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On the Solvability of a Self-Reference Functional and Quadratic Functional Integral Equations

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Abstract. In this paper we study the existence of solutions of a self-reference functional integral equation and functional quadratic integral equation. Some examples will be given.

1. Introduction

Differential (integral) equations with deviating arguments that depends on both the state variable x and the time t, are called self-reference differential (integral) equations. These types of equations play an important role in nonlinear analysis, and have many applications (for example see [22]).

Buicá [12] proved the existence and uniqueness of the solution of the initial value problem

$$\begin{aligned} x'(t) &= f(t, x(x(t))), \quad t \in [a, b] \\ x(0) &= x_0 \end{aligned}$$

which is equivalent to integral equation

$$x(t) = x_0 + \int_0^t f(s, x(x(s)))ds,$$

where $f \in C([a, b] \times [a, b])$ and satisfied Lipshitz condition,

$$|f(t, x) - f(t, y)| \leq k|x - y|, \ k > 0.$$

Banaś and Cabrera [4] studied the existence and asymptotic behavior of solutions of the functional integral equation

$$x(t) = f\Big(t, \ \int_0^t x(s)ds, \ \int_0^t x\big(h(s, x(s))\big)ds\Big), \quad t \geq 0,$$

by using measure of noncompactness technique where $f : R_+ \times R \times R \to R$ is continuous. For other works (see [1], [11], [14] and [19]).

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Our aim in this work firstly, is to relax the assumptions of Buicá [12] and generalized their results. We study the existence of solutions $x \in C[0,T]$ of the self-reference functional integral equation

$$x(t) \;=\; f\Big(t,\; \int_0^t g\Big(s, x(x(s))\Big)ds\Big), \qquad t \in [0,T]. \tag{1}$$

where the function g satisfies Carathéodory condition. Moreover we study the existence of a unique solution for this equation.
Secondly, we study the existence of solutions $x \in C[0,T]$ of the self-reference quadratic functional integral equation.

$$x(t) \;=\; f\Big(t,\; \int_0^t f_1\Big(s, x(x(s))\Big)ds \int_0^t f_2\Big(s, x(x(s))\Big)ds\Big), \qquad t \in [0,T]. \tag{2}$$

where $f_1,\ f_2$ satisfy Carathéodory condition. The Uniqueness of the solution will be studied also.

2. Functional integral equation

2.1. Existence of solution

Consider the functional integral equation (1) under the following assumptions:

(*i*) $f : [0,T] \times R \to R$ is continuous such that

$$|f(t_2, x) - f(t_1, y)| \leq K_1|t_2 - t_1| + K_2|x - y|$$

where $K_1,\ K_2$ are two positive constants.

(*ii*) $g : [0,T] \times [0,T] \to R$ satisfies Carathéodory condition i.e g are measurable in t for all $x \in [0,T]$ and continuous in x for almost all $t \in [0,T]$,

(*iii*) there exist a measurable bounded function $m(t)$ and a constant $b > 0$ such that

$$|g(t,x)| \leq m(t) + b|x|.$$

(*iv*) $LT + |x(0)| \leq T$ and $L = K_1 + K_2M < 1$, where $M = A + bT$ and A is a positive constant such that $|m(t)| \leq A$.

Remark 2.1. *Using assumption* (*i*) *we have*

$$|f(t,x) - f(t,0)| \quad \leq \quad K_2|x|,$$

then

$$|f(t,x)| \quad \leq \quad K_2|x| + |f(t,0)|.$$

Theorem 2.2. *Let the assumptions (i)-(iv) be satisfied, then the functional integral equation (1) has at least one solution $x \in C[0,T]$.*

Proof. Define the set S_L by

$$S_L = \Big\{x \in C[0,T] : |x(t) - x(s)| \leq L|t - s|\Big\} \subset C[0,T],$$

It is clear that S_L is nonempty ,closed, bounded and convex subset of $C[0,T]$. Now define the operator G associated with equation (1) (as in [4] and [12]) by

$$Gx(t) = f\Big(t, \int_0^t g(s, x(x(s)))ds\Big), \quad t \in [0,T].$$

Clear that G is makes sense and well-defined. Now, let $x \in C[0,T]$, then for $t \in [0,T]$, we get

$$\begin{aligned}
|Gx(t)| &= \left|f\Big(t, \int_0^t g(s,x(x(s)))ds\Big)\right| \\
&\leq K_2\left|\int_0^t g(s,x(x(s)))ds\right| + |f(t,0)| \\
&\leq |f(t,0)| + K_2\int_0^t |g(s,x(x(s)))|ds \\
&\leq |f(t,0)| + K_2\int_0^t \{m(s) + b|x(x(s))|\}ds \\
&\leq |f(t,0)| + K_2\Big[AT + b\int_0^t \{L|x(s)| + |x(0)|\}ds\Big] \\
&\leq |f(t,0)| + K_2\Big[A + b(LT + |x(0)|)\Big]T \\
&\leq |f(t,0)| + K_2\big(A + bT\big)T \\
&\leq |f(t,0)| + K_2MT
\end{aligned}$$

But

$$\begin{aligned}
|f(t,0)| &\leq |f(t,0) - f(0,0)| + |f(0,0)| \\
&\leq K_1\, t + |f(0,0)| \\
&\leq K_1\, T + |x(0)|,
\end{aligned}$$

then

$$\begin{aligned}
|(Gx)(t)| &\leq K_1T + |x(0)| + K_2MT \\
&\leq (K_1 + K_2M)T + |x(0)| \\
&= L\,T + |x(0)| \leq T
\end{aligned}$$

which proves that the class $\{Gx\}$ is uniformly bounded on S_L. Now let $x \in S_L$ and $t_1,\ t_2 \in [0,T]$ with $t_1 < t_2$ such that $|t_2, -t_1| < \delta$, then

$$\begin{aligned}
|Gx(t_2) - Gx(t_1)| &= \left|f\Big(t_2, \int_0^{t_2} g(s,x(x(s)))ds\Big) - f\Big(t_1, \int_0^{t_1} g(s,x(x(s)))ds\Big)\right| \\
&\leq K_1\,|t_2 - t_1| + K_2\,\left|\int_0^{t_2} g(s,x(x(s)))ds - \int_0^{t_1} g(s,x(x(s)))ds\right|, \\
&\leq K_1\,|t_2 - t_1| + K_2\,\int_{t_1}^{t_2} |g(s,x(x(s)))|ds \\
&\leq K_1\,|t_2 - t_1| + K_2\,\int_{t_1}^{t_2} \Big\{m(s) + b|x(x(s))|\Big\}ds \\
&\leq K_1\,|t_2 - t_1| + K_2\Big[A(t_2 - t_1) + b\int_{t_1}^{t_2} \{L|x(s)| + |x(0)|\}ds\Big] \\
&\leq K_1|t_2 - t_1| + K_2\Big(A + b\{L\,T + |x(0)|\}\Big)(t_2 - t_1) \\
&\leq K_1|t_2 - t_1| + K_2\Big(A + bT\Big)(t_2 - t_1) \\
&\leq K_1|t_2 - t_1| + K_2M(t_2 - t_1) \\
&\leq L|t_2 - t_1|
\end{aligned}$$

This proves that $Gx(t) \in S_L$, $G : S_L \to S_L$ and the class of functions $\{Gx\}$ is equicontinuous. By Arzela-Ascoli Theorem ([21] page (54)), we find that G is compact.
Now we will show that G is continuous. Let $\{x_n\} \subset S_L$ such that $x_n \to x_0$ uniformly on $[0, T]$, (i.e, $|x_n(t) - x_0(t)| \leq \epsilon_1$ (say)) this implies also $|x_n(x_0(t)) - x_0(x_0(t))| \leq \epsilon_2$ for arbitrary $\epsilon_1, \epsilon_2 \geq 0$, then

$$\begin{aligned}
|g(t, x_n(x_n(t)))| &\leq m(t) + b|x_n(x_n(t))| \\
&\leq m(t) + b\Big[L|x_n(s)| + |x_n(0)|\Big] \\
&\leq m(t) + bT.
\end{aligned}$$

and

$$\begin{aligned}
|x_n(x_n(t)) - x_0(x_0(t))| &= |x_n(x_n(t)) - x_n(x_0(t)) + x_n(x_0(t)) - x_0(x_0(t))| \\
&\leq |x_n(x_n(t)) - x_n(x_0(t))| + |x_n(x_0(t)) - x_0(x_0(t))| \\
&\leq L|x_n(t) - x_0(t)| + |x_n(x_0(t)) - x_0(x_0(t))| \\
&\leq L\epsilon_1 + \epsilon_2
\end{aligned}$$

which implies that

$$x_n(x_n(t))) \to (x_0(x_0(t))) \ in \ S_L.$$

Now the function g is continuous in the second argument, then

$$g\Big(t, x_n(x_n(t))\Big) \to g\Big(t, x_0(x_0(t))\Big).$$

Using Lebesgues dominated convergence theorem ([13] page(151)) we have

$$\lim_{n\to\infty} \int_0^t g\Big(s, x_n(x_n(s))\Big)ds = \int_0^t g\Big(s, x_0(x_0(s))\Big)ds$$

and from the continuity of f we obtain

$$\begin{aligned}
\lim_{n\to\infty} \Big(Gx_n\Big)(t) &= \lim_{n\to\infty} f\Big(t, \int_0^t g\Big(s, x_n(x_n(s))\Big)ds\Big), \\
&= f\Big(t, \lim_{n\to\infty} \int_0^t g\Big(s, x_n(x_n(s))\Big)ds\Big), \\
&= f\Big(t, \int_0^t g\Big(s, x_0(x_0(s))\Big)ds\Big). \\
&= \Big(Gx_0\Big)(t).
\end{aligned}$$

Then G is continuous.
Now all conditions of Schauder fixed point Theorem ([20] page (482)), are satisfied, then the operator G has at least one fixed point $x \in S_L$. Consequently there exist at leat one solution $x \in C[0, T]$ of equation (1) which completes the proof.

Now, as in Banaś [10] (page 247) we can prove the following corollaries:

Corollary 2.3. *Let the assumptions* $(ii) - (iv)$ *of Theorem 2.2 be satisfied. Let* $a : [0, T] \to R$ *such that*

$$|a(t_2) - a(t_1)| \leq a|t_2 - t_1|,$$

then the integral equation

$$x(t) = a(t) + \int_0^t g\Big(s, x(x(s))\Big)ds, \qquad t \in [0, T] \tag{3}$$

has at leat one solution $x \in C[0, T]$.

Corollary 2.4. *Let the assumption of corollary (2.3) be satisfied, assume $a(t) = x_0$, then the initial value problem*

$$\frac{d}{dt}x(t) \; = g\Big(\big(t, x(x(t))\big) \; a.e, \tag{4}$$

$$x(0) = x_0 \tag{5}$$

has at leat one solution $x \in C[0, T]$.

2.2. Uniqueness of the solution

In this section we prove the uniqueness of the solution of the functional integral equation (1). For this aim we assume that

(i')

$$|g(t, x) - g(t, y)| \leq b\,|x - y|$$

(ii')

$$|g(t, 0)| \leq A,$$

Theorem 2.5. *Let the assumptions (i), (ii), (iv) of Theorem 2.2 and (i'), (ii') be satisfied, if*

$$b\,T\,K_2(L + 1) < 1,$$

then the solution $x \in C[0, T]$ of equation (1) is unique.

Proof. Assumption (iii) of Theorem 2.2 can be deduced from assumption (i') *and* (ii') if we put $y = 0$ in (i') we get

$$\begin{aligned} |g(t, x)| &\leq b\,|x| + |g(t, 0)| \\ &\leq b\,|x| + A \end{aligned} \tag{6}$$

hence we deduce that all assumptions of Theorem 2.2 are satisfied. Then the solution of equation (1) exists. Now let $x,\ y$ be two solutions of (1), then

$$\begin{aligned} |x(t) - y(t)| &= \left| f\Big(t, \int_0^t g\big(s, x(x(s))\big)ds\Big) - f\Big(t, \int_0^t g\big(s, y(y(s))\big)ds\Big)\right|, \\ &\leq K_2 \left| \int_0^t g\big(s, x(x(s))\big)ds - \int_0^t g\big(s, y(y(s))\big)ds \right| \\ &\leq K_2 \int_0^t |g\big(s, x(x(s))\big) - g\big(s, y(y(s))\big)|\,ds \\ &\leq K_2\,b \int_0^t |x(x(s)) - y(y(s))|ds \\ &\leq K_2\,b \Big[\int_0^t L|x(s) - y(s)|ds + \int_0^t |x(y(s)) - y(y(s))|ds\Big] \\ &\leq K_2\,b\,T\Big[L\|x - y\| + \|x - y\|\Big], \end{aligned} \tag{7}$$

thus we have

$$\|x - y\| \leq K_2\,b\,T(L + 1)\|x - y\|,$$

hence

$$\big(1 - K_2\,b\,T(L + 1)\big)\|x - y\| \leq 0.$$

Since $K_2\,b\,T(L + 1) < 1$, then we get $x = y$ and the solution of (1) is unique.

Corollary 2.6. *Let the assumptions (ii), (iv), (i′) and (ii′) of Theorem 2.5 be satisfied. Let* $a : [0, T] \to R$ *such that* $a : [0, T] \to R$ *is continuous such that*

$$|a(t_2) - a(t_1)| \le a|t_2 - t_1|,$$

then the solution of the integral equation (3) is unique. Consequently if $a(t) = x_0$*, then the solution of the initial value problem (4) and (5) is unique.*

Remark 2.7. *In the Theorems 2.2 and 2.5 we generalized the results of Buicá [12] and relaxed their assumptions.*

Example 2.8. *Consider the following equation*

$$x(t) = \frac{1}{5}(1 + t) + \int_0^t \Big(\frac{1}{7 - s} + \frac{e^{-s}}{16} x(x(s))\Big) ds \tag{8}$$

where $t \in [0, 2]$ *here we have*

$$g\big(t, x(x(t))\big) = \frac{1}{7 - t} + \frac{e^{-t}}{16} x(x(t)),$$

thus

$$|g\big(t, x) - g\big(t, y)| \le \frac{1}{16}|x - y|,$$

so we have $b = \frac{1}{16}$*,* $K_2 = 1$*,* $a(t) = \frac{1}{5}(1 + t)$*, thus* $K_1 = a = \frac{1}{5}$*,* $g(t, 0) = \frac{1}{7-t}$*, and* $A = \frac{1}{5}$*, thus we get* $M = \frac{13}{40}$ *and* $L = 0.525 < 1$*, hence* $b\, T(L + 1) = 0.190625 < 1$.
Now clear that all assumptions of Corollary 2.6 are satisfied, then equation (8) has a unique solution.

3. Quadratic integral equation

Quadratic integral equations have many applications in the theory of radiative transfer, kinetic theory of gases and in the traffic theory, this applications was introduced by several authors (see for example [2], [3], [6], [7], [9], [5], [8], [15], [17], [16] and [18]).

3.1. Existence of solution

Consider now the quadratic functional integral equation (2) under the following assumptions:

(1) $f : [0, T] \times R \to R$ satisfies the Lipshitz condition

$$|f(t_2, x) - f(t_1, y)| \le k_1|t_2 - t_1| + k_2|x - y|$$

k_1, k_2 are two positive constants.

(2) $f_i : [0, T] \times [0, T] \to R$ satisfy Carathéodory condition i.e f_i are measurable in t for all $x \in C[0, T]$ and continuous in x for almost all $t \in [0, T]$, $i = 1, 2$.

(3) There exist two measurable bounded functions m_1, m_2 and constants b_1 , $b_2 > 0$ such that

$$|f_i(t, x)| \le m_i(t) + b_i|x|, \ i = 1, 2.$$

(4) $L_1 T + |x(0)| \le T$ and $L_1 = k_1 + 2k_2 M_1 M_2 T < 1$, where

$$M_1 = A_1 + b_1 T$$

and

$$M_2 = A_2 + b_2 T$$

where A_i, $i = 1, 2$ are two positive constants such that $|m_i(t)| \le A_i$, $i = 1, 2$.

Theorem 3.1. *Let the assumptions* (1) − (4) *be satisfied, then the quadratic functional integral equation (2) has at least one solution* $x \in C[0,T]$.

Proof. Define the set S_{L_1} by

$$S_{L_1} = \Big\{x \in C[0,T] : |x(t) - x(s)| \leq L_1|t-s|\Big\} \subset C[0,T],$$

It is clear that S_{L_1} is nonempty ,closed, bounded and convex subset of $C[0,T]$.

Define the operator F associated with equation (2) by

$$Fx(t) = f\Big(t, \int_0^t f_1(s, x(x(s)))ds \int_0^t f_2(s, x(x(s)))ds\Big), \qquad t \in [0,T].$$

Now, let $x \in C[0,T]$, then for $t \in [0,T]$ we can get

$$\begin{aligned}
|Fx(t)| &= \Big|f\Big(t, \int_0^t f_1(s, x(x(s)))ds \int_0^t f_2(s, x(x(s)))ds\Big)\Big| \\
&\leq k_2\Big|\int_0^t f_1(s, x(x(s)))ds \int_0^t f_2(s, x(x(s)))ds\Big| + |f(t,0)| \\
&\leq |f(t,0)| + k_2 \int_0^t |f_1(s, x(x(s)))|ds \int_0^t |f_2(s, x(x(s)))|ds \\
&\leq |f(t,0)| + k_2 \int_0^t \{m_1(s) + b_1|x(x(s))|\}ds \int_0^t \{m_2(s) + b_2|x(x(s))|\}ds \\
&\leq |f(t,0)| + k_2\Big[A_1 T + b_1 \int_0^t \{L_1|x(s)| + |x(0)|\}ds\Big]\Big[A_2 T + b_2 \int_0^t \{L_1|x(s)| + |x(0)|\}ds\Big] \\
&\leq |f(t,0)| + k_2\Big[A_1 + b_1(L_1 T + |x(0)|)\Big]\Big[A_2 + b_2(L_1 T + |x(0)|)\Big]T^2 \\
&\leq |f(t,0)| + k_2\Big[A_1 + b_1 T\Big]\Big[A_2 + b_2 T\Big]T^2 \\
&\leq |f(t,0)| + 2k_2 M_1 M_2 T^2.
\end{aligned}$$

But

$$\begin{aligned}
|f(t,0)| &\leq |f(t,0) - f(0,0)| + |f(0,0)| \\
&\leq k_1\, t + |f(0,0)| \\
&\leq k_1\, T + |x(0)|.
\end{aligned}$$

Then we get

$$\begin{aligned}
|(Fx)(t)| &\leq k_1 T + |x(0)| + 2k_2 M_1 M_2 T^2 \\
&= L_1\, T + |x(0)| \leq T.
\end{aligned}$$

This proves that the class $\{Fx\}$ is uniformly bounded on S_{L_1}.

Now let $x \in S_{L_1}$ and $t_1,\ t_2 \in [0,T]$ with $t_1 < t_2$ such that $|t_2, -t_1| < \delta$, then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= \left|f\Big(t_2, \int_0^{t_2} f_1(s, x(x(s)))ds \int_0^{t_2} f_2(s, x(x(s)))ds\Big) - f\Big(t_1, \int_0^{t_1} f_1(s, x(x(s)))ds \int_0^{t_1} f_2(s, x(x(s)))ds\Big)\right| \\
&\leq k_1|t_2 - t_1| + k_2 \left| \int_0^{t_2} f_1(s, x(x(s)))ds \int_0^{t_2} f_2(s, x(x(s)))ds - \int_0^{t_1} f_1(s, x(x(s)))ds \int_0^{t_1} f_2(s, x(x(s)))ds \right|, \\
&\leq k_1|t_2 - t_1| \\
&+ k_2 \left|\Big(\int_0^{t_1} f_1(s, x(x(s)))ds + \int_{t_1}^{t_2} f_1(s, x(x(s)))ds\Big)\Big(\int_0^{t_1} f_2(s, x(x(s)))ds + \int_{t_1}^{t_2} f_2(s, x(x(s)))ds\Big)\right. \\
&- \left. k_2 \int_0^{t_1} f_1(s, x(x(s)))ds \int_0^{t_1} f_2(s, x(x(s)))ds\right| \\
&\leq k_1|t_2 - t_1| + k_2 \left| \int_0^{t_1} f_1(s, x(x(s)))ds \int_{t_1}^{t_2} f_2(s, x(x(s)))ds\right| \\
&+ k_2 \left| \int_{t_1}^{t_2} f_1(s, x(x(s)))ds \int_0^{t_1} f_2(s, x(x(s)))ds\right| + k_2 \left| \int_{t_1}^{t_2} f_1(s, x(x(s)))ds \int_{t_1}^{t_2} f_2(s, x(x(s)))ds\right| \\
&\leq k_1|t_2 - t_1| + k_2 \int_0^{t_1} |f_1(s, x(x(s)))|ds \int_{t_1}^{t_2} |f_2(s, x(x(s)))|ds \\
&+ k_2 \int_{t_1}^{t_2} |f_1(s, x(x(s)))|ds \int_0^{t_1} |f_2(s, x(x(s)))|ds + k_2 \int_{t_1}^{t_2} |f_1(s, x(x(s)))|ds \int_{t_1}^{t_2} |f_2(s, x(x(s)))|ds \\
&= k_1|t_2 - t_1| + k_2 \int_0^{t_2} |f_1(s, x(x(s)))|ds \int_{t_1}^{t_2} |f_2(s, x(x(s)))|ds \\
&+ k_2 \int_{t_1}^{t_2} |f_1(s, x(x(s)))|ds \int_0^{t_1} |f_2(s, x(x(s)))|ds \\
&\leq k_1|t_2 - t_1| + k_2 \Big(\int_0^{t_2} \{m_1(s) + b_1|x(x(s))|\}ds\Big)\Big(\int_{t_1}^{t_2} \{m_2(s) + b_2|x(x(s))|\}ds\Big) \\
&+ k_2 \Big(\int_{t_1}^{t_2} \{m_1(s) + b_1|x(x(s))|\}ds\Big)\Big(\int_0^{t_1} \{m_2(s) + b_2|x(x(s))|\}ds\Big) \\
&\leq k_1|t_2 - t_1| + k_2\Big[A_1T + b_1 \int_0^{t_2} \{L_1|x(s)| + |x(0)|\}ds\Big]\Big[A_2(t_2 - t_1) + b_2 \int_{t_1}^{t_2} \{L_1|x(s)| + |x(0)|\}ds\Big] \\
&+ k_2\Big[A_1(t_2 - t_1) + b_1 \int_{t_1}^{t_2} \{L_1|x(s)| + |x(0)|\}ds\Big]\Big[A_2T + b_2 \int_0^{t_1} \{L_1|x(s)| + |x(0)|\}ds\Big] \\
&\leq k_1|t_2 - t_1| + 2Tk_2\Big(A_1 + b_1\{L_1\, T + |x(0)|\}\Big)\Big(A_2 + b_2\{L_1\, T + |x(0)|\}\Big)(t_2 - t_1) \\
&\leq k_1|t_2 - t_1| + (2k_2TM_1M_2)(t_2 - t_1) \\
&\leq L_1|t_2 - t_1|
\end{aligned}$$

This proves that $Fx(t) \in S_{L_1}$ hence $F : S_L \to S_{L_1}$, and the class of functions $\{Fx\}$ is equi-continuous. Since the class of functions $\{Fx\}$ is uniformly bounded and equicontinuous on $[0, T]$, by Arzela-Ascoli Theorem [21], we find that F is compact.

Now we will show that F is continuous. Let $\{x_n\} \subset S_{L_1}$ such that $x_n \to x_0$ uniformly on $[0, T]$, then

$$\begin{aligned}
|f_i(t, x_n(x_n(t)))| &\leq m_i(t) + b_i|x_n(x_n(t))| \leq m_i(t) + b_i\Big[L_1T + |x_n(0)|\Big] \\
&= m_i(t) + b_iT \quad i = 1,\ 2.
\end{aligned}$$

and

$$\begin{aligned}
|x_n(x_n(t)) - x_0(x_0(t))| &= |x_n(x_n(t)) - x_n(x_0(t)) + x_n(x_0(t)) - x_0(x_0(t))| \\
&\leq |x_n(x_n(t)) - x_n(x_0(t))| + |x_n(x_0(t)) - x_0(x_0(t))| \\
&\leq L_1|x_n(t) - x_0(t)| + |x_n(x_0(t)) - x_0(x_0(t))| \\
&\leq L_1\epsilon_1 + \epsilon_2.
\end{aligned}$$

This implies that

$$x_n(x_n(t))) \to (x_0(x_0(t)).$$

Now $f_i, \ \ i = 1, \ 2$ continuous in the second argument, then

$$f_i\Big(t, x_n(x_n(t))\Big) \to f_i\Big(t, x_0(x_0(t))\Big), \ \ i = 1, \ 2.$$

By Lebesgues dominated convergence ([13] page(151)) theorem we have,

$$\lim_{n\to\infty} \int_0^t f_1\Big(s, x_n(x_n(s))\Big)ds \int_0^t f_2\Big(s, x_n(x_n(s))\Big)ds = \int_0^t f_1\Big(s, x_0(x_0(s))\Big)ds \int_0^t f_2\Big(s, x_0(x_0(s))\Big)ds$$

and from the continuity of f we have

$$\begin{aligned}
\lim_{n\to\infty} \Big(Fx_n\Big)(t) &= \lim_{n\to\infty} f\Big(t, \int_0^t f_1\Big(s, x_n(x_n(s))\Big)ds \int_0^t f_2\Big(s, x_n(x_n(s))\Big)ds\Big), \\
&= f\Big(t, \lim_{n\to\infty} \int_0^t f_1\Big(s, x_n(x_n(s))\Big)ds \int_0^t f_2\Big(s, x_n(x_n(s))\Big)ds\Big), \\
&= f\Big(t, \int_0^t f_1\Big(s, x_0(x_0(s))\Big)ds \int_0^t f_2\Big(s, x_0(x_0(s))\Big)ds\Big). \\
&= \Big(Fx_0\Big)(t).
\end{aligned}$$

Then F is continuous. Now all conditions of Schauder fixed point Theorem ([20] page (482)), are satisfied, then the operator F has at least one fixed point $x \in S$. Consequently there exist at leat one solution $x \in C[0, T]$ of equation (2) which completes the proof.

Corollary 3.2. *Let the assumptions* (2) – (4) *of Theorem 3.1 be satisfied. Let $a : [0, T] \to R$ is continuous such that*

$$|a(t_2) - a(t_1)| \leq a|t_2 - t_1|,$$

then the quadratic integral equation

$$x(t) \ = \ a(t) \ + \int_0^t f_1\Big(s, x(x(s))\Big)ds \int_0^t f_2\Big(s, x(x(s))\Big)ds, \qquad t \in [0, T]. \tag{9}$$

has at leat one solution $x \in C[0, T]$.

Corollary 3.3. *Let the assumptions of Corollary 3.2 be satisfied, then the quadratic integral equation*

$$x(t) \ = \ a(t) \ + \Big(\int_0^t g\Big(s, x(x(s))\Big)ds\Big)^2 \ ds, \qquad t \in [0, T]. \tag{10}$$

has at leat one solution $x \in C[0, T]$.

Proof. If we put $f_1 = f_2 = g$, in equation (9) we get, the quadratic integral equation (10) has at leat one solution $x \in C[0, T]$.

Example 3.4. *Consider the following quadratic integral equation*

$$x(t) = \Big(\frac{2}{3}t + \frac{1}{4}\Big) + \Big(\int_0^t \Big[\frac{1}{32}s + \frac{3}{32}x\big(x(s)\big)\Big]ds\Big)\Big(\int_0^t \Big[\frac{1}{12}s + \frac{3}{12}x\big(x(s)\big)\Big]ds\Big) \tag{11}$$

where $t \in [0, 1]$. Here we have:

$x(0) = 1/4$, $a(t) = \frac{2}{3}t + \frac{1}{4}$ *then* $a = 2/3$

$$f_1\big(t, x(x(t))\big) = \frac{1}{32}t + \frac{3}{32}x\big(x(t)\big), \text{ hence } m_1(t) = \frac{t}{32},\ b_1 = \frac{3}{32},\ A_1 = \frac{1}{32},$$

$$f_2\big(t, x(x(t))\big) = \frac{1}{12}t + \frac{3}{12}x\big(x(t)\big), \text{ hence } m_2(t) = \frac{t}{12},\ b_2 = \frac{3}{12},\ A_2 = \frac{1}{12},$$

thus we have $M_1 = \frac{1}{8}$ *and* $M_2 = \frac{1}{3}$ *then* $L_1 = 3/4 < 1$.

Now its easy to verify all the assumptions of Corollary 3.2, then the previous quadratic integral equation has at least one solution $x \in C[0, T]$.

Example 3.5. *Consider the following quadratic integral equation*

$$x(t) = \frac{28 + 3t}{49} + \int_0^t \Big(\frac{1}{7}(s + e^{-s}) + \frac{x(x(s))^2}{14(1 + |x(x(s))|)}\Big)ds \int_0^t \Big(\frac{1}{7 + 2s}\sin(3(s + 1)) + \frac{1}{14}e^{-s}x(x(s))\Big)ds \tag{12}$$

where $t \in [0, 2]$. Here we have:

$x(0) = 4/7$, $a(t) = \frac{28+3t}{49}$ *then* $a = 3/49$

$$f_1\big(t, x(x(t))\big) = \frac{1}{7}(t + e^{-t}) + \frac{x(x(t))^2}{14(1 + |x(x(t))|)}, \text{ hence } m_1(t) = \frac{1}{7}(t + e^{-t}),\ b_1 = \frac{1}{14},\ A_1 = \frac{3}{7},$$

$$f_2\big(t, x(x(t))\big) = \frac{1}{7 + 2t}\sin(3(t + 1)) + \frac{1}{14}e^{-t}x(x(t)), \text{ hence } m_2(t) = \frac{1}{7 + 2t},\ b_2 = \frac{1}{14},\ A_2 = \frac{1}{7},$$

hence we have $M_1 = \frac{4}{7}$ *and* $M_2 = \frac{2}{7}$ *then* $L_1 = 5/7 < 1$.

Now its easy to verify all the assumptions of Corollary 3.2, then the quadratic integral equation (12) has at least one solution $x \in C[0, T]$.

3.2. Uniqueness of the solution

In this section we prove the existence of a unique solution $x \in C[0, T]$ of the quadratic integral equation (2). For the uniqueness of the solution we assume that

(1′)

$$|f_i(t, x) - f_i(t, y)| \le b_i\, |x - y| \quad i = 1,\ 2$$

(2′)

$$|f_i(t, 0)| \le A_i$$

where $b_i,\ A_i$ are a positive constants, $i = 1,\ 2$

Theorem 3.6. *Let the assumptions* (1), (2), (4), (1′) *and* (2′) *be satisfied, if*

$$(N_1\, b_2 + N_2\, b_1)\, k_2\, T\, (L_1 + 1) \le 1,$$

then equation (2) has a unique solution $x \in C[0, T]$.

Proof. Assumption (3) can be deduced from assumption (1′) *and* (2′) if we put $y = 0$ in (1′) we get

$$|f_i(t,x)| \leq b_i\,|x| + |f_i(t,0)| \quad i = 1,2. \tag{13}$$

hence we deduce that all assumptions of theorem (3.1) are satisfied. Then the solution of equation (2) exists. Now let $x,\ y$ be two solutions of (2), then

$$\begin{aligned}
|x(t)-y(t)| &= \Big|f\Big(t, \int_0^t f_1\big(s,x(x(s))\big)ds \int_0^t f_2\big(s,x(x(s))\big)ds\Big)\\
&- f\Big(t, \int_0^t f_1\big(s,y(y(s))\big)ds \int_0^t f_2\big(s,y(y(s))\big)ds\Big)\Big|,\\
&\leq k_2\Big|\int_0^t f_1\big(s,x(x(s))\big)ds \int_0^t f_2\big(s,x(x(s))\big)ds - \int_0^t f_1\big(s,y(y(s))\big)ds \int_0^t f_2\big(s,y(y(s))\big)ds\Big|\\
&= k_2\Big|\int_0^t f_1\big(s,x(x(s))\big)\,ds\Big[\int_0^t \Big\{f_2\big(s,x(x(s))\big) - f_2\big(s,y(y(s))\big)\Big\}\,ds\Big]\\
&+ k_2\int_0^t f_2\big(s,y(y(s))\big)\,ds\Big[\int_0^t \Big\{f_1\big(s,x(x(s))\big) - f_1\big(s,y(y(s))\big)\Big\}\,ds\Big]\Big|\\
&\leq k_2\int_0^t \big|f_1\big(s,x(x(s))\big)\big|\,ds \int_0^t \big|f_2\big(s,x(x(s))\big) - f_2\big(s,y(y(s))\big)\big|\,ds\\
&+ k_2\int_0^t \big|f_2\big(s,y(y(s))\big)\big|\,ds \int_0^t \big|f_1\big(s,x(x(s))\big) - f_1\big(s,y(y(s))\big)\big|\,ds.\\
&\leq k_2\int_0^t \big|f_1\big(s,x(x(s))\big)\big|\,ds\; b_2\int_0^t |x(x(s))-y(y(s))|ds\\
&+ k_2\int_0^t \big|f_2\big(s,y(y(s))\big)\big|\,ds\; b_1\int_0^t |x(x(s))-y(y(s))|ds
\end{aligned} \tag{14}$$

Now using (13) we obtain,

$$\begin{aligned}
\int_0^t \big|f_i\big(s,x(x(s))\big)\big|ds &\leq b_i\int_0^t |x(x(s))|ds + \int_0^t |f_i(t,0)|ds\\
&\leq b_i\int_0^t \Big\{L_1\,T + |x(0)|\Big\}ds \;+\; A_iT\\
&= b_i\,T^2 \;+\; A_iT = N_i\ (say).
\end{aligned} \tag{15}$$

Moreover we have,

$$\begin{aligned}
|x(x(s))-y(y(s))| &= |x(x(s))-y(y(s))+x(y(s))-x(y(s))|\\
&\leq |x(x(s))-x(y(s))| + |x(y(s))-y(y(s))|\\
&\leq L_1|x(s))-y(s)| + |x(y(s))-y(y(s))|
\end{aligned} \tag{16}$$

Substituting by (15) and (16) in (14) we get,

$$\begin{aligned}
|x(t)-y(t)| &\leq k_2N_1\,b_2\,(L_1+1)\,\|x-y\|\int_0^t ds \;+ k_2N_2\,b_1\,(L_1+1)\,\|x-y\|\int_0^t ds,\\
&\leq k_2N_1\,b_2\,\|x-y\|\,(L_1+1)\,T \;+ k_2N_2\,b_1\,\|x-y\|\,T\,(L_1+1)
\end{aligned}$$

then

$$\|x-y\| \leq k_2(N_1\,b_2 + k_2N_2\,b_1)\,T\,(L_1+1)\,\|x-y\|,$$

thus

$$\Big[1 - (N_1\ b_2 + N_2\ b_1)\ k_2\ T\ (L_1 + 1)\Big]\ ||x - y|| \leq 0$$

since

$$(N_1\ b_2 + N_2\ b_1)\ k_2\ T\ (L_1 + 1) < 1,$$

then we get $x = y$ and the solution of equation (2) is unique solution.

Corollary 3.7. *Let the assumptions* (2), (4), (1′) *and* (2′) *of Theorem 3.6 be satisfied, if* $f(t, x) = a(t) + x$ *where* $a : [0, T] \to R$ *is continuous such that*

$$|a(t_2) - a(t_1)| \leq a|t_2 - t_1|,$$

then the quadratic integral equation

$$x(t) \ = \ a(t) \ + \int_0^t f_1\Big(s, x(x(s))\Big)ds \int_0^t f_2\Big(s, x(x(s))\Big)ds, \qquad t \in [0, T]. \tag{17}$$

has a unique solution $x \in C[0, T]$.

Example 3.8. *Consider Example 3.4, we have*

$$|f_1\Big(t, x) - f_1\Big(t, y)| = \frac{3}{32}|x - y|,$$

$$|f_2\Big(t, x) - f_2\Big(t, y)| \leq \frac{3}{12}|x - y|,$$

also $|f_1\Big(t, 0)| = \frac{t}{32} \leq \frac{1}{32}$, *and* $|f_2\Big(t, 0)| = \frac{t}{12} \leq \frac{1}{12}$ *thus we get* $N_1 = \frac{1}{8}$, $N_2 = \frac{1}{3}$ *hence*

$$(N_1\ b_2 + N_2\ b_1)\ k_2\ T\ (L_1 + 1) = 0.109 < 1,$$

Now clear that all assumptions of Corollary 3.7 are satisfied, then equation (11) has a unique solution.

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