



# On the Rapidly Convergence in Capacity of the Sequence of Holomorphic Functions

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**Abstract.** In this paper, we are interested in finding sufficient conditions on a Borel set  $X$  lying either inside a bounded domain  $D \subset \mathbb{C}^n$  or in the boundary  $\partial D$  so that if  $\{r_m\}_{m \geq 1}$  is a sequence of rational functions and  $\{f_m\}_{m \geq 1}$  is a sequence of bounded holomorphic functions on  $D$  with  $\{f_m - r_m\}_{m \geq 1}$  is convergent fast enough to 0 in some sense on  $X$  then the convergence occurs on the whole domain  $D$ . The main result is strongly inspired by Theorem 3.6 in [3] whether the  $\{f_m\}$  is a constant sequence.

## 1. Introduction

Let  $D$  be a bounded domain in  $\mathbb{C}^n$ ,  $\{f_m\}_{m \geq 1}$  be a sequence of holomorphic functions defined on  $D$  and  $\{r_m\}_{m \geq 1}$  be a sequence of rational functions on  $\mathbb{C}^n$ . Throughout this paper, we always assume that  $1 \leq \deg r_m \leq m$  i.e., the numerator and the denominator of  $r_m$  are non-constant polynomials of degree at most  $m$ . Let  $X$  be a Borel subset which lies either in  $D$  or  $\partial D$ . In this paper, we will refine the techniques in [3] concerning with sufficient conditions that guarantees convergence in some sense of  $\{f_m - r_m\}_{m \geq 1}$  to 0 on  $D$  as soon as the convergence occurs pointwise on  $X$ . By a classical theorem of Vitali, under the additional conditions that  $\{r_m\}_{m \geq 1}$  and  $\{f_m\}_{m \geq 1}$  are uniformly bounded on compact sets of  $D$  and  $X$  is not contained in any proper analytic subset of  $D$ , the sequence  $\{f_m - r_m\}_{m \geq 1}$  converges uniformly to 0 on compact sets of  $D$  provided that the convergence holds pointwise only on  $X$ . Motivated by the problem of finding local conditions for single-valuedness of holomorphic continuation, Gonchar proved in Theorem 2 of [4] the following remarkable result which deals with the special case where  $f_m = f$  for every  $m \geq 1$ .

**Theorem 1.1.** Let  $\{r_m\}_{m \geq 1}$  be a sequence of rational functions converges rapidly in measure on an open set  $X$  to a holomorphic function  $f$  defined on a domain  $D(X \subset D)$  i.e., for every  $\varepsilon > 0$

$$\lim_{m \rightarrow \infty} \lambda_{2n}(z \in X : |r_m(z) - f(z)|^{1/m} > \varepsilon) = 0.$$

Here  $\lambda_{2n}$  is the Lebesgue measure in  $\mathbb{R}^{2n}$ . Then  $\{r_m\}_{m \geq 1}$  must converge rapidly in measure to  $f$  on the whole domain  $D$ .

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We should also mention a related theorem of Siciak in [8], where it is proved that fast polynomial approximation yields extension of the holomorphic function to a larger domain dependent on the approximation speed. Much later, by using techniques of pluripotential theory, Bloom was able to prove an analogous result in which rapidly convergence in measure is replaced by rapidly convergence in capacity and the set  $X$  is only required to be Borel and non-pluripolar (see Theorem 2.1 in [1]). More precisely, we have

**Theorem 1.2.** *Let  $f$  be a holomorphic function defined on a domain  $D \subset \mathbb{C}^n$ . Let  $\{r_m\}_{m \geq 1}$  be a sequence of rational functions with degree  $r_m \leq m$  converging rapidly in capacity to  $f$  on a non-pluripolar Borel subset  $X$  of  $D$ . Then  $\{r_m\}_{m \geq 1}$  converges to  $f$  rapidly in capacity on  $D$ .*

Using a standard result which relates convergence in capacity and pointwise convergence (cf. Lemma 2.2), it is not hard to check that Theorem 1.1 follows from Theorem 1.2 (see Theorem 2.2 in [1]). Our first result is an analogue of Theorem 1.2, in which rapid convergence occurs in some sense on a subset lying in the boundary of the domain and the function  $f$  is replaced by a sequence of bounded holomorphic functions. Namely, we have

**Theorem 1.3.** *Let  $\{r_m\}_{m \geq 1}$  be a sequence of rational functions on  $\mathbb{C}^n$ . Let  $\{f_m\}_{m \geq 1}$  be a sequence of bounded holomorphic functions defined on a bounded domain  $D \subset \mathbb{C}^n$  such that*

$$(a) \sup_{m \geq 1} \frac{1}{m} \log \|f_m\|_D < \infty.$$

*Assume that there exists a Borel non-pluripolar set  $X \subset \partial D$  satisfying the following properties:*

*(b) The pluriharmonic measure of  $X$  relative to  $D$*

$$\omega(z, X, D) := \sup\{\varphi(z) : \varphi \in \text{PSH}(D), \varphi < 0, \limsup_{\xi \rightarrow x, \xi \in D} \varphi(\xi) \leq -1 \forall x \in X\}$$

*is negative on  $D$ .*

*(c) For every  $z \in X$  and every sequence  $\{z_m\}_{m \geq 1} \subset D$  with  $z_m \rightarrow z$  we have*

$$\lim_{m \rightarrow \infty} |f_m(z_m) - r_m(z_m)|^{1/m} = 0.$$

*Then the following assertions hold:*

*(i) The sequence  $|f_m - r_m|^{1/m}$  is convergent in capacity to 0 on  $D$ .*

*(ii) There exists a pluripolar subset  $E$  of  $\mathbb{C}^n$  with the following property: For every  $z_0 \in D \setminus E$  and every affine complex subspace  $L$  of  $\mathbb{C}^n$  passing through  $z_0$ , there exists a subsequence  $\{r_{m_j}\}_{j \geq 1}$  (dependent only on  $z_0$ ) such that  $|f_{m_j} - r_{m_j}|^{1/m_j}|_{D_{z_0}}$  is convergent in capacity (with respect to  $D_{z_0}$ ) to 0, where  $D_{z_0}$  is the connected component of  $D \cap L$  that contains  $z_0$ .*

We observe that the non-pluripolarity of  $X$  is not sufficient to guarantee the assumption (a). Indeed, let  $D \subset \mathbb{C}$  be the unit disk and  $X$  be the circular Cantor middle-third set. Then  $X$  is non-polar, but of harmonic measure zero. See Exercise 5.3.7 in [7]. On the other hand, if  $D$  is the unit ball in  $\mathbb{C}^n$  and  $X$  is an open subset of  $\partial D$  then by the maximum principle we can see that  $X$  satisfies the assumption (a).

The proof of Theorem 1.3 runs roughly as follows. First by adding suitable plurisubharmonic function to  $\frac{1}{m} \log |f_m - r_m|$  and using Bernstein-Markov's inequality together with the assumption (a), we obtain a sequence  $\{u_m\}_{m \geq 1}$  of plurisubharmonic functions on  $D$  which is uniformly bounded from above. Next, using the assumptions (b), (c) and the compactness in topology  $L_{\text{loc}}^1$  of the cone of plurisubharmonic functions, we infer that the  $\{u_m\}_{m \geq 1}$  converges uniformly to  $-\infty$  on compact sets of  $D$ . The last step consists in invoking an inequality of Chern-Levine-Nirenberg type to derive the desired convergence in capacity. It should be remarked that the original proof of Theorem 1.2 relies heavily on a comparison theorem of Alexander and Taylor on comparing the relative capacity of a Borel subset of a domain and its global (or Siciak) capacity. Thus this method does not seem tractable in the case where the set under consideration sits in the boundary of the domain. Our note ends up with a version of Theorem 1.3 in which the set  $X$  is only supposed to be non-pluripolar but lies inside the domain  $D$  and the assumption (a) of Theorem 3.3 is required to hold on compact subsets of  $D$ . We will leave the details of the proof to the interested reader.

## 2. Preliminaries

For the reader convenience, first we briefly recall standard elements of pluripotential theory that will be needed later on. Let  $D$  be a domain in  $\mathbb{C}^n$ . An upper semicontinuous function  $u : D \rightarrow [-\infty, \infty)$  is said to be plurisubharmonic if for every complex line  $l$ , the restriction of  $u$  on each connected component of  $D \cap l$  is either subharmonic or identically  $-\infty$ . The cone of plurisubharmonic function on  $D$  is denoted by  $PSH(D)$ . A subset  $E$  of  $D$  is said to be pluripolar if for every  $z_0 \in E$  there exists an open connected neighbourhood  $U$  of  $z_0$  and  $u \in PSH(U), u \not\equiv -\infty$  such that  $u \equiv -\infty$  on  $E \cap U$ . According to a classical theorem of Josefson, if  $E$  is pluripolar then there exists a plurisubharmonic function  $u$  which defined globally on  $D$  such that  $u \equiv -\infty$  on  $E$ . Clearly a proper complex subvariety of  $D$  is pluripolar. On the other hand, it is not hard to show that any subset of  $\mathbb{R}^n$  with positive Lebesgue measure is not pluripolar (in  $\mathbb{C}^n$ ).

For a Borel subset  $E$  in a bounded domain  $D \subset \mathbb{C}^n$ , following Bedford and Taylor (see [6] p.120) we let  $\text{cap}(E, D)$  be the relative capacity of a Borel subset  $E$  in  $D$  which is defined as

$$\text{cap}(E, D) = \sup \left\{ \int_E (dd^c u)^n : u \in PSH(D), -1 < u < 0 \right\}.$$

It is well known that relative capacity enjoys some important properties such as sub-additivity and monotonicity under increasing sequences. Moreover, a deep result in Bedford-Taylor’s theory states that pluripolar subsets of  $D$  are exactly those with vanishing relative capacity. We will frequently appeal to the following estimate (Bernstein-Walsh’s inequality) which bounds the sup norm of a polynomial on an arbitrary compact set in terms of its sup norm on a given non-pluripolar set: *For every compact  $K, X \subset \mathbb{C}^n$  with  $X$  is non-pluripolar and every polynomial  $p$  we have*

$$\frac{1}{\deg p} \log \|p\|_K \leq \frac{1}{\deg p} \log \|p\|_X + C_{K,X} \tag{1}$$

Here  $C_{K,X} > 0$  is a constant dependent only on  $K, X$ . We recall below several types of convergence of measurable functions.

**Definition 2.1.** ([3]) Let  $\{f_m\}_{m \geq 1}, f$  be Borel, complex valued measurable functions defined on a domain  $D \subset \mathbb{C}^n$  and  $X$  be a Borel subset of  $D$ . We say that the sequence  $\{f_m\}_{m \geq 1}$  (i) is rapidly pointwise convergent to  $f$  on  $X$  if

$$\lim_{m \rightarrow \infty} |f_m(x) - f(x)|^{1/m} = 0 \quad \forall x \in X;$$

(ii) is rapidly convergent in capacity to  $f$  on  $X$  if for every  $\varepsilon > 0$  we have

$$\lim_{m \rightarrow \infty} \text{cap}(X_{m,\varepsilon}, D) = 0,$$

where  $X_{m,\varepsilon} := \{x \in X : |f_m(x) - f(x)|^{1/m} > \varepsilon\}$ ;

(iii) is called rapidly convergent in capacity to  $f$  on  $D$  if the property (ii) holds true for every Borel subset  $X$  of  $D$ ;

(iv) is rapidly uniformly convergent to  $f$  on  $X$  if

$$\lim_{m \rightarrow \infty} \sup_{x \in X} |f_m(x) - f(x)|^{1/m} = 0.$$

The following relation between convergence in capacity and pointwise convergence is somewhat standard.

**Lemma 2.2.** ([3]) *Let  $\{f_m\}_{m \geq 1}$  and  $f$  be Borel measurable functions defined on a bounded domain  $D \subset \mathbb{C}^n$ . If  $\{f_m\}_{m \geq 1}$  converges in capacity to  $f$  on a Borel subset  $X$  of  $D$ , then there exists a subsequence  $\{f_{m_j}\}_{j \geq 1}$  and a pluripolar subset  $E \subset X$  such that  $\{f_{m_j}\}_{j \geq 1}$  converges pointwise to  $f$  on  $X \setminus E$ .*

We should remark that there exist pointwise convergent sequences that contains no subsequence that converges in capacity. Indeed, let  $\{A_m\}_{m \geq 1}$  be a sequence of pairwise disjoint subsets of the unit disk  $\Delta \subset \mathbb{C}$  such that  $\inf_{m \geq 1} \text{cap}(A_m, \Delta) > 0$ . Then the sequence  $\{\chi_{A_m}\}_{m \geq 1}$  provides the desired example.

Finally, we have the following sort of compactness in  $PSH(D)$ .

**Lemma 2.3.** ([3]) *Let  $\{u_m\}_{m \geq 1}$  be a sequence of plurisubharmonic functions defined on a domain  $D$  in  $\mathbb{C}^n$ . Suppose that the sequence is uniformly bounded from above on compact subsets of  $D$  and does not converge to  $-\infty$  uniformly on some compact subset of  $D$ . Then the following assertions hold:*

- (a) *There exists a subsequence  $\{u_{m_j}\}_{j \geq 1}$  converging in  $L^1_{\text{loc}}(D)$  to a function  $u \in PSH(D)$ ,  $u \not\equiv -\infty$ .*
- (b)  *$\limsup_{j \rightarrow \infty} u_{m_j} \leq u$  on  $D$ .*
- (c)  *$\limsup_{j \rightarrow \infty} u_{m_j} = u$  outside a pluripolar subset of  $D$ .*
- (d) *The set  $\{z \in D : \lim_{j \rightarrow \infty} u_{m_j}(z) = -\infty\}$  is pluripolar.*

### 3. Proof of the main theorem

We need the following auxiliary fact giving a sufficient condition for a sequence of measurable functions converging in capacity to 0. We would like to repeat a reasoning due to [3] for the reader convenience.

**Lemma 3.1.** ([3]) *Let  $\{u_m\}_{m \geq 1}$  be a sequence of plurisubharmonic functions and  $\{v_m\}_{m \geq 1}$  be a sequence of measurable functions defined on a bounded domain  $D \subset \mathbb{C}^n$ . Assume that the following conditions are satisfied:*

- (a)  *$\{u_m\}_{m \geq 1}$  is uniformly bounded from above;*
- (b) *There exists a compact subset  $X$  of  $D$  such that*

$$\inf_{m \geq 1} \sup_{z \in X} u_m(z) > -\infty;$$

- (c)  *$u_m + v_m$  converges to  $-\infty$  uniformly on compact subsets of  $D$ .*

*Then the sequence  $\{e^{v_m}\}_{m \geq 1}$  converges to 0 in capacity.*

**Proof of Theorem 1.3.** (i) First, after removing from  $X$  a pluripolar subset (possibly empty), we may assume that  $r_m(z) \in \mathbb{C}$  for every  $z \in X$  and  $m \geq 1$ . In view of the assumptions (a) and (c) we deduce that

$$\sup_{m \geq 1} \frac{1}{m} \log |r_m(z)| < \infty \quad \forall z \in X.$$

For  $N \geq 1$  we let

$$X_N := \{z \in X : \sup_{m \geq 1} \frac{1}{m} \log |r_m(z)| \leq N\}.$$

It follows that  $X = \cup_{N \geq 1} X_N$ . Since  $X$  is non-pluripolar, we deduce that there exists  $N_0 \geq 1$  such that  $X' := X_{N_0}$  is non-pluripolar. Now we write  $r_m = p_m/q_m$  with  $\|q_m\|_{X'} = 1$ . By the choice of  $X'$  we also have

$$\frac{1}{m} \log \|p_m\|_{X'} \leq N_0.$$

Therefore, by Bernstein-Markov's inequality (1), we obtain the following estimates for every compact set  $K \subset \mathbb{C}^n$

$$\sup_{m \geq 1} \max \left\{ \frac{1}{m} \log \|p_m\|_K, \frac{1}{m} \log \|q_m\|_K \right\} < \infty. \tag{2}$$

By combining (2) and the assumption (a) we get

$$\sup_{m \geq 1} \frac{1}{m} \log \|q_m f_m - p_m\|_D < \infty \tag{3}$$

Now we set

$$u_m := \frac{1}{m} \log |f_m - r_m| + \frac{1}{m} \log |q_m| \quad \forall m \geq 1. \tag{4}$$

Clearly  $u_m$  is plurisubharmonic on  $D$  for every  $m \geq 1$ . Moreover, using (3), (4) we see that the sequences  $\{u_m\}_{m \geq 1}$  and  $\{v_m\}_{m \geq 1}$ , where  $v_m := \frac{1}{m} \log |q_m|$ , are uniformly bounded from above on  $D$  and on compact sets of  $\mathbb{C}^n$ , respectively. Next, we show that the sequence  $\{u_m\}_{m \geq 1}$  converges to  $-\infty$  uniformly on compact subset of  $D$ . Suppose otherwise, then by Lemma 2.3, there exists a subsequence  $\{u_{m_j}\}_{j \geq 1}$  and a function  $u \in PSH(D)$ ,  $u \not\equiv -\infty$  such that the set

$$Y := \{z \in D : \limsup_{j \rightarrow \infty} u_{m_j}(z) \neq u(z)\}$$

is pluripolar. Fix  $z \in X$  and a sequence  $\{z_j\}_{j \geq 1} \subset D$  that converges to  $z$ , we claim that

$$\lim_{m \rightarrow \infty} u(z_j) = -\infty. \tag{5}$$

Assume the claim is false, then  $u$  is bounded from below on a subsequence of  $\{z_j\}_{j \geq 1}$ . For simplicity of exposition, we can suppose that  $u$  is bounded from below on the whole sequence  $\{z_j\}_{j \geq 1}$ . Thus there exists a constant  $A$  such that

$$u(z_j) > A \quad \forall j \geq 1.$$

Fix  $j \geq 1$ . Since  $Y$  is pluripolar, we can find a complex line  $l$  passing through  $z_j$  such that  $l \cap Y$  is polar (in  $l$ ). Then by Theorem 5.4.2 in [7], the set  $l \setminus (Y \cup \{z_j\})$  is non-thin at the point  $z_j$ . Thus we can choose  $z'_j \in D$  such that

$$|z_j - z'_j| < 1/j, u(z'_j) > A \text{ and } z'_j \notin Y.$$

By the definition of  $Y$ , we can find a sequence  $\{l(j)\}_{j \geq 1} \uparrow \infty$  such that

$$u_{m_{l(j)}}(z'_j) > A \quad \forall j \geq 1.$$

On the other hand, since  $\{v_m\}_{m \geq 1}$  is uniformly bounded from above on compact sets of  $\mathbb{C}^n$  we have

$$\sup_{m \geq 1, j \geq 1} v_m(z'_j) < \infty.$$

So there exists a constant  $\delta > 0$  so that

$$|f_{m_{l(j)}}(z'_j) - r_{m_{l(j)}}(z'_j)|^{1/m_{l(j)}} > \delta \quad \forall j \geq 1.$$

This contradicts the assumption (c). Thus the claim (5) follows. On the other hand, since  $\{u_m\}_{m \geq 1}$  is uniformly bounded from above on  $D$ , we have  $\sup_D u = c < \infty$ . Thus, for every  $N > 0$  we get the following estimate

$$u(z) \leq c + N\omega(z, X, D) \quad \forall z \in D. \tag{6}$$

By letting  $N \rightarrow \infty$  in (6) and using the assumption (b) we obtain  $u(z) = -\infty$  for every  $z \in D$ , which is absurd. Thus we have proved that  $\{u_m\}_{m \geq 1}$  converges to  $-\infty$  uniformly on some compact subset of  $D$ .

Finally, note that by the normalization  $\|q_m\|_{X'} = 1$  we get

$$\sup_{X'} v_m = 0, \quad \forall m \geq 1.$$

Thus  $v_m$  does not tend to  $-\infty$  uniformly on  $X'$ . In view of the relation (4), we may invoke Lemma 3.1 to conclude that  $|f_m - r_m|^{1/m}$  converges to 0 in capacity on  $D$ .

(ii) By what we have proved in (i), the sequence  $\{v_m\}_{m \geq 1}$  is uniformly bounded from above on compact sets of  $\mathbb{C}^n$ . Moreover,  $\{v_m\}_{m \geq 1}$  does not tend to  $-\infty$  uniformly on some compact set of  $\mathbb{C}^n$ . Thus, by Lemma 2.3 (c), there exists a pluripolar subset  $E$  of  $\mathbb{C}^n$  such that

$$\limsup_{j \rightarrow \infty} v_m > -\infty \text{ on } D \setminus E.$$

We will show that  $E$  has the desired property. For this, fix a point  $z_0 \in D \setminus E$  and an affine complex subspace  $L$  that contains  $z_0$ . Choose a subsequence  $\{v_{m_j}\}_{j \geq 1}$  such that

$$\lim_{j \rightarrow \infty} v_{m_j}(z_0) > -\infty.$$

Let  $v'_j$  and  $v''_j$  be the restrictions to  $D \cap L$  of the two sequences  $v_{m_j}$  and  $\frac{1}{m_j} \log |f_{m_j} - r_{m_j}|$ . According to the results provided in (i), we have  $v'_j + v''_j$  converges uniformly to  $-\infty$  on compact sets of  $D_{z_0}$ . Now in view of the choice of  $v_{m_j}$ , we may apply Lemma 3.1 to reach the conclusion that the sequence  $e^{v''_j} = |f_{m_j} - r_{m_j}|^{1/m_j}$  converges in capacity to 0 in  $D_{z_0}$ . We are done.

**Remarks.** (a) The assumption (a) in Theorem 1.3 is essential for the proof. We do not know if the theorem is still valid without this hypothesis.

(b) The main difficulty that leads to the passage into subsequence in (ii) lies in the fact that the complex subspace  $L$  may be disjoint from the non-pluripolar set  $X$ , thus a direct application of (a) is not possible then.

The above proof actually gives the following variant of Theorem 1.3 in the special case where  $\{r_m\}_{m \geq 1}$  is a sequence of polynomials.

**Proposition 3.2.** *Let  $\{p_m\}_{m \geq 1}$  be a sequence of polynomials on  $\mathbb{C}^n$  ( $1 \leq \deg p_m \leq m$ ) and  $\{f_m\}_{m \geq 1}$  be a sequence of bounded holomorphic functions defined on a bounded domain  $D \subset \mathbb{C}^n$  that satisfies the condition (a) of Theorem 1.3. Assume that there exists a Borel non-pluripolar set  $X \subset \partial D$  satisfying the properties (b) and (c) given in Theorem 1.3 with  $r_m$  is replaced by  $p_m$ . Then  $\{f_m - p_m\}_{m \geq 1}$  is rapidly uniformly convergent to 0 on compact sets of  $D$ .*

*Proof.* First, we define the following sequence of plurisubharmonic functions on  $D$

$$u_m := \frac{1}{m} |f_m - p_m|, \quad \forall m \geq 1.$$

Then by the same reasoning as the proof of Theorem 1.3 we see that  $\{u_m\}_{m \geq 1}$  converges to  $-\infty$  uniformly on compact sets of  $D$ . The desired conclusion follows.  $\square$

By the same reasoning as in the proof of Theorem 1.3, we also have the following result which is truly a version of Bloom’s theorem for sequence of holomorphic functions. The details of the proof are therefore omitted.

**Theorem 3.3.** *Let  $\{r_m\}_{m \geq 1}$  be a sequence of rational functions on  $\mathbb{C}^n$ . Let  $\{f_m\}_{m \geq 1}$  be a sequence of holomorphic functions defined on a bounded domain  $D \subset \mathbb{C}^n$  such that*

$$\sup_{m \geq 1} \frac{1}{m} \log \|f_m\|_K < \infty,$$

for every compact subsets  $K$  of  $D$ . Assume that there exists a Borel non-pluripolar set  $X \subset D$  satisfying the following condition: For every  $z \in X$  we have

$$\lim_{m \rightarrow \infty} |f_m(z) - r_m(z)|^{1/m} = 0.$$

Then the following assertions hold:

(i) The sequence  $|f_m - r_m|^{1/m}$  is convergent in capacity to 0 on  $D$ .

(ii) There exists a pluripolar subset  $E$  of  $\mathbf{C}^n$  with the following property: For every  $z_0 \in D \setminus E$  and every affine complex subspace  $L$  of  $\mathbf{C}^n$  passing through  $z_0$ , there exists a subsequence  $\{r_{m_j}\}_{j \geq 1}$  (dependent only on  $z_0$ ) such that  $|f_{m_j} - r_{m_j}|^{1/m_j}|_{D_{z_0}}$  is convergent in capacity (with respect to  $D_{z_0}$ ) to 0, where  $D_{z_0}$  is the connected component of  $D \cap L$  that contains  $z_0$ .

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## References

- [1] Bloom T. On the convergence in capacity of rational approximants. *Constr. Approx.* 2001; 17: 91-102.
- [2] Blocki Z. Lecture notes in pluripotential theory. <http://gamma.im.uj.edu.pl/blocki/publ/ln/wykl.pdf>.
- [3] Dieu N. Q., Manh P. V., Bang P. H. and Hung L. T. Vitali's theorem without uniform boundedness. *Publ. Mat.* 2016; 60(2): 311-334.
- [4] A. Gonchar, A. A local condition for the single valuedness of analytic functions of several variables. *Math. USSR Sbornik.* 1974; 22: 305-322.
- [5] Hörmander L. *Notions of Convexity.* Birkhäuser Press. 1993.
- [6] Klimek M. *Pluripotential Theory.* Oxford Academic Press. 1991.
- [7] Ransford T. *Potential theory in the complex plane.* Cambridge University Press. 1995.
- [8] Siciak J. On some extremal functions and their applications in the theory of analytic functions of several complex variables. *Transactions of Amer. Math. Soc.* 1962; 105: 322-357.