\perp = d_1, \dim M_\theta = d_2$ and $\dim M_T = d_3$. If $d_2 = 0$ then M is a CR-warped product of the form $M = M_\perp \times_f M_T$ which have been studied in [40].

Now, we construct an example of a non-trivial warped product skew CR-submanifold of order 1 of the form $M = M_2 \times_f M_T$.

Example 4.1. Consider the Kenmotsu manifold $M = \mathbb{R} \times_f C^4$ with the structure (ϕ, ξ, η, g) is given by

$$\phi\left(\sum_{i=1}^5 (X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i}) + Z \frac{\partial}{\partial t}\right) = Y_i \frac{\partial}{\partial x^i} - X_i \frac{\partial}{\partial y^i},$$

$$\xi = 3e^{-t} \frac{\partial}{\partial t}, \eta = \frac{1}{3} e^t dt \text{ and } g = \eta \otimes \eta + \frac{e^{3t}}{9} \sum_{i=1}^5 (dx^i \otimes dx^i + dy^i \otimes dy^i)$$

Now, we consider a submanifold M of \bar{M} defined by the immersion χ as follows:

$$\chi(u, v, w, s, \theta, \phi, t) = 3(e^{-t}u, 0, w, 0, 2\theta + 3\phi, 0, e^{-t}v, s, 0, 3\theta + 2\phi, t).$$

Then the local orthonormal frame of TM is spanned by the following:

$$Z_1 = \frac{3}{e^t} \left(\frac{\partial}{\partial x^1}\right), \quad Z_2 = \frac{3}{e^t} \left(\frac{\partial}{\partial y^2}\right), \quad Z_3 = 3 \frac{\partial}{\partial x^3}, \quad Z_4 = 3 \frac{\partial}{\partial y^3}$$

$$Z_5 = 3\left(2 \frac{\partial}{\partial x^5} + 3 \frac{\partial}{\partial y^5}\right), \quad Z_6 = 3\left(3 \frac{\partial}{\partial x^5} + 2 \frac{\partial}{\partial y^5}\right), \quad Z_7 = 3 \frac{\partial}{\partial t}.$$

Also, we have

$$\phi Z_1 = -\frac{3}{e^t} \left(\frac{\partial}{\partial y^1}\right), \phi Z_2 = \frac{3}{e^t} \left(\frac{\partial}{\partial x^2}\right), \quad \phi Z_3 = -3 \frac{\partial}{\partial y^3}, \phi Z_4 = 3 \frac{\partial}{\partial x^3},$$

$$\phi Z_5 = 3\left(-2 \frac{\partial}{\partial y^5} + 3 \frac{\partial}{\partial x^5}\right), \quad \phi Z_6 = 3\left(-3 \frac{\partial}{\partial y^5} + 2 \frac{\partial}{\partial x^5}\right), \quad \phi Z_7 = 0.$$

If we define $D^\perp = \text{span}\{Z_1, Z_2, Z_7\}$, $D^\theta = \text{span}\{Z_5, Z_6\}$ and $D^T = \text{span}\{Z_3, Z_4\}$ then by simple calculations we can say that D^T is an invariant distribution and D^θ is a slant distribution with slant angle $\cos^{-1} \frac{5}{13}$. Hence M is a proper skew CR-submanifold of \bar{M} of order 1. Also, it is clear that $D^\perp \oplus D^\theta$ and D^T both are integrable. If we denote the integral manifolds of $D^\perp \oplus D^\theta$ and D^T by M_2 and M_T respectively, then the metric tensor g_M of M is given by

$$\begin{aligned} g_M &= (du^2 + dv^2) + 13(d\theta^2 + d\phi^2) + e^{3t}(dw^2 + ds^2) \\ &= g_{M_2} + e^{3t}(dw^2 + ds^2). \end{aligned}$$

Thus $M = M_2 \times_f M_T$ is a warped product skew CR-submanifold of \bar{M} with the warping function $f = \sqrt{e^{3t}}$.

Now, we prove the followings:

Lemma 4.2. Let $M = M_2 \times_f M_T$ be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that ξ is tangent to $M_2 = M_\perp \times M_\theta$, then we have

$$\xi \ln f = 1, \tag{25}$$

$$g(h(X, Z), \phi W) = 0, \tag{26}$$

$$g(h(X, U), \phi Z) = g(h(X, Z), QU) = 0, \tag{27}$$

and

$$g(h(X, U), QV) = 0 \tag{28}$$

for every $X \in \Gamma(M_T)$, $Z, W \in \Gamma(M_\perp)$ and $U, V \in \Gamma(M_\theta)$.

Proof. The proof of (25) is similar as in [32].

Now, for $X \in \Gamma(M_T)$ and $Z \in \Gamma(M_\perp)$, we have

$$\begin{aligned} g(h(X, Z), \phi W) &= g(\bar{\nabla}_Z X, \phi W) \\ &= -g(\bar{\nabla}_Z \phi X, W) + g((\bar{\nabla}_Z \phi)X, W). \end{aligned}$$

Using (5) and (15) in the above equation, we obtain

$$g(h(X, Z), \phi W) = -(Z \ln f)g(\phi X, W) = 0. \quad (29)$$

Thus, we get (26). Again, for $X \in \Gamma(M_T)$, $Z \in \Gamma(M_\perp)$ and $U \in \Gamma(M_\theta)$, we have

$$g(h(X, U), \phi Z) = g(\bar{\nabla}_U X, \phi Z) = -g(\bar{\nabla}_U \phi X, Z) + g((\bar{\nabla}_U \phi)X, Z).$$

Using (5) and (15), the above equation reduces to

$$g(h(X, U), \phi Z) = -(U \ln f)g(\phi X, Z) = 0. \quad (30)$$

Also,

$$\begin{aligned} g(h(X, U), \phi Z) &= g(\bar{\nabla}_X U, \phi Z), \\ &= -g(\bar{\nabla}_X \phi U, Z) + g((\bar{\nabla}_X \phi)U, Z), \\ &= -g(\bar{\nabla}_X P U, Z) - g(\bar{\nabla}_X Q U, Z) + g((\bar{\nabla}_X \phi)U, Z). \end{aligned}$$

Using (5), (7) and (15) in the above equation, we obtain

$$g(h(X, U), \phi Z) = g(h(X, Z), Q U). \quad (31)$$

From (30) and (31) we get (27). Again, for $X \in \Gamma(M_T)$ and $U, V \in \Gamma(M_\theta)$ we have

$$\begin{aligned} g(h(X, U), Q V) &= g(\bar{\nabla}_U X, Q V) \\ &= g(\bar{\nabla}_U X, \phi V) - g(\bar{\nabla}_U X, P V) \\ &= -g(\bar{\nabla}_U \phi X, V) + g((\bar{\nabla}_U \phi)X, V) - g(\bar{\nabla}_U X, P V). \end{aligned}$$

By virtue of (5) and (15), the above equation yields

$$\begin{aligned} g(h(X, U), Q V) &= -(U \ln f)g(\phi X, V) + \eta(V)g(\phi U, X) - (U \ln f)g(X, P V) \\ &= 0. \end{aligned}$$

Thus we get (28). \square

Proposition 4.3. Let $M = M_2 \times_f M_T$ be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that ξ is tangent to $M_2 = M_\perp \times M_\theta$, then we have $h(X, E) \in \nu$ for every $X \in \Gamma(M_T)$ and $E \in \Gamma(M_2)$

Proof. The proof is obvious from (26), (27), (28) and the fact that $h(X, \xi) = 0$, for every $X \in \Gamma(M_T)$. \square

Lemma 4.4. Let $M = M_2 \times_f M_T$ be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that ξ is tangent to $M_2 = M_\perp \times M_\theta$, then we have

$$g(h(X, Y), \phi Z) = \{(Z \ln f) - \eta(Z)\}g(X, \phi Y), \quad (32)$$

$$g(h(X, Y), Q U) = \{\eta(U) - (U \ln f)\}g(\phi X, Y) + (P U \ln f)g(X, Y) \quad (33)$$

and

$$g(h(X, Y), Q P U) = \cos^2 \theta \{\eta(U) - (U \ln f)\}g(X, Y) - (P U \ln f)g(\phi X, Y), \quad (34)$$

for every $X, Y \in \Gamma(M_T)$, $Z \in \Gamma(M_\perp)$ and $U \in \Gamma(M_\theta)$.

Proof. For every $X, Y \in \Gamma(M_T)$ and $Z \in \Gamma(M_\perp)$, we have

$$\begin{aligned} g(h(X, Y), \phi Z) &= g(\bar{\nabla}_X Y, \phi Z) \\ &= -g(\bar{\nabla}_X \phi Y, Z) + g((\bar{\nabla}_X \phi)Y, Z) \\ &= g(\phi Y, \bar{\nabla}_X Z) + \eta(Z)g(\phi X, Y). \end{aligned}$$

Using (15) in the above equation, we obtain

$$g(h(X, Y), \phi Z) = (Z \ln f)g(X, \phi Y) + \eta(Z)g(\phi X, Y),$$

from which the relation (32) follows.

Also, for every $X, Y \in \Gamma(M_T)$ and $U \in \Gamma(M_\theta)$, we have

$$\begin{aligned} g(h(X, Y), QU) &= g(\bar{\nabla}_X Y, \phi U) - g(\bar{\nabla}_X Y, PU) \\ &= -g(\bar{\nabla}_X \phi Y, U) + g((\bar{\nabla}_X \phi)Y, U) + g(\bar{\nabla}_X PU, Y). \end{aligned}$$

Using (5) and (15) in the above equation, we obtain

$$g(h(X, Y), QU) = (U \ln f)g(X, \phi Y) + \eta(U)g(\phi X, Y) + (PU \ln f)g(X, Y),$$

from which the relation (33) follows. Also, replacing U by PU in (33) and using (12), we get (34). \square

Now, replacing X by ϕX and Y by ϕY in (32), we obtain the following:

$$g(h(\phi X, Y), \phi Z) = \{(Z \ln f) - \eta(Z)\}g(X, Y), \quad (35)$$

$$g(h(X, \phi Y), \phi Z) = \{\eta(Z) - (Z \ln f)\}g(X, Y) \quad (36)$$

and

$$g(h(\phi X, \phi Y), \phi Z) = \{(Z \ln f) - \eta(Z)\}g(X, \phi Y). \quad (37)$$

Also, replacing X by ϕX and Y by ϕY in (33), we get the following:

$$g(h(\phi X, Y), QU) = \{\eta(U) - (U \ln f)\}g(X, Y) + (PU \ln f)g(\phi X, Y). \quad (38)$$

$$g(h(X, \phi Y), QU) = -\{\eta(U) - (U \ln f)\}g(X, Y) - (PU \ln f)g(\phi X, Y) \quad (39)$$

and

$$g(h(\phi X, \phi Y), QU) = \{\eta(U) - (U \ln f)\}g(\phi X, Y) + (PU \ln f)g(X, Y). \quad (40)$$

Corollary 4.5. Let $M = M_2 \times_f M_T$ be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that ξ is tangent to M_2 and $M_2 = M_\perp \times M_\theta$, then we have

- (i) $g(h(\phi X, Y), \phi Z) = -g(h(X, \phi Y), \phi Z)$,
- (ii) $g(h(\phi X, \phi Y), \phi Z) = g(h(X, Y), \phi Z)$,
- (iii) $g(h(\phi X, Y), QU) = -g(h(X, \phi Y), QU)$,
- and (iv) $g(h(\phi X, \phi Y), QU) = g(h(X, Y), QU)$.

Proof. The relation (i) follows from (35) and (36).

The relation (ii) follows from (32) and (37).

The relation (iii) follows from (38) and (39).

The relation (iv) follows from (33) and (40). \square

5. Characterization of Skew CR-warped products of the form $M_2 \times_f M_T$

Now, we obtain a characterization for a proper skew CR-warped product submanifold of order 1 of the form $M = M_2 \times_f M_T$ such that $M_2 = M_\perp \times M_\theta$ of a Kenmotsu manifold \bar{M} .

Theorem 5.1. *Let M be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that ξ is orthogonal to the invariant distribution \mathcal{D}^T , then M is locally a warped product skew CR-submanifold if and only if*

$$A_{\phi Z}X = \{\eta(Z) - (Z\mu)\}\phi X, \tag{41}$$

$$A_{QU}X = \{\eta(U) - (U\mu)\}\phi X + (PU\mu)X, \tag{42}$$

and

$$(\xi\mu) = 1 \tag{43}$$

for every $X \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$, $U \in \Gamma(\mathcal{D}^\theta)$ and for some smooth function μ on M satisfying $Y(\mu) = 0$, for any $Y \in \Gamma(\mathcal{D}^T)$.

Proof. Let $M = M_2 \times_f M_T$ be a proper warped product skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that $M_2 = M_\perp \times M_\theta$. We denote the tangent space of M_T , M_\perp and M_θ by \mathcal{D}^T , \mathcal{D}^\perp and \mathcal{D}^θ , respectively. Then from (26) and from (27), we have

$$A_{\phi Z}X \perp \mathcal{D}^\perp \tag{44}$$

and

$$A_{\phi Z}X \perp \mathcal{D}^\theta \tag{45}$$

for every $X \in \Gamma(\mathcal{D}^T)$ and $Z \in \Gamma(\mathcal{D}^\perp)$ respectively. Also since $h(B, \xi) = 0$, for every $B \in \Gamma(TM)$, we have

$$g(A_{\phi Z}X, \xi) = g(h(X, \xi), \phi Z) = 0. \tag{46}$$

From (44), (45) and (46), we can say that

$$A_{\phi Z}X \in \Gamma(\mathcal{D}^T). \tag{47}$$

From (32) and (47), we get (41). Also from (27), we have

$$A_{QU}X \perp \mathcal{D}^\perp, \tag{48}$$

for every $X \in \Gamma(\mathcal{D}^T)$ and $U \in \Gamma(\mathcal{D}^\theta)$, and from (28), we have

$$A_{QU}X \perp \mathcal{D}^\theta \tag{49}$$

for every $X \in \Gamma(\mathcal{D}^\perp)$ and $U \in \Gamma(\mathcal{D}^\theta)$.

From (46), (48) and (49), we can say that

$$A_{QU}X \in \Gamma(\mathcal{D}^T), \tag{50}$$

for every $X \in \Gamma(\mathcal{D}^T)$ and $U \in \Gamma(\mathcal{D}^\theta)$. The relation (42) follows from (33) and (50) and also (43) follows from (25).

Conversely, let M be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that (41)-(43) holds. Then from (20), (22) and (41), we have

$$g(\nabla_Z W, X) = 0 \tag{51}$$

and

$$g(\nabla_U Z, X) = 0 \tag{52}$$

for every $X \in \Gamma(\mathcal{D}^T)$ and $Z, W \in \Gamma(\mathcal{D}^\perp)$. Also, from (21), (23) and (42), we have

$$g(\nabla_Z U, X) = 0 \tag{53}$$

and

$$g(\nabla_U V, X) = 0 \tag{54}$$

for every $X \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$ and $U, V \in \Gamma(\mathcal{D}^\theta)$. Hence from (51)-(54), we can conclude that

$$g(\nabla_E F, X) = 0$$

for every $E, F \in \Gamma(\mathcal{D}^\perp \oplus \mathcal{D}^\theta + \{\xi\})$ and $X \in \Gamma(\mathcal{D}^T)$.

Therefore, the leaves of $\mathcal{D}^\perp \oplus \mathcal{D}^\theta + \{\xi\}$ are totally geodesic in M .

Now, from (18) and (41), we have

$$g([X, Y], Z) = 0 \tag{55}$$

for every $X, Y \in \Gamma(\mathcal{D}^T)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, Also from (19) and (42), we have

$$g([X, Y], U) = 0 \tag{56}$$

for every $X, Y \in \Gamma(\mathcal{D}^T)$ and $U \in \Gamma(\mathcal{D}^\theta)$.

Since $h(A, \xi) = 0$ for every $A \in \Gamma(TM)$, we have from (55) and (56) that

$$g([X, Y], E) = 0$$

for every $X, Y \in \Gamma(\mathcal{D}^T)$ and $E \in \Gamma(\mathcal{D}^\perp \oplus \mathcal{D}^\theta + \{\xi\})$. Consequently the distribution \mathcal{D}^T is integrable.

Next, we consider the integrable manifold M_T of \mathcal{D}^T and let h^T be the second fundamental form of M_T in M . Then for any $X, Y \in \Gamma(\mathcal{D}^T)$, we have from (16) that

$$\begin{aligned} g(h^T(X, Y), Z) &= g(\nabla_X Y, Z) \\ &= g(A_{\phi Z} X, \phi Y) - \eta(Z)g(X, Y). \end{aligned} \tag{57}$$

By virtue of (41), (57) yields

$$g(h^T(X, Y), Z) = -(Z\mu)g(X, Y). \tag{58}$$

Similarly for any $X, Y \in \Gamma(\mathcal{D}^T)$ and $U \in \Gamma(\mathcal{D}^\theta)$, we have from (17) that

$$\begin{aligned} g(h^T(X, Y), U) &= g(\nabla_X Y, U) \\ &= \csc^2 \theta [g(A_{QU} X, \phi Y) - g(A_{QU} X, Y)] - \eta(U)g(X, Y). \end{aligned} \tag{59}$$

In view of (42), (59) reduces to

$$\begin{aligned} g(h^T(X, Y), U) &= \csc^2 \theta [\{\eta(U) - (U\mu)\}g(\phi X, \phi Y) + (PU\mu)g(X, \phi Y) \\ &\quad - \cos^2 \theta \{\eta(U) - (U\mu)\}g(X, Y) + (PU\mu)g(\phi X, Y)] - \eta(U)g(X, Y) \\ &= -(U\mu)g(X, Y). \end{aligned} \tag{60}$$

Also for any $X, Y \in \Gamma(\mathcal{D}^T)$, we have

$$g(h^T(X, Y), \xi) = g(\nabla_X Y, \xi) = -g(Y, \bar{\nabla}_X \xi) = -g(X, Y).$$

Using (43) in the above equation we obtain

$$g(h^T(X, Y), \xi) = -(\xi\mu)g(X, Y). \tag{61}$$

From (58), (60) and (61), we conclude that

$$g(h^T(X, Y), E) = -g(\nabla\mu, E)g(X, Y)$$

for every $X, Y \in \Gamma(\mathcal{D}^T)$ and $E \in \Gamma(\mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \{\xi\})$. Consequently, M_T is totally umbilical in \bar{M} with mean curvature vector $H^T = -\nabla\mu$.

Finally, we show that H^T is parallel with respect to the normal connection D^N of M_T in M . We take $E \in \Gamma(\mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \{\xi\})$ and $X \in \Gamma(\mathcal{D}^T)$, then we have

$$g(D_X^N \nabla\mu, E) = g(\nabla_X \nabla^\perp \mu, Z) + g(\nabla_X \nabla^\theta \mu, U) + g(\nabla_X \nabla^\xi \mu, \xi),$$

where $\nabla^\perp \mu$, $\nabla^\theta \mu$ and $\nabla^\xi \mu$ are the gradient components of μ on M along \mathcal{D}^\perp , \mathcal{D}^θ and $\{\xi\}$ respectively. Then by the property of Riemannian metric, the above equation reduces to

$$\begin{aligned} g(D_X^N \nabla\mu, E) &= Xg(\nabla^\perp \mu, Z) - g(\nabla^\perp \mu, \nabla_X Z) + Xg(\nabla^\theta \mu, U) \\ &\quad - g(\nabla^\theta \mu, \nabla_X U) + Xg(\nabla^\xi \mu, \xi) - g(\nabla^\xi \mu, \nabla_X \xi) \\ &= X(Z\mu) - g(\nabla^\perp \mu, [X, Z]) - g(\nabla^\perp \mu, \nabla_Z X) \\ &\quad + X(U\mu) - g(\nabla^\theta \mu, [X, U]) - g(\nabla^\theta \mu, \nabla_U X) \\ &\quad + X(\xi\mu) - g(\nabla^\xi \mu, [X, \xi]) - g(\nabla^\xi \mu, \nabla_\xi X) \\ &= Z(X\mu) + g(\nabla_Z \nabla^\perp \mu, X) + U(X\mu) + g(\nabla_U \nabla^\theta \mu, X) \\ &\quad + \xi(X\mu) + g(\nabla_\xi \nabla^\xi \mu, X) \\ &= 0, \end{aligned}$$

since $(X\mu) = 0$, for any $X \in \Gamma(\mathcal{D}^T)$ and $\nabla_Z \nabla^\perp \mu + \nabla_U \nabla^\theta \mu + \nabla_\xi \nabla^\xi \mu = \nabla_E \nabla\mu$ is orthogonal to \mathcal{D}^T for any $E \in \Gamma(\mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \{\xi\})$ as $\nabla\mu$ is the gradient along M_2 and M_2 is totally geodesic in \bar{M} . Therefore, the mean curvature vector H^T of M_T is parallel. Thus, M_T is an extrinsic sphere in M . Hence by Theorem 2.3, M is locally a warped product submanifold. Thus the proof is complete. \square

Corollary 5.2. *Let M be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that ξ is tangent to the anti-invariant distribution \mathcal{D}^\perp , then M is locally a warped product submanifold if and only if*

- (i) $A_{\phi Z} X = \{(Z\mu) - \eta(Z)\}\phi X$,
- (ii) $A_{QU} X = (PU\mu)X - (U\mu)\phi X$
- (iii) $(\xi\mu) = 1$,

for every $X \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$, $U \in \Gamma(\mathcal{D}^\theta)$ and for some smooth function μ on M satisfying $Y(\mu) = 0$, for any $Y \in \Gamma(\mathcal{D}^T)$.

Corollary 5.3. *Let M be a proper Skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that ξ is tangent to the slant distribution \mathcal{D}^θ , then M is locally a warped product submanifold if and only if*

- (i) $A_{\phi Z} X = (Z\mu)\phi X$,
- (ii) $A_{QU} X = \{\eta(U) - (U\mu)\}\phi X + (PU\mu)X$
- (iii) $(\xi\mu) = 1$,

for every $X \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$, $U \in \Gamma(\mathcal{D}^\theta)$ and for some smooth function μ on M satisfying $Y(\mu) = 0$ for any $Y \in \Gamma(\mathcal{D}^T)$.

6. Generalized inequalities on warped product skew CR-submanifolds

In this section, we establish two inequalities on a warped product skew CR-submanifold $M = M_2 \times_f M_T$ of a Kenmotsu manifold \bar{M} such that $M_2 = M_\perp \times M_\theta$. We take $\dim M_T = 2p$, $\dim M_\perp = q$, $\dim M_\theta = 2s + 1$ and their corresponding tangent spaces are \mathcal{D}^T , \mathcal{D}^\perp and $\mathcal{D}^\theta \oplus \{\xi\}$ respectively.

Assume that $\{e_1, e_2, \dots, e_p, e_{p+1} = \phi e_1, \dots, e_{2p} = \phi e_p\}$, $\{e_{2p+1} = e_1^*, \dots, e_{2p+q} = e_q^*\}$ and $\{e_{2p+q+1} = \hat{e}_1, e_{2p+q+2} = \hat{e}_2, \dots, e_{2p+q+s} = \hat{e}_s, e_{2p+q+s+1} = \hat{e}_{s+1} = \sec \theta P \hat{e}_1, \dots, e_{2p+q+2s} = \hat{e}_{2s} = \sec \theta P \hat{e}_s, e_{2p+q+2s+1} = \hat{e}_{2s+1} = \xi\}$ are local orthonormal frames of \mathcal{D}^T, D^\perp and $D^\theta \oplus \{\xi\}$ respectively. Then the local orthonormal frames for $\phi D^\perp, QD^\theta$ and ν are $\{e_{n+1} = \bar{e}_1 = \phi e_1^*, \dots, e_{n+q} = \bar{e}_q = \phi e_q^*\}$, $\{e_{n+q+1} = \bar{e}_{q+1} = \csc \theta Q \hat{e}_1, \dots, e_{n+q+s} = \bar{e}_{q+s} = \csc \theta Q \hat{e}_s, e_{n+q+s+1} = \bar{e}_{q+s+1} = \csc \theta \sec \theta Q P \hat{e}_1, \dots, e_{n+q+2s} = \bar{e}_{q+2s} = \csc \theta \sec \theta Q P \hat{e}_s\}$ and $\{e_{n+q+2s+1}, \dots, e_{2m+1}\}$, respectively. Clearly $\dim \nu = (2m + 1 - n - q - 2s)$.

Now, we have the following inequalities:

Theorem 6.1. *Let $M = M_2 \times_f M_T$ be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that ξ is tangent to M_θ , where $M_2 = M_\perp \times M_\theta$, then the squared norm of the second fundamental form satisfies*

$$\|h\|^2 \geq 2p[\|\nabla^\perp \ln f\|^2 + (\csc^2 \theta + \cot^2 \theta)(\|\nabla^\theta \ln f\|^2 - 1)], \tag{62}$$

where $\nabla^\perp \ln f$ and $\nabla^\theta \ln f$ are the gradient of $\ln f$ along M_\perp and M_θ , respectively and for the case of equality, M_2 becomes totally geodesic and M_T becomes totally umbilical in \bar{M} .

Proof. From (8), we have

$$\|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^n g(h(e_i, e_j), e_r)^2.$$

Decomposing the above relation for our constructed frames, we get

$$\begin{aligned} \|h\|^2 &= \sum_{r=n+1}^{2m+1} \sum_{i,j=2p+1}^q g(h(e_i^*, e_j^*), e_r)^2 + 2 \sum_{r=n+1}^{2m+1} \sum_{i=2p+1}^q \sum_{j=1}^{2s+1} g(h(e_i^*, \hat{e}_j), e_r)^2 \\ &+ \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{2s+1} g(h(\hat{e}_i, \hat{e}_j), e_r)^2 + 2 \sum_{r=n+1}^{2m+1} \sum_{i=1}^q \sum_{j=1}^{2p} g(h(e_i^*, e_j), e_r)^2 \\ &+ 2 \sum_{r=n+1}^{2m+1} \sum_{i=1}^{2s+1} \sum_{j=1}^{2p} g(h(\hat{e}_i, e_j), e_r)^2 + \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{2p} g(h(e_i, e_j), e_r)^2. \end{aligned} \tag{63}$$

Now, again decomposing (63) along the normal subbundles $\phi D^\perp, QD^\theta$ and ν , we get

$$\begin{aligned} \|h\|^2 &= \sum_{r=n+1}^{n+q} \sum_{i,j=2p+1}^q g(h(e_i^*, e_j^*), e_r)^2 \\ &+ \sum_{r=n+q+1}^{n+2s} \sum_{i,j=2p+1}^q g(h(e_i^*, e_j^*), e_r)^2 + \sum_{r=n+2s+1}^{2m+1} \sum_{i,j=2p+1}^q g(h(e_i^*, e_j^*), e_r)^2 \\ &+ 2 \sum_{r=n+1}^{n+q} \sum_{i=2p+1}^q \sum_{j=1}^{2s+1} g(h(e_i^*, \hat{e}_j), e_r)^2 + 2 \sum_{r=n+q+1}^{n+2s} \sum_{i=2p+1}^q \sum_{j=1}^{2s+1} g(h(e_i^*, \hat{e}_j), e_r)^2 \\ &+ 2 \sum_{r=n+2s+1}^{2m+1} \sum_{i=2p+1}^q \sum_{j=1}^{2s+1} g(h(e_i^*, \hat{e}_j), e_r)^2 + \sum_{r=n+1}^{n+q} \sum_{i,j=1}^{2s+1} g(h(\hat{e}_i, \hat{e}_j), e_r)^2 \\ &+ \sum_{r=n+q+1}^{n+2s} \sum_{i,j=1}^{2s+1} g(h(\hat{e}_i, \hat{e}_j), e_r)^2 + \sum_{r=n+2s+1}^{2m+1} \sum_{i,j=1}^{2s+1} g(h(\hat{e}_i, \hat{e}_j), e_r)^2 \end{aligned} \tag{64}$$

$$\begin{aligned}
 &+2 \sum_{r=n+1}^{n+q} \sum_{i=1}^q \sum_{j=1}^{2p} g(h(e_i^*, e_j), e_r)^2 + 2 \sum_{r=n+q+1}^{n+2s} \sum_{i=1}^q \sum_{j=1}^{2p} g(h(e_i^*, e_j), e_r)^2 \\
 &+2 \sum_{r=n+2s+1}^{2m+1} \sum_{i=1}^q \sum_{j=1}^{2p} g(h(e_i^*, e_j), e_r)^2 + 2 \sum_{r=n+1}^{n+q} \sum_{i=1}^{2s+1} \sum_{j=1}^{2p} g(h(\hat{e}_i, e_j), e_r)^2 \\
 &+2 \sum_{r=n+q+1}^{n+2s} \sum_{i=1}^{2s+1} \sum_{j=1}^{2p} g(h(\hat{e}_i, e_j), e_r)^2 + 2 \sum_{r=n+2s+1}^{2m+1} \sum_{i=1}^{2s+1} \sum_{j=1}^{2p} g(h(\hat{e}_i, e_j), e_r)^2 \\
 &+ \sum_{r=n+1}^{n+q} \sum_{i,j=1}^{2p} g(h(e_i, e_j), e_r)^2 + \sum_{r=n+q+1}^{n+2s} \sum_{i,j=1}^{2p} g(h(e_i, e_j), e_r)^2 \\
 &+ \sum_{r=n+2s+1}^{2m+1} \sum_{i,j=1}^{2p} g(h(e_i, e_j), e_r)^2.
 \end{aligned}$$

Now, by Proposition 4.3, the tenth, eleventh, thirteenth and fourteenth terms of (64) are equal to zero. Also, we can not find any relation for a warped product in the form $g(h(E, F), v)$ for any $E, F \in \Gamma(TM)$. So, leaving the positive third, sixth, ninth, twelfth, fifteenth and eighteenth terms of (64) we get

$$\begin{aligned}
 \|h\|^2 &\geq \sum_{r=1}^q \sum_{i,j=2p+1}^q g(h(e_i^*, e_j^*), \phi e_r^*)^2 + \sum_{r=1}^{2s} \sum_{i,j=2p+1}^q g(h(e_i^*, e_j^*), \tilde{e}_r)^2 \\
 &+ 2 \sum_{r=1}^q \sum_{i=2p+1}^q \sum_{j=1}^{2s+1} g(h(e_i^*, \hat{e}_j), \phi e_r^*)^2 + 2 \sum_{r=1}^s \sum_{i=2p+1}^q \sum_{j=1}^{2s+1} g(h(e_i^*, \hat{e}_j), \tilde{e}_r)^2 \\
 &+ \sum_{r=1}^q \sum_{i,j=1}^{2s+1} g(h(\hat{e}_i, \hat{e}_j), \phi e_r^*)^2 + \sum_{r=1}^s \sum_{i,j=1}^{2s+1} g(h(\hat{e}_i, \hat{e}_j), \tilde{e}_r)^2 \\
 &+ \sum_{r=1}^q \sum_{i,j=1}^{2p} g(h(e_i, e_j), \phi e_r^*)^2 + \sum_{r=1}^{2s} \sum_{i,j=1}^{2p} g(h(e_i, e_j), \tilde{e}_r)^2.
 \end{aligned} \tag{65}$$

Also, we have no relation for a warped product of the forms $g(h(Z, W), \phi \mathcal{D}^\perp)$, $g(h(Z, W), Q\mathcal{D}^\theta)$, $g(h(Z, U), \phi \mathcal{D}^\perp)$, $g(h(Z, U), Q\mathcal{D}^\theta)$, $g(h(U, V), \phi Z)$ and $g(h(U, V), Q\mathcal{D}^\theta)$ for any $Z, W \in \Gamma(\mathcal{D}^\perp)$, $U \in \Gamma(\mathcal{D}^\theta \oplus \{\xi\})$. So, we leave these terms from (65) and obtain

$$\|h\|^2 \geq \sum_{r=1}^q \sum_{i,j=1}^{2p} g(h(e_i, e_j), \phi e_r^*)^2 + \sum_{r=1}^{2s} \sum_{i,j=1}^{2p} g(h(e_i, e_j), \tilde{e}_r)^2. \tag{66}$$

Now,

$$\begin{aligned}
 \sum_{r=1}^q \sum_{i,j=1}^{2p} g(h(e_i, e_j), \phi e_r^*)^2 &= \sum_{r=1}^q \sum_{i=1}^p g(h(e_i, \phi e_j), \phi e_r^*)^2 + \sum_{r=1}^q \sum_{i=1}^p g(h(\phi e_i, e_j), \phi e_r^*)^2 \\
 &+ \sum_{r=1}^q \sum_{i=1}^p g(h(\phi e_i, \phi e_j), \phi e_r^*)^2.
 \end{aligned}$$

Using Corollary 4.5 ((i) and (ii)), the above relation reduces to

$$\sum_{r=1}^q \sum_{i,j=1}^{2p} g(h(e_i, e_j), \phi e_r^*)^2 = 2 \sum_{r=1}^q \sum_{i=1}^p g(h(\phi e_i, e_j), \phi e_r^*)^2 + 2 \sum_{r=1}^q \sum_{i=1}^p g(h(e_i, e_j), \phi e_r^*)^2. \tag{67}$$

By virtue of (32), (67) yields

$$\sum_{r=1}^q \sum_{i,j=1}^{2p} g(h(e_i, e_j), \phi e_r^*)^2 = 2 \sum_{r=1}^q \sum_{i,j=1}^p \{\eta(e_r^*) - e_r^* \ln f\}^2 g(e_i, e_j)^2 + 2 \sum_{r=1}^q \sum_{i,j=1}^p \{\eta(e_r^*) - e_r^* \ln f\}^2 g(e_i, \phi e_j)^2. \quad (68)$$

Now, since $\eta(e_r^*) = 0$ for every $r = 1, 2, \dots, q$ and $g(e_i, \phi e_j) = 0$ for every $i, j = 1, 2, \dots, p$ so (68) turns into

$$\sum_{r=1}^q \sum_{i,j=1}^{2p} g(h(e_i, e_j), \phi e_r^*)^2 = 2p \sum_{r=1}^q (e_r^* \ln f)^2 = 2p \|\nabla^+ \ln f\|^2. \quad (69)$$

On the other hand,

$$\begin{aligned} \sum_{r=1}^{2s} \sum_{i,j=1}^{2p} g(h(e_i, e_j), \tilde{e}_r)^2 &= \csc^2 \theta \sum_{r=1}^s \sum_{i,j=1}^{2p} g(h(e_i, e_j), Q\hat{e}_r)^2 + \sec^2 \theta \csc^2 \theta \sum_{r=1}^s \sum_{i,j=1}^{2p} g(h(e_i, e_j), QP\hat{e}_r)^2 \\ &= \csc^2 \theta \sum_{r=1}^s \sum_{i,j=1}^p g(h(e_i, e_j), Q\hat{e}_r)^2 + \csc^2 \theta \sum_{r=1}^s \sum_{i,j=1}^p g(h(e_i, \phi e_j), Q\hat{e}_r)^2 \\ &\quad + \csc^2 \theta \sum_{r=1}^s \sum_{i,j=1}^p g(h(\phi e_i, e_j), Q\hat{e}_r)^2 + \csc^2 \theta \sum_{r=1}^s \sum_{i,j=1}^p g(h(\phi e_i, \phi e_j), Q\hat{e}_r)^2 \\ &\quad + \sec^2 \theta \csc^2 \theta \sum_{r=1}^s \sum_{i,j=1}^p g(h(e_i, e_j), QP\hat{e}_r)^2 + \sec^2 \theta \csc^2 \theta \sum_{r=1}^s \sum_{i,j=1}^p g(h(e_i, \phi e_j), QP\hat{e}_r)^2 \\ &\quad + \sec^2 \theta \csc^2 \theta \sum_{r=1}^s \sum_{i,j=1}^p g(h(\phi e_i, e_j), QP\hat{e}_r)^2 + \sec^2 \theta \csc^2 \theta \sum_{r=1}^s \sum_{i,j=1}^p g(h(\phi e_i, \phi e_j), QP\hat{e}_r)^2. \end{aligned}$$

Using Corollary 4.5, ((iii) and (iv)), (33), (34) and the fact that $g(e_i, \phi e_j) = 0$ for every $i, j = 1, 2, \dots, p$ in the above relation, we obtain

$$\begin{aligned} \sum_{r=1}^{2s} \sum_{i,j=1}^{2p} g(h(e_i, e_j), \tilde{e}_r)^2 &= 2p \csc^2 \theta \sum_{r=1}^s (P\hat{e}_r \ln f)^2 + 2p \csc^2 \theta \sum_{r=1}^s \{\eta(\hat{e}_r) - (\hat{e}_r \ln f)\}^2 \\ &\quad + 2p \sec^2 \theta \csc^2 \theta \cos^4 \theta \sum_{r=1}^s \{\eta(\hat{e}_r) - (\hat{e}_r \ln f)\}^2 + 2p \sec^2 \theta \csc^2 \theta \sum_{r=1}^s (P\hat{e}_r \ln f)^2 \end{aligned}$$

Since $\eta(\hat{e}_r) = 0$ for every $r = 1, 2, \dots, s$, the above equation reduces to

$$\begin{aligned} \sum_{r=1}^{2s} \sum_{i,j=1}^{2p} g(h(e_i, e_j), \tilde{e}_r)^2 &= 2p \cot^2 \theta \sum_{r=1}^s (\sec \theta P\hat{e}_r \ln f)^2 + 2p \csc^2 \theta \sum_{r=1}^s (\hat{e}_r \ln f)^2 \\ &\quad + 2p \cot^2 \theta \sum_{r=1}^s (\hat{e}_r \ln f)^2 + 2p \csc^2 \theta \sum_{r=1}^s (\sec \theta P\hat{e}_r \ln f)^2 \\ &= 2p \cot^2 \theta \sum_{r=1}^{2s} (\hat{e}_r \ln f)^2 + 2p \csc^2 \theta \sum_{r=1}^{2s} (\hat{e}_r \ln f)^2 \\ &= 2p(\csc^2 \theta + \cot^2 \theta) \left\{ \sum_{r=1}^{2s+1} (\hat{e}_r \ln f)^2 - (\xi \ln f)^2 \right\}. \end{aligned}$$

Using (10) and (25) in the above equation, we get

$$\sum_{r=1}^{2s} \sum_{i,j=1}^{2p} g(h(e_i, e_j), \tilde{e}_r)^2 = 2p(\csc^2 \theta + \cot^2 \theta) \{ \|\nabla^\theta \ln f\|^2 - 1 \}. \tag{70}$$

Again using (69) and (70) in (66), we get the inequality (62).

If the inequality of (62) holds, then by leaving third term of (64), we get $g(h(\mathcal{D}^\perp, \mathcal{D}^\perp), \nu) = 0$, which implies that

$$h(\mathcal{D}^\perp, \mathcal{D}^\perp) \perp \nu. \tag{71}$$

Also, by leaving the first and second term of (65), we get $h(\mathcal{D}^\perp, \mathcal{D}^\perp) \perp \phi \mathcal{D}^\perp$ and $h(\mathcal{D}^\perp, \mathcal{D}^\perp) \perp Q\mathcal{D}^\theta$ respectively. Therefore

$$h(\mathcal{D}^\perp, \mathcal{D}^\perp) \subseteq \nu. \tag{72}$$

From (71) and (72), we obtain

$$h(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0. \tag{73}$$

Similarly by leaving sixth term of (64), we get

$$h(\mathcal{D}^\perp, \mathcal{D}^\theta) \perp \nu. \tag{74}$$

Also, leaving the third and fourth term of (65), we get $h(\mathcal{D}^\perp, \mathcal{D}^\theta) \perp \phi \mathcal{D}^\perp$ and $h(\mathcal{D}^\perp, \mathcal{D}^\theta) \perp Q\mathcal{D}^\theta$ respectively. Therefore,

$$h(\mathcal{D}^\perp, \mathcal{D}^\theta) \subseteq \nu. \tag{75}$$

From (74) and (75), we obtain

$$h(\mathcal{D}^\perp, \mathcal{D}^\theta) = 0. \tag{76}$$

Again, by leaving ninth term of (64), we get

$$h(\mathcal{D}^\theta, \mathcal{D}^\theta) \perp \nu. \tag{77}$$

Also, leaving fifth and sixth term of (65), we get $h(\mathcal{D}^\theta, \mathcal{D}^\theta) \perp \phi \mathcal{D}^\perp$ and $h(\mathcal{D}^\theta, \mathcal{D}^\theta) \perp Q\mathcal{D}^\theta$ respectively. Therefore,

$$h(\mathcal{D}^\theta, \mathcal{D}^\theta) \subseteq \nu. \tag{78}$$

From (77) and (78), we obtain

$$h(\mathcal{D}^\theta, \mathcal{D}^\theta) = 0. \tag{79}$$

Next by leaving the twelfth term of (64), we get

$$h(\mathcal{D}^\perp, \mathcal{D}^T) \perp \nu. \tag{80}$$

From (80) and Proposition 4.3, we get

$$h(\mathcal{D}^\perp, \mathcal{D}^T) = 0. \tag{81}$$

Also, leaving fifteenth term of (64), we get

$$h(\mathcal{D}^\theta, \mathcal{D}^T) \perp \nu. \tag{82}$$

From (82) and Proposition 4.1, we get

$$h(\mathcal{D}^\theta, \mathcal{D}^T) = 0. \quad (83)$$

Thus from (73), (76), (79), (81), (83) and the fact that M_2 is totally geodesic in M ([6], [10]), we conclude that M_2 is totally geodesic in \bar{M} . Next by leaving the eighteenth term of (64), we get

$$h(\mathcal{D}^T, \mathcal{D}^T) \perp v. \quad (84)$$

Then from (58), (60), (84) and the fact that M_T is totally umbilical in M ([6], [10]), we conclude that M_T is totally umbilical in \bar{M} . This completes the proof of the theorem. \square

Theorem 6.2. Let $M = M_2 \times_f M_T$ be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold \bar{M} such that ξ is tangential to M_\perp , where $M_2 = M_\perp \times M_\theta$, then the squared norm of the second fundamental form satisfies

$$\|h\|^2 \geq 2p[\|\nabla^\perp \ln f\|^2 - 1 + (\csc^2 \theta + \cot^2 \theta)\|\nabla^\theta \ln f\|^2]. \quad (85)$$

If the equality of (85) holds, then M_2 is totally geodesic and M_T is totally umbilical in \bar{M} .

Proof. For this theorem, we take $\dim M_\perp = q + 1$ and $\dim M_\theta = 2s$. So, orthonormal frames of $\mathcal{D}^\perp \oplus \{\xi\}$ and \mathcal{D}^θ will be $\{e_{2p+1} = e_1^*, \dots, e_{2p+q} = e_q^*, e_{2p+q+1} = \xi\}$ and $\{e_{2p+q+2} = \hat{e}_1, \dots, e_{2p+q+s+1} = \hat{e}_s, e_{2p+q+s+2} = \hat{e}_{s+1} = \sec \theta P \hat{e}_1, \dots, e_{2p+q+2s+1} = \hat{e}_{2s} = \sec \theta P \hat{e}_s\}$, respectively. Then the proof of the theorem is similar as Theorem 6.1. \square

Remark: If we take $\dim M_\theta = 0$ in a warped product skew CR-submanifold $M = M_2 \times_f M_T$ of a Kenmotsu manifold \bar{M} such that $M_2 = M_\perp \times M_\theta$, then it turns into CR-warped product $M = M_\perp \times_f M_T$ which was studied in [40]. Therefore, Theorem 5.1 and Theorem 6.2 are the generalizations of results of [40] as follows:

Corollary 6.3. (Theorem 3.1 of [40]) A proper contact CR-submanifold of a Kenmotsu manifold \bar{M} is locally a contact CR-warped product of the form $M_\perp \times_f M_T$ if and only if

$$A_{\phi Z} X = \{\eta(Z) - (Z\mu)\} \phi X,$$

for every $X \in \Gamma(\mathcal{D}^T)$ and $Z \in \Gamma(\mathcal{D}^\perp \oplus \{\xi\})$, for some function μ on M satisfying $(Y\mu) = 0$ for any $Y \in \Gamma(\mathcal{D}^T)$.

Corollary 6.4. (Theorem 3.2 of [40]) Let \bar{M} be a $(2m + 1)$ -dimensional Kenmotsu manifold and $M = M_\perp \times_f M_T$ an n -dimensional contact CR-warped product submanifold, such that M_\perp is a $(q + 1)$ -dimensional anti-invariant submanifold tangent to ξ and M_T is a $2p$ -dimensional invariant submanifold of \bar{M} , then the squared norm of the second fundamental form of M satisfies

$$\|h\|^2 \geq 2p[\|\nabla^\perp \ln f\|^2 - 1] \quad (86)$$

where $\nabla^\perp \ln f$ is the gradient of $\ln f$. If the equality of (86) holds, then M_\perp is totally geodesic and M_T is totally umbilical in \bar{M} .

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