



Supporting and Separating for Rough Convex Set

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Abstract. In this article, we are concerned with a new kind of convexity which is called rough convexity with respect to an equivalence relation. We established a necessary and sufficient condition for a rough set to be rough convex set. Some geometrical and topological properties for this kind of sets are deeply studied.

1. Introduction

The study of convexity and its generalizations play an important role in many fields such as engineering, analysis, optimization theory, differential geometry and fractal mathematics. In the last three decades, a lot of mathematicians generalized the convexity notion, in Euclidian space E^n , to D-convexity [8], E-convexity [5], invexity [9], relative convexity [12], strict convexity [10], convex fuzzy bodies [7], weak convexity [6], starshaped fuzzy sets [1], symmetric fuzzy numbers [3], fuzzy differential equations [4] and some algebraic properties of the quotient space of fuzzy numbers [2].

In this paper, we firstly introduce some results of convex set, supporting and separating hyperplanes in Euclidian space and rough set. Secondly, we introduce a new concept of convexity which is called rough convex set and discuss some geometrical and topological properties for this kind as well as its supporting and separating hyperplanes.

2. Preliminaries

In this section, we introduce some definitions and well-known results of convex and rough sets, which helps us throughout this article. We refer to [11, 14] for the standard material on differential geometry.

Definition 2.1. [8, 11] A set $A \subseteq E^n$ is convex if for each pair of points x and y in A , the line segment joining x and y , denoted by $\sigma(x, y)$, lies completely in A .

Definition 2.2. [11] The intersection of all convex sets including the set A in E^n is called the convex hull of A . In other words, the convex hull of A is the smallest convex set including A .

Definition 2.3. [8] A body B in Euclidean space is called convex body, if it is a closed convex set with the maximal dimension (it has non-empty interior).

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Definition 2.4. [9, 14] A hyperplane H in the Euclidean space E^n is said to be supporting hyperplane to a set B , if H intersects the closure \bar{B} of B and a closed side of H contains \bar{B} . The points in $H \cap \bar{B}$ are called contact points of H with \bar{B} .

Definition 2.5. [14] Two subsets A and B of E^n are separated by a hyperplane H if they are contained in opposite sides of H .

Proposition 2.6. [11] Let A be a nonempty subset of E^n with nonempty interior. If there is a supporting hyperplane for every point $a \in \partial A$, where ∂A is the boundary of A , then A is convex.

Theorem 2.7. [15] Let A and B be two non-empty disjoint convex sets. Then there exists a hyperplane such that A and B are on opposite sides of the hyperplane.

Now, let us introduce and discuss the concept of the upper and lower approximations of a subset A of the universal set $U \subset E^n$. If R is an equivalence relation over U , then U/R denotes to the family of all equivalence classes of R . We will say that A is R -definable if A is the union of some basic categories, otherwise A is R -undefinable. The R -definable sets are called R -exact sets and R -undefinable sets are called R -inexact or R -rough sets [16].

Rough sets can however be defined approximately and to this end we will employ two exact sets referred to as a lower and upper approximations of the set.

Definition 2.8. [16] If (U, R) is an information system, where U is a universe set and R is an equivalence relation. Then the R -lower and the R -upper approximations of any subset A of U with respect to R are defined, respectively, as follows:

$$\begin{aligned} \underline{R}(A) &= \cup\{Y \in U/R : Y \subseteq A\} \\ \bar{R}(A) &= \cup\{Y \in U/R : Y \cap A \neq \phi\}, \end{aligned}$$

Proposition 2.9. [16]

1. The set A is R -definable if and only if $\underline{R}(A) = \bar{R}(A)$.
2. The set A is rough with respect to R if and only if $\underline{R}(A) \neq \bar{R}(A)$.

Remark 2.10. [16] The lower and the upper approximations of A with respect to R can be also presented in an equivalence form as follow:

$$\begin{aligned} \underline{R}(A) &= \{x \in U : [x]_R \subseteq A\} \\ \bar{R}(A) &= \{x \in U : [x]_R \cap A \neq \phi\}, \end{aligned}$$

where $[x]_R$ is the equivalence class containing an element $x \in A$.

Remark 2.11. [16] The R -lower and the R -upper approximations of A are interior and closure operations respectively.

Definition 2.12. [16] The set $\bar{R}(A) - \underline{R}(A)$ is called the R -boundary region of A which is denoted by $BN_R(A)$. Also, the set $(U - \bar{R}(A))$ is called the R -negative region of A and denoted by $NEG_R(A)$. Finally, the lower approximation of A , $\underline{R}(A)$, is called the R -positive region of A and denoted by $POS_R(A)$.

Notation 2.13. [16] The upper approximation of A , $\bar{R}(A)$, is the union of positive and boundary regions of A .

Proposition 2.14. [16] If X and Y are two subsets of the universal set U . Then the following properties are satisfied:

1. $\underline{R}(X) \subseteq X \subseteq \bar{R}(X)$
2. $\underline{R}(\phi) = \bar{R}(\phi) = \phi$. $\underline{R}(U) = \bar{R}(U) = U$
3. $\bar{R}(X \cup Y) = \bar{R}(X) \cup \bar{R}(Y)$

4. $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$
5. If $X \subseteq Y$, then $\underline{R}(X) \subseteq \underline{R}(Y)$ and $\overline{R}(X) \subseteq \overline{R}(Y)$
6. $\overline{R}(X \cup Y) \supseteq \overline{R}(X) \cup \overline{R}(Y)$
7. $\overline{R}(X \cap Y) \subseteq \overline{R}(X) \cap \overline{R}(Y)$
8. $\underline{R}(X^c) = (\overline{R}(X))^c, \overline{R}(X^c) = (\underline{R}(X))^c$, where X^c is the complement of X .
9. $\underline{R}(\underline{R}(X)) = \overline{R}(\underline{R}(X)) = \underline{R}(X)$
10. $\overline{R}(\overline{R}(X)) = \underline{R}(\overline{R}(X)) = \overline{R}(X)$.

Definition 2.15. [13] The set A is said to be roughly-included (rough subset) in the set B , denoted by $A \subseteq_R B$, if $\underline{R}(A) \subseteq \underline{R}(B)$ and $\overline{R}(A) \subseteq \overline{R}(B)$.

Definition 2.16. [13] If (U, R) is an information system and $A \subseteq U$. Then, x is said to be surely (possibly) belongs to A with respect to R , if x belongs to $\underline{R}(A)$ (x belongs to $\overline{R}(A)$), which is denoted by $x \in_s A$ ($x \in_p A$), respectively.

Definition 2.17. [13] The set A is called surely (possibly) subset of B if for all $x \in_s A$ ($x \in_p A$), then $x \in_s B$ ($x \in_p B$) and it is denoted by $A \subseteq_s B$ ($A \subseteq_p B$), respectively.

Corollary 2.18. [13] If A, B, C and D are subsets of U , then it is easy to see that:

1. $A \subseteq_s B$ if and only if $\underline{R}(A) \subseteq \underline{R}(B)$,
2. $A \subseteq_p B$ if and only if $\overline{R}(A) \subseteq \overline{R}(B)$,
3. If $A \subseteq_s B$ and $C \subseteq_s D$, then $A \cap C \subseteq_s B \cap D$,
4. If $A \subseteq_p B$ and $C \subseteq_p D$, then $A \cup C \subseteq_p B \cup D$.

Remark 2.19. [13] In the above Corollary and proposition (2.14), we can see that :

1. (3) is not true for the union and (4) is not true for the intersection, in general.
2. If A is a subset of B , $\underline{R}(A) \subseteq \underline{R}(B)$ and $\overline{R}(A) \subseteq \overline{R}(B)$, then A is surely and possibly subset of B , i.e., $A \subseteq_s B$ and $A \subseteq_p B$.

3. Rough convex set

In this section, we will introduce a new concept of convexity and discuss the necessary and sufficient conditions for rough set to be a rough convex set with some topological and geometrical properties for this kind.

Definition 3.1. A rough set A is called rough convex set with respect to an equivalence relation R if for all pair of points $x, y \in_s A$ ($x, y \in_p A$), there exists a line segment $\sigma(x, y)$ joining x and y and $\sigma(x, y) \subseteq \overline{R}(A)$, see figure (1) and (2)

Remark 3.2. It is clear that, from the above definition, every nonempty convex set is rough convex set, but the converse is not true see figure (1).

Theorem 3.3. The rough set A is convex set if and only if its upper approximation $\overline{R}(A)$ is convex.

Proof. Firstly, Let A be a rough set with respect to an equivalence relation R , and the upper approximation $\overline{R}(A)$ be convex. Then, for all two points x and y are surely (possibly) belong to A we have:

1. If $x, y \in_s A$, then $x, y \in \underline{R}(A)$. Since $\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$, we have $x, y \in \overline{R}(A)$. But $\overline{R}(A)$ is convex from the assumption. Then, the line segment from x to y lies wholly in $\overline{R}(A)$. Thus, A is rough convex set with respect to R .

2. If $x, y \in_p A$ implies that $x, y \in \bar{R}(A)$. From the assumption and $A \subset \bar{R}(A)$ we obtain the line segment from x to y lies wholly in $\bar{R}(A)$, i.e., $\sigma(x, y) \subset \bar{R}(A)$ and hence A is rough convex set with respect to R .

Secondly, Let the set A be a rough convex set with respect to R . Then, from definition (3.1) we obtain that $\bar{R}(A)$ is convex and the proof is complete. \square

Example 3.4. Let a universal set U be a liver in human body. This liver contains a set of cancer focuses A , see figure (3). The surgent wants to make good control on these focuses, so he must determine the minimum area contains these focuses, i.e., the upper approximation must be convex set that enables the surgent to isolate the focus exactly.

Corollary 3.5. The upper approximation of a rough convex set with respect to R is the convex hull of it.

Corollary 3.6. The interior of a rough convex set with respect to R does not necessary be convex, see figure (2).

Corollary 3.7. The closure of a rough convex set must be convex as it is the upper approximation of the rough convex set.

Theorem 3.8. Let A and B be two rough convex sets with respect to R . If $\bar{R}(A \cap B) = \bar{R}(A) \cap \bar{R}(B)$, then the non-empty intersection of A and B is rough convex set with respect to R .

Proof. If $A \cap B \neq \phi$, then for all pair of points $x, y \in_s (A \cap B)$ or $x, y \in_p (A \cap B)$, we have $x, y \in_s A$ ($x, y \in_p A$) and $x, y \in_s B$ ($x, y \in_p B$). Since A is rough convex set, then the line segment $\sigma(x, y)$ from x to y is contained in $\bar{R}(A)$ (i.e. $\sigma(x, y) \subset \bar{R}(A)$). Also, for B , we have $\sigma(x, y) \subset \bar{R}(B)$. Hence, $\sigma(x, y) \subset \bar{R}(A) \cap \bar{R}(B)$ see Figure (4). From $\bar{R}(A) \cap \bar{R}(B) = \bar{R}(A \cap B)$, we obtain the line segment $\sigma(x, y)$ is included in $\bar{R}(A \cap B)$. Hence, $A \cap B$ is rough convex set and the proof is complete. \square

Remark 3.9. If A and B are two rough convex sets with respect to R , then the union $(A \cup B)$ is not necessarily rough convex set, see figure (5) and figure (6).

4. Internal and external supporting hyperplanes of convex rough body

In this section, the supporting and separating hyperplanes of the rough convex set are discussed.

Definition 4.1. A rough set B is called rough convex body if the lower approximation $\underline{R}(B) \neq \phi$ and the upeer approximation $\bar{R}(B)$ is closed convex set.

Definition 4.2. Let A be a rough convex set with respect to an equivalence relation R and $H = \{H_1, H_2\}$ be the set of all supporting hyperplanes of A . Consider the following two cases:

$$\begin{aligned}
 H_1 &= \{h : h \text{ is supporting at any point of } \underline{R}(A) \} \\
 H_2 &= \{h : h \text{ is supporting at any point of } \bar{R}(A) \text{ and } h \cap A = \phi \}
 \end{aligned}$$

Each element $h \in H$ is called rough supporting of A , if H is a rough set.

Definition 4.3. A supporting hyperplane of $\underline{R}(A)$ is called an internal supporting hyperplane of A and a supporting hyperplane of $\bar{R}(A)$ is called an external supporting hyperplane of A .

Theorem 4.4. Let $B \subset U$ be a rough convex body with respect to an equivalence relation R and the point p be an arbitrary point in B . Then one of the following is satisfied:

1. B has an external supporting.
2. B has an internal supporting.

Proof. Since B is rough convex body with respect to R , then $int(B) = \underline{R}(B) \neq \phi$. Hence there is a point $p \in B$ such that $p \in_s B$ or $p \in_p B$, we have $p \in \underline{R}(B)$ or $p \in \bar{R}(B)$.

1. If $p \in \underline{R}(B)$, B is a rough convex body with respect to R and $\text{int}(B) = \underline{R}(B) \neq \phi$, then there is a point $r \in \underline{R}(B)$ such that the line segment $\sigma(p, r) \subset B$ and there is a hyperplane $H(r)$ pass through r . Since $H(r) \cap \underline{R}(B) \neq \phi$ and using definition (4.3), then B has an internal supporting at r .
2. If $p \in \overline{R}(B)$, B is a rough convex body with respect to R and by using theorem (3.3), then $\overline{R}(B)$ is convex. Hence there is a point $q \in \overline{R}(B)$ such that the line segment $\sigma(p, q) \subset \overline{R}(B)$ and there is a hyperplane $H(q)$ pass through q such that $H(q) \cap \sigma(p, q) = q$. Hence, $H(q) \cap \overline{R}(B) \neq \phi$, then B has an external supporting at q .

□

Theorem 4.5. Let A be a rough set with respect to R with non-empty interior. If there is an external supporting hyperplane of A at every point p of the boundary of $\overline{R}(A)$, which is denoted by $\partial\overline{R}(A)$. Then A is rough convex set with respect to R .

Proof. Since A is a rough set with respect to R with non-empty interior, then $\underline{R}(A) \neq \phi$.

Let $p \in \partial\overline{R}(A)$, then there is a tangent hyperplane $T(p)$ for $\partial\overline{R}(A)$ passes through p . From definition (4.3), and $T(p) \cap \partial\overline{R}(A) \neq \phi$, then the hyperplane $T(p)$ is an external supporting of the set A at p .

Assume that q is any point in $\overline{R}(A)$, hence there is a line segment $\sigma(p, q)$, such that $\sigma(p, q) \cap T(p) = p$, which implies that $\sigma(p, q) \subset \overline{R}(A)$. Hence $\overline{R}(A)$ is convex and using theorem (3.3), we have the set A is rough convex with respect to R . □

Theorem 4.6. Let A and B be two rough convex sets with respect to R and the upper approximation of A intersection the upper approximation of B is empty. Then there is a hyperplane separating A and B .

Proof. Since $\overline{R}(A) \cap \overline{R}(B) = \phi$, $A \subset \overline{R}(A)$, and $B \subset \overline{R}(B)$, then $A \cap B = \phi$, i.e; the sets A and B are disjoint. From theorem (3.3) and theorem (2.7), there is a hyperplane H separates A and B . □

5. Figures

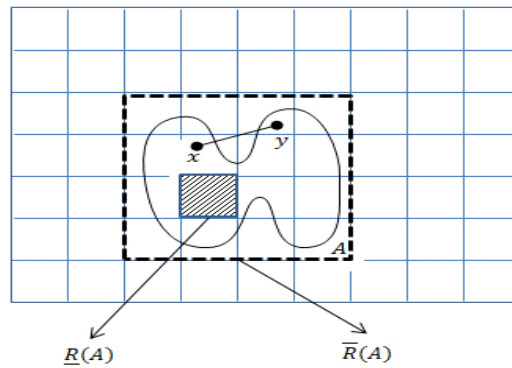


Figure 1: The set A is rough convex set with respect to R .

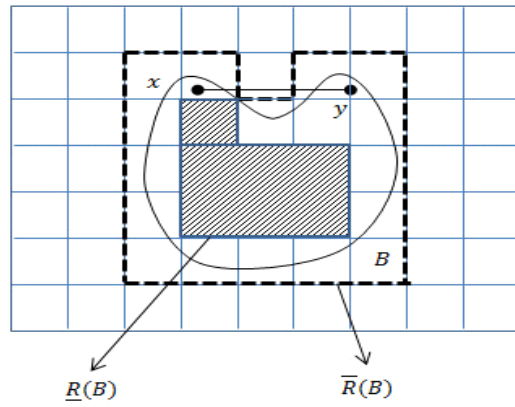


Figure 2: The set B is not rough convex with respect to R .

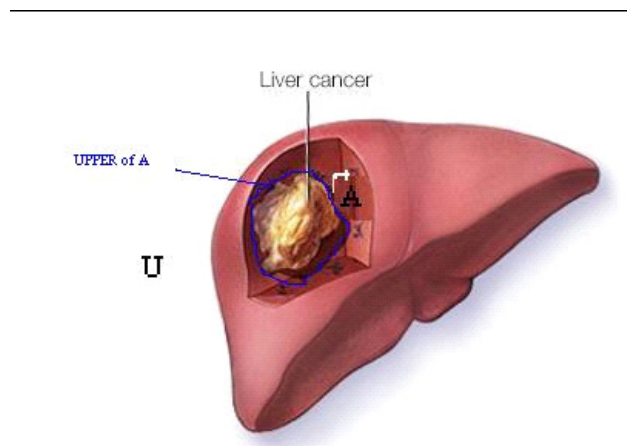


Figure 3: The liver cancer is the rough set A .

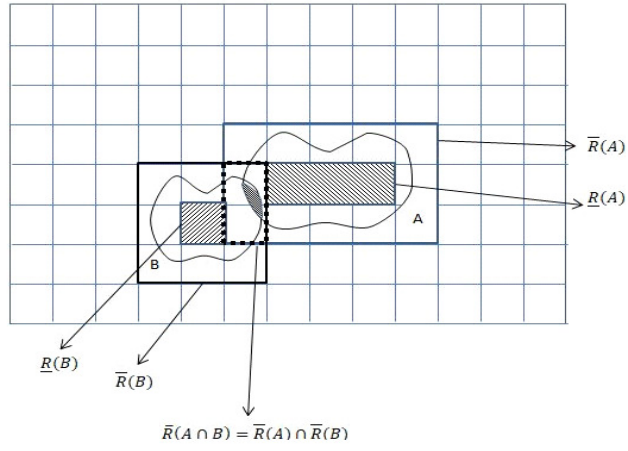


Figure 4: $A \cap B$ is rough convex with respect to R

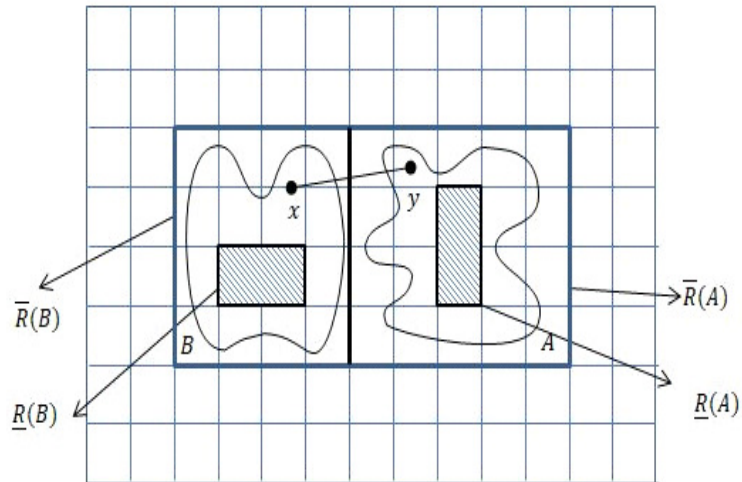
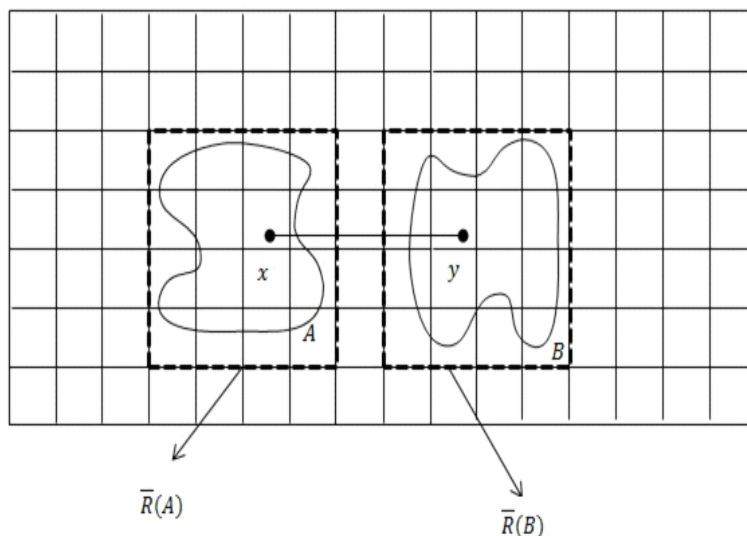


Figure 5: $A \cup B$ is rough convex with respect to R

Figure 6: $A \cup B$ is not rough convex with respect to R

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