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Criteria of Saddle Points for the General Form of Vector Optimization Problem in Complex Space

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Abstract. In this paper, the criteria of the saddle point type for a general form of complex multi-objective programming problems are derived where the objective function of the problem is considered full complex. The criteria with/without differentiability assumptions imposed in the involved functions are developed. It is shown that the efficient solutions can be characterized in terms of saddle points whether the functions are differentiable or not. The obtained theoretical results are generalizations of their real correspondents in literature and complete the results in complex space.

1. Introduction

By a nonlinear muli-objective programming problem in complex space, we understand two or more objective complex functions to be minimized with respect to a complex domination cone over a complex feasible region. The concepts of efficient solutions and saddle points play the crucial roles in solving such problems. In the last decades, a number of papers has extended different aspects of the optimization theory from real to complex space being its important in variant applied fields of the electrical engineering and networks.

In [1], Abrams had given Kuhn-Tucker type saddle point optimality criteria for a scalar complex programming problem, having a real part objective function, under analyticity hypotheses for the objective and constraints functions. In [4], some saddle points optimality criteria were given for that problem without differentiability.

The multi-objective programming problem in complex space had been formulated in [5] by Duca where the definitions of efficient and properly efficient solutions were introduced and characterized in terms of optimal solutions of a related scalar optimization problem. Necessary and sufficient efficiency criteria were established with/without differentiable hypotheses about the functions and many of the criteria were characterized in terms of saddle points.

However, the entries of the objective function in literatures were expressed as real. In [15], we extended the complex problem to contain a full complex objective function, and the optimality conditions were derived. In [7], we introduced the concept of efficient solutions, for a generalized complex vector

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optimization problem in which the objective function was considered complex. The solutions were characterized by optimal solutions of a related scalar problem and the efficiency conditions were investigated. Recently, some concepts of proper efficiency were extended for vector optimization problems in [16]. For more details concerning this type of optimization problems, the attentive reader can refer additional works like [2, 3, 6, 10, 12–14] in real space, and [8, 9, 11] in complex space.

In this work, we introduce the Lagrangian function and saddle point concepts for the extended form of multi-objective programming problems in complex space, taking also into account both the real and the imaginary parts of the objective functions, and investigate saddle point theorems due to Kuhn-Tucker and Fritz John in the cases of differentiability and non-differentiability. The analysis carried out here is based on a complex domination cone.

2. Notations and preliminaries

This section is concerned with gathering basic definitions that will be used throughout the paper. Let A^T, $\overline{A}$ and A^H denote the transpose, conjugate and conjugate transpose for a matrix $A \in \mathbb{C}^{m\times n}$, respectively. A vector $z \in \mathbb{C}^n$ is considered as a column matrix.

Definition 2.1. *A non-empty set $S \subset \mathbb{C}^m$ is said to be*

(1) convex iff $\lambda S + (1-\lambda)S \subseteq S$, for $0 \leqq \lambda \leqq 1$,

(2) cone iff $\lambda S \subseteq S$, for $\lambda \geqq 0$,

(3) polyhedral cone iff $S := \bigcap_{k=1}^{p} H_{u_k}$, where $H_{u_k} := \left\{z \in \mathbb{C}^m : Re\ u_k^H z \geqq 0\right\}$, for some vectors $u_1, \ldots, u_p \in \mathbb{C}^m$ and integer $p > 0$.

Definition 2.2. *The dual S^* of a non-empty set $S \subset \mathbb{C}^m$ is defined as*

$$S^* := \left\{z \in \mathbb{C}^m : w \in S \Longrightarrow Re\ \langle z,\ w\rangle = Re\ z^H w \geqq 0\right\}.$$

Lemma 2.3. *The set $S \subset \mathbb{C}^m$ is closed iff $S = (S^*)^*$.*

Definition 2.4. *A convex cone $S \subset \mathbb{C}^m$ is pointed iff $S \cap (-S) = \{0\}$, and it is solid iff int $S \neq \emptyset$.*

If $S \subset \mathbb{C}^m$ is a pointed closed convex cone, S generates a partial order in $\mathbb{C}^m$ by $z \leqq w$ iff $w - z \in S$.

Definition 2.5. *For a closed convex cone $S \in \mathbb{C}^m$ and $z_0 \in S$, $S(z_0)$, cone S at z_0, is defined as*

$$S(z_0) := \left\{z \in \mathbb{C}^m : Re\ w^H z_0 = 0,\ w \in S^* \Longrightarrow Re\ w^H z \geqq 0\right\}.$$

In the case that S is polyhedral, $S(z_0)$ is then the intersection of closed halfspaces containing z_0 on their boundaries. Note that $z_0 \in S(z_0)$, $-z_0 \in S(z_0)$ and $S \subseteq S(z_0)$.

Definition 2.6. *For a closed convex cone S in $\mathbb{C}^m$, the function $g : \mathbb{C}^n \to \mathbb{C}^m$ is said to be convex with respect to S iff*

$$(1-\lambda)g(z_0) + \lambda g(z) - g((1-\lambda)z_0 + \lambda z) \in S,$$

for all $z_0,\ z \in \mathbb{C}^n$ and $0 \leqq \lambda \leqq 1$.

As g is differentiable, the above condition is equivalent to

$$g(z) - g(z_0) - \nabla_z\, g(z_0)(z - z_0) - \nabla_{\overline{z}}\, g(z_0)(\overline{z} - \overline{z_0}) \in S,$$

where $(j,\ k)^{th}$ elements of the matrices $\nabla_z\, g(z_0)$ and $\nabla_{\overline{z}}\, g(z_0)$ are $\frac{\partial g_j(z_0)}{\partial w_k}$ and $\frac{\partial g_j(z_0)}{\partial \overline{w_k}}$, respectively.

The function g is said to be concave with respect to S iff $-g$ is convex with respect to S.

3. Vector optimization problems in complex space

We consider the generalized vector optimization problem:

$$\begin{aligned} &\textit{min } f(z) = (f_1(z), \ ..., \ f_p(z)) \\ &\textit{subject to } z \in M := \{z \in X : g(z) \in S\}, \end{aligned} \tag{TP}$$

where X is a non-empty subset of $\mathbb{C}^n$, $f : X \to \mathbb{C}^p$ and $g : X \to \mathbb{C}^m$ are two vector-valued functions, and S is a cone in $\mathbb{C}^m$.

The following definitions of complex constraint qualifications due to Kuhn-Tucker and Slater are given in [5].

Definition 3.1. *Let S be a polyhedral cone in $\mathbb{C}^m$. The function g ia said to satisfy complex Kuhn-Tucker constraint qualification at a point $z_0 \in X$ with respect to M (briefly, z_0 is a qualified point) iff for all $z \in \mathbb{C}^n \setminus \{0\}$ satisfying*

$$\nabla_z \, g(z_0)z + \nabla_{\overline{z}} \, g(z_0)\overline{z} \in S(g(z_0)),$$

there exist $\varepsilon > 0$ and a differentiable arc $\beta : [0, \varepsilon[\to \mathbb{C}^n$ such that $\beta(0) = z_0$, $\nabla_\theta \beta(0) = kz$ for some $k > 0$, $\beta(\theta) \in X$ and $g(\beta(\theta)) \in S$ for all $\theta \in [0, \varepsilon[$.

Definition 3.2. *Let S be a subset of $\mathbb{C}^m$. The function g ia said to satisfy complex Slater constraint qualification with respect to M iff int $S \neq \emptyset$ and there exists a point $z \in X$ satisfying $g(z) \in$ int S.*

3.1. Efficient solutions criteria

The concept of efficiency of problem (TP), relying on the decision maker's preferences, is introduced in [7] as the following.

Definition 3.3. *Let $T \subset \mathbb{C}^p$ be a pointed closed convex cone and z_0 be feasible. The point z_0 is called an efficient solution of (TP) with respect to the cone T iff there is no other feasible $z \neq z_0$ such that*

$$f(z_0) - f(z) \in T \setminus \{0\}.$$

In other words, z_0 is an efficient solution of (TP) with respect to T iff

$$[f(M) - f(z_0)] \cap (-T) = \{0\}.$$

Remark 3.4. *If $T = \mathbb{R}^p_+ + i\mathbb{R}^p$, where $\mathbb{R}^p_+$ is the non-negative orthant of $\mathbb{R}^p$, Definition 3.3 descends definition of Duca [5] which recaptures the definition of Pareto point in real space.*

Extensions to Kuhn-Tucker and Fritz John conditions in complex space may be stated as follows.

Definition 3.5. *A vector $(z, \tau, v) \in X \times T^* \times S^*$ is said to satisfy*

(i) complex Fritz John efficiency conditions for problem (TP) iff

$$\tau^H \nabla_z \, f(z) + \tau^T \overline{\nabla_{\overline{z}} \, f(z)} - v^H \nabla_z \, g(z) - v^T \overline{\nabla_{\overline{z}} \, g(z)} = 0, \tag{1}$$

and

$$Re \, v^H g(z) = 0, \tag{2}$$

with $(\tau, v) \neq (0, 0)$.

(ii) complex Kuhn-Tucker efficiency conditions for problem (TP) iff (1) and (2) are satisfied with $\tau \neq 0$.

A complex versions of a Fritz John necessary conditions theorem for efficiency can be provided as follows.

Theorem 3.6. *Let T be a pointed closed convex cone in $\mathbb{C}^p$, S be a polyhedral cone in $\mathbb{C}^m$, f and g be differentiable functions at a point $z_0 \in M$. If z_0 is an efficient solution of problem (TP) with respect to T, then there are $\tau_0 \in T^*$ and $v_0 \in S^*$ such that (z_0, τ_0, v_0) satisfies complex Fritz John efficiency conditions for problem (TP).*

Proof. Equation (2) can be written as

$$v^T \overline{g(z)} + v^H g(z) = 0,$$

or

$$[g(z)]^H v + [g(z)]^T \overline{v} = 0.$$

Assume that there is no $\tau \in T^*$ and $v \in S^*$ satisfying complex Fritz John efficiency conditions. It follows that the system

$$\begin{bmatrix} \nabla_z f(z_0) & 0 \\ \overline{\nabla_{\overline{z}} f(z_0)} & 0 \\ -\nabla_z g(z_0) & g(z_0) \\ -\overline{\nabla_{\overline{z}} g(z_0)} & \overline{g(z_0)} \end{bmatrix}^H \begin{bmatrix} \tau \\ \overline{\tau} \\ v \\ \overline{v} \end{bmatrix} = 0 \,;\; 0 \neq \begin{bmatrix} \tau \\ \overline{\tau} \\ v \\ \overline{v} \end{bmatrix} \in T^* \times \overline{T^*} \times S^* \times \overline{S^*},$$

has no nonzero solution.
By complex Gordan's alternative theorem, and closedness of both S and T, there exist $p \in \mathbb{C}^n$ and $q \in \mathbb{C}$ such that

$$-\begin{bmatrix} \nabla_z f(z_0) & 0 \\ \overline{\nabla_{\overline{z}} f(z_0)} & 0 \\ -\nabla_z g(z_0) & g(z_0) \\ -\overline{\nabla_{\overline{z}} g(z_0)} & \overline{g(z_0)} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \in int\, (T \times \overline{T} \times S \times \overline{S}),$$

i.e.,

$$-\nabla_z f(z_0)\, p \in int\, T, \tag{3}$$

$$-\overline{\nabla_{\overline{z}} f(z_0)}\, p \in int\, \overline{T}, \tag{4}$$

$$\nabla_z g(z_0)\, p - g(z_0)\, q \in int\, S, \tag{5}$$

and

$$\overline{\nabla_{\overline{z}} g(z_0)}\, p - \overline{g(z_0)}\, q \in int\, \overline{S}. \tag{6}$$

Conjugating (6) and adding to (5) yields

$$\nabla_z g(z_0)\, p + \nabla_{\overline{z}} g(z_0)\, \overline{p} - g(z_0)[q + \overline{q}] \in int\, S.$$

Now, since g is differentiable at z_0, then we can choose $t > 0$ sufficiently small, such that $z_0 + tp \in X$ and

$$g(z_0 + tp) = g(z_0) + t\nabla_z g(z_0)\, p + t\nabla_{\overline{z}} g(z_0)\, \overline{p} + o(t),$$

where $\frac{o(t)}{t} \longrightarrow 0$, as $t \longrightarrow 0$.
Then

$$\begin{aligned} g(z_0 + tp) &= [1 + t(q + \overline{q})]g(z_0) + t[\nabla_z g(z_0)\, p + \nabla_{\overline{z}} g(z_0)\, \overline{p} - g(z_0)(q + \overline{q})] + o(t) \\ &\in S + int\, S \subset S, \end{aligned}$$

thus $z_0 + tp$ is feasible to problem (TP).
Similarly, conjugating (4) and adding to (3) gives

$$-[\nabla_z f(z_0)\, p + \nabla_{\overline{z}} f(z_0)\, \overline{p}] \in int\ T.$$

The differentiability of f at z_0 implies, for sufficiently small $t > 0$,

$$f(z_0 + tp) = f(z_0) + t\nabla_z f(z_0)\, p + t\nabla_{\overline{z}} f(z_0)\, \overline{p} + o(t),$$

and so

$$f(z_0 + tp) - f(z_0) = t[\nabla_z f(z_0)\, p + \nabla_{\overline{z}} f(z_0)\, \overline{p}] + o(t) \in -int\ T,$$

which means

$$f(z_0) - f(z_0 + tp) \in int\ T \subset T\backslash\{0\}.$$

This contradicts the assumption that z_0 is an efficient solution of (TP) with respect to T, and whence the proof is completed. □

The following is a generalization of a Kuhn-Tucker necessary conditions theorem for efficiency in complex space which illustrates that the cases at which $\tau_0 = 0$, are excluded from the above theorem.

Theorem 3.7. *Let T be a pointed closed convex cone in $\mathbb{C}^p$, S be a solid polyhedral cone in $\mathbb{C}^m$, f and g be differentiable functions at a feasible qualified point z_0. If z_0 is an efficient solution of (TP) with respect to T, then there are $\tau_0 \in T^*$ and $v_0 \in (S(g(z_0)))^* \subset S^*$ such that (z_0, τ_0, v_0) satisfies complex Kuhn-Tucker efficiency conditions for problem (TP).*

Proof. Since $g(z_0) \in S(g(z_0))$ and $-g(z_0) \in S(g(z_0))$, then $v_0 \in (S(g(z_0)))^*$ implies $Re\, v_0^H g(z_0) \geqq 0$ and $Re\, v_0^H g(z_0) \leqq 0$, and thus (2) holds.
In order to prove (1), assume inversely that there is no $0 \neq \tau \in T^*$ and $v \in (S(g(z_0)))^*$ satisfying (1), it follows that there is no a solution for the system

$$\begin{bmatrix} \nabla_z f(z_0) \\ \nabla_{\overline{z}} f(z_0) \end{bmatrix}^H \begin{bmatrix} \tau \\ \overline{\tau} \end{bmatrix} - \begin{bmatrix} \nabla_z g(z_0) \\ \nabla_{\overline{z}} g(z_0) \end{bmatrix}^H \begin{bmatrix} v \\ \overline{v} \end{bmatrix} = 0,$$

$$0 \neq \begin{bmatrix} \tau \\ \overline{\tau} \end{bmatrix} \in T^* \times \overline{T^*}, \begin{bmatrix} v \\ \overline{v} \end{bmatrix} \in (S(g(z_0)))^* \times \overline{(S(g(z_0)))^*}.$$

By using complex Motzkin's alternative theorem, it follows that there exists a solution $p \in \mathbb{C}^n$ to the system

$$-\begin{bmatrix} \nabla_z f(z_0) \\ \nabla_{\overline{z}} f(z_0) \end{bmatrix} p \in int\ (T \times \overline{T}), \begin{bmatrix} \nabla_z g(z_0) \\ \nabla_{\overline{z}} g(z_0) \end{bmatrix} p \in S(g(z_0)) \times \overline{S(g(z_0))},$$

hence

$$-\nabla_z f(z_0)\, p - \nabla_{\overline{z}} f(z_0)\, \overline{p} \in int\ T. \tag{7}$$

Similarly,

$$\nabla_z g(z_0)\, p + \nabla_{\overline{z}} g(z_0)\, \overline{p} \in S(g(z_0)). \tag{8}$$

Since z_0 is qualified, then (8) leads to the existence of a differentiable vector function $\beta(\theta)$ defined on $[0, \varepsilon[$ such that $\beta(0) = z_0$, $\nabla_\theta \beta(0) = kp$, $\beta(\theta) \in X$ and $g(\beta(\theta)) \in S$ for $\theta \in [0, \varepsilon[$, ε, $k > 0$. From feasibility of $\beta(\theta)$ for $\theta \in [0, \varepsilon[$ and efficiency of z_0 to (TP), we have for $\theta \in [0, \varepsilon[$,

$$f(\beta(0)) - f(\beta(\theta)) \notin T\backslash\{0\}.$$

The differentiability of f at $z_0 = \beta(0)$ yields

$$-\nabla_z f(\beta(0))(\beta(\theta) - \beta(0)) - \nabla_{\bar{z}} f(\beta(0))(\overline{\beta(\theta)} - \overline{\beta(0)}) - o(\theta) \notin T\backslash\{0\},$$

where $\frac{o(\theta)}{\theta} \longrightarrow 0$, as $\theta \longrightarrow 0$.
Also the differentiability of β at $\theta = 0$ implies

$$-\nabla_z f(\beta(0))(\nabla_\theta \beta(0)\theta) - \nabla_{\bar{z}} f(\beta(0))(\overline{\nabla_\theta \beta(0)}\theta) - o(\theta) \notin T\backslash\{0\}.$$

Thus

$$-\nabla_z f(\beta(0))(\nabla_\theta \beta(0)) - \nabla_{\bar{z}} f(\beta(0))(\overline{\nabla_\theta \beta(0)}) \notin T\backslash\{0\}.$$

Since $\beta(0) = z_0$ and $\nabla_\theta \beta(0) = kp$ for some $k > 0$, we get

$$-\nabla_z f(z_0)\, p - \nabla_{\bar{z}} f(z_0)\, \bar{p} \notin T\backslash\{0\}.$$

This contradiction with (7) ensures that there exist $0 \neq \tau_0 \in T^*$ and $v_0 \in (S(g(z_0)))^*$ such that (1) is satisfied. □

3.2. Saddle point criteria

The saddle point concept gives an alternative way to characterize the efficient solutions and to establish efficiency conditions. In this subsection we give the saddle point criteria for nonlinear complex multi-objective programming problem (TP) with and without differentiability hypothesis of both f and g. We start off by defining the Lagrangian function $L(z, \tau, v)$ corresponding to problem (TP):

$$L(z, \tau, v) = Re\, \tau^H f(z) - Re\, v^H g(z)$$

for $z \in X$, $\tau \in \mathbb{C}^p$ and $v \in \mathbb{C}^m$.

Definition 3.8. *A vector $(z_0, \tau_0, v_0) \in X \times T^* \times S^*$ is said to be a*

(i) complex Fritz John saddle point for the Lagrangian function $L(z, \tau, v)$ of problem (TP) iff

$$L(z_0, \tau_0, v) \leqq L(z_0, \tau_0, v_0) \leqq L(z, \tau_0, v_0), \tag{9}$$

for all $z \in X$ and $v \in S^*$ with $(\tau_0, v_0) \neq (0, 0)$,

(ii) complex Kuhn-Tucker saddle point for the Lagrangian function $L(z, \tau, v)$ of problem (TP) iff (9) is satisfied for all $z \in X$ and $v \in S^*$ with $\tau_0 \neq 0$.

3.2.1. Saddle point conditions without differentiability

We establish the basic saddle point theorems for non-differentiable multi-objective programming problem (TP). A complex version of a Fritz John saddle point necessary conditions theorem for efficiency can be stated as follows.

Theorem 3.9. *Let X be a non-empty convex subset of $\mathbb{C}^n$, T be a pointed closed convex cone in $\mathbb{C}^p$ and S be a solid polyhedral cone in $\mathbb{C}^m$. Assume f is convex with respect to T and g is concave with respect to S. If z_0 is an efficient solution of problem (TP) with respect to T, then there exist $\tau_0 \in T^*$ and $v_0 \in S^*$ such that (z_0, τ_0, v_0) is a complex Fritz John saddle point for the Lagrangian function $L(z, \tau, v)$ of problem (TP).*

Proof. If z_0 is an efficient solution of (TP) with respect to T, it follows that the system

$$\begin{cases} f(z_0) - f(z) \in int\, T, \\ g(z) \in S, \\ z \in X \end{cases}$$

is inconsistent.
In view of Theorem 5.1 of [8], there exist $\tau_0 \in T^*$ and $v_0 \in S^*$ with $(\tau_0, v_0) \neq (0, 0)$ such that

$$Re\ \tau_0^H[f(z_0) - f(z)] + Re\ v_0^H g(z) \leqq 0, \tag{10}$$

for all $z \in X$.
By letting $z = z_0$, the above inequality yields $Re\ v_0^H g(z_0) \leqq 0$.
But since $v_0 \in S^*$ and $g(z_0) \in S$, we have $Re\ v_0^H g(z_0) \geqq 0$.
It follows that

$$Re\ v_0^H g(z_0) = 0. \tag{11}$$

From (10) and (11),

$$\begin{aligned} L(z_0, \tau_0, v_0) - L(z, \tau_0, v_0) &= Re\ \tau_0^H[f(z_0) - f(z)] - Re\ v_0^H[g(z_0) - g(z)] \\ &= Re\ \tau_0^H[f(z_0) - f(z)] - Re\ v_0^H g(z) \leqq 0. \end{aligned}$$

This proves the second inequality of (9).
On the other hand, using (11), we have for all $v \in S^*$,

$$L(z_0, \tau_0, v) - L(z_0, \tau_0, v_0) = Re\ (v_0^H - v^H)g(z_0) = -Re\ v^H g(z_0) \leqq 0.$$

This proves the first inequality of (9). □

In the cases where $\tau_0 = 0$, it is naturally obvious that the objective function f has moved out from the conditions. Consequently, we state complex versions of Kuhn-Tucker saddle point necessary and sufficient conditions theorems for efficiency.

Theorem 3.10. *Let X, T, S, f, g be as in Theorem 3.9 above, and further g satisfy Slater constraint qualification with respect to M. If z_0 is an efficient solution of (TP) with respect to T, then there exist $\tau_0 \in T^*$ and $v_0 \in S^*$ such that (z_0, τ_0, v_0) is a complex Kuhn-Tucker saddle point for the Lagrangian function $L(z, \tau, v)$ of problem (TP).*

Proof. If z_0 is an efficient solution of (TP) with respect to T, it follows from Theorem 3.9 above that there exist $\tau_0 \in T^*$ and $v_0 \in S^*$ with $(\tau_0, v_0) \neq (0, 0)$ such that (9) is satisfied and moreover, $Re\ v_0^H g(z_0) = 0$.
If $\tau_0 = 0$, then $v_0 \neq 0$. The second inequality of (9) implies

$$Re\ v_0^H g(z) \leqq 0, \tag{12}$$

for all $z \in X$.
Since g satisfies Slater constraint qualification with respect to M, there exists a point $z_1 \in X$ such that $g(z_1) \in int\ S$ and so $Re\ v_0^H g(z_1) > 0$ which contradicts (12). Therefore $\tau_0 \neq 0$. □

The following theorem provides us that if we find a saddle point (z_0, τ_0, v_0) of the Lagrangian function $L(z, \tau, v)$ of problem (TP), then z_0 is an efficient solution of the problem. The theorem does not require the convexity assumptions on the complex problem.

Theorem 3.11. *Let S be a closed convex cone in $\mathbb{C}^m$ and T be a pointed closed convex cone in $\mathbb{C}^p$. If a vector (z_0, τ_0, v_0) is a complex Kuhn- Tucker saddle point for the Lagrangian function $L(z, \tau, v)$ of problem (TP) with $\tau_0 \in int\ T^*$, then z_0 is an efficient solution of (TP) with respect to T.*

Proof. Since (z_0, τ_0, v_0) is a complex Kuhn- Tucker saddle point for the Lagrangian function $L(z, \tau, v)$ of problem (TP), then for all $v_0 \in S^*$, the first inequality of (9) implies

$$Re\ (v_0^H - v^H)g(z_0) \leqq 0. \tag{13}$$

By letting $v = w + v_0$, the above inequality yields $Re\ w^H g(z_0) \geqq 0$ for all $w \in S^*$.
Therefore, $g(z_0) \in (S^*)^*$, which means, from $(S^*)^* = S$, that z_0 is a feasible point to problem (TP).

On the second hand, putting $v = 0$ in (13) gives $Re\ v_0^H g(z_0) \leqq 0$.
Because $v_0 \in S^*$ and $g(z_0) \in S$, we get $Re\ v_0^H g(z_0) \geqq 0$.
Thus $Re\ v_0^H g(z_0) = 0$.
Consequently, for any feasible point z, the second inequality of (9) leads to

$$Re\ \tau_0^H[f(z_0) - f(z)] \leqq -Re\ v_0^H g(z) \leqq 0. \tag{14}$$

If we suppose that z_0 is not an efficient solution, then there exists $\hat{z} \in M$ such that $f(z_0) - f(\hat{z}) \in T\backslash\{0\}$.
Since $\tau_0 \in int\ T^*$, we get

$$Re\ \tau_0^H[f(z_0) - f(\hat{z})] > 0,$$

which contradicts the inequality (14). Hence, z_0 is an efficient solution of (TP) with respect to T. □

3.2.2. Saddle point conditions with differentiability

In this subsection, necessary and sufficient conditions for a saddle point for the Lagrangian function $L(z, \tau, v)$ of problem (TP) are established in the presence of convexity and differentiability.

The following is a generalization of a Fritz John sufficient conditions theorem for a saddle point.

Theorem 3.12. *Let T be a pointed closed convex cone in $\mathbb{C}^p$, S be a solid polyhedral cone in $\mathbb{C}^m$, f and g be differentiable functions at a feasible point z_0 at which f is convex with respect to T and g is concave with respect to S. If (z_0, τ_0, v_0) satisfies complex Fritz John efficiency conditions for problem (TP), then (z_0, τ_0, v_0) is a complex Fritz John saddle point for the Lagrangian function $L(z, \tau, v)$ of problem (TP).*

Proof. If (z_0, τ_0, v_0) satisfies complex Fritz John efficiency conditions for problem (TP), then equation (2) leads, directly for all $v \in S^*$, to

$$L(z_0, \tau_0, v) - L(z_0, \tau_0, v_0) = Re\ (v_0^H - v^H)g(z_0) = -Re\ v^H g(z_0) \leqq 0.$$

This proves the first inequality of (10).
In order to prove the second inequality, we find, for any $z \in M$, that

$$L(z_0, \tau_0, v_0) - L(z, \tau_0, v_0) = -Re\ \tau_0^H[f(z) - f(z_0)] + Re\ v_0^H[g(z) - g(z_0)].$$

By using the definition of differentiability and convexity of both f and g, we obtain

$$\begin{aligned} L(z, \tau_0, v_0) - L(z_0, \tau_0, v_0) \quad \leqq \quad & -Re\ [\tau_0^H \nabla_z f(z_0) - \tau_0^T \overline{\nabla_{\bar{z}} f(z_0)}](z - z_0) \\ & +Re\ [v_0^H \nabla_z g(z_0) - v_0^T \overline{\nabla_{\bar{z}} g(z_0)}](z - z_0), \end{aligned}$$

for any z.
Thus by equation (1), we have

$$L(z_0, \tau_0, v_0) \leqq L(z, \tau_0, v_0).$$

□

Of course Theorem 3.9 still true in the case that f and g are differentiable.

Corollary 3.13. *Let T, S, f and g be as in Theorem 3.12 above. If z_0 is an efficient solution of (TP) with respect to T, then there exist $\tau_0 \in T^*$ and $v_0 \in (S(g(z_0)))^* \subset S^*$ such that (z_0, τ_0, v_0) is a complex Fritz John saddle point for the Lagrangian function $L(z, \tau, v)$ of problem (TP).*

Proof. The proof follows directly from Theorems 3.6 and 3.12 above. □

Now the theorem of Kuhn-Tucker sufficient conditions is introduced.

Theorem 3.14. *Let T, S, f and g be as in Theorem 3.12 above. If (z_0, τ_0, v_0) satisfies complex Kuhn-Tucker efficiency conditions for problem (TP), then (z_0, τ_0, v_0) is a complex Kuhn-Tucker saddle point for the Lagrangian function $L(z, \tau, v)$ of problem (TP).*

Proof. The proof is analogous as of Theorem 3.12 above. □

Corollary 3.15. *Let T, S, f, g be as in Theorem 3.12 above, and moreover z_0 be qualified. If z_0 is an efficient solution of (TP) with respect to T, then there exist $\tau_0 \in T^*$ and $v_0 \in (S(g(z_0)))^* \subset S^*$ such that (z_0, τ_0, v_0) is a complex Kuhn-Tucker saddle point for the Lagrangian function $L(z, \tau, v)$ of problem (TP).*

Proof. The proof follows from Theorems 3.7 and 3.14 above. □

Theorem 3.16. *Let T and S be as in Theorem 3.12 above. If (z_0, τ_0, v_0) is a complex Kuhn-Tucker saddle point for the Lagrangian function $L(z, \tau, v)$ of problem (TP), then (z_0, τ_0, v_0) satisfies complex Kuhn-Tucker efficiency conditions for problem (TP).*

Proof. If (z_0, τ_0, v_0) is a complex Kuhn-Tucker saddle point for the Lagrangian function $L(z, \tau, v)$ of problem (TP), it follows, as in the proof of Theorem 3.11 above, that z_0 is feasible to (TP) and for any feasible point z,

$$Re\ \tau_0^H[f(z) - f(z_0)] \geqq 0.$$

This means that z_0 solves the scalar problem

$$\begin{aligned} &min\ Re\ \tau_0^H f(z) \\ &subject\ to\ z \in M, \end{aligned}$$

with $0 \neq \tau_0 \in T^*$.
This implies, by using of Theorem 2.4.11 in [5] the existence of $v_0 \in (S(g(z_0)))^*$ such that

$$\nabla_z\ \tau_0^H f(z_0) + \overline{\nabla_{\bar{z}}\ \tau_0^H f(z_0)} - v_0^H \nabla_z\ g(z_0) - v_0^T \overline{\nabla_{\bar{z}}\ g(z_0)} = 0,$$

and

$$Re\ v_0^H g(z_0) = 0.$$

Since τ_0 is a constant vector, then equations (1) and (2) follow. Hence (z_0, τ_0, v_0) satisfies complex Kuhn-Tucker efficiency conditions for problem (TP). □

4. Conclusion

In this paper, the notions of Fritz John and Kuhn-Tucker saddle point have been introduced, and the criteria of the saddle point have been established for a generalized form of the multi-objective programming problem in complex space taking into consideration the two parts of the objective vector. The efficient solutions of the problem has been characterized in terms of saddle points whether the functions are differentiable or not. The results generalize the parallel ones in real and complex spaces.

References

[1] R.A. Abrams, Nonlinear programming in complex space: sufficient conditions and duality, J. Math. Anal. Appl. 38 (1972) 619-632.
[2] T. Antczak, Sufficient optimality conditions for semi-infinite multiobjective fractional programming under (ϕ, ρ)-V-invexity and generalized (ϕ, ρ)-V-invexity, Filomat 30(14)(2016)3649-3665.
[3] T. Antczak, Saddle point criteria in semi-infinite minimax fractional programming under (ϕ, ρ)-invexity, Filomat 31(9) (2017) 2557-2574.
[4] D.I. Duca, Saddle point optimality criteria of nonlinear programming in complex space without differentiability, Studia Univ. Babes-Bolyai Math. 25(4) (1980) 39-46.
[5] D.I. Duca, Multicriteria Optimization in Complex Space, Casa Cartii de Stiinta, Cluj-Napoca 2005.

[6] D.I. Duca, L. Lupsa, Saddle points for vector valued functions: existence, necessary and sufficient theorems, J. Glob. Optim. 53 (2012) 431–440.
[7] M.E Elbrolosy, Efficiency for a generalized form of vector optimization problems in complex space, Optimization DOI: 10.1080/02331934.2015.1104680, 65(6) (2016) 1245-1257.
[8] O. Ferrero, On nonlinear programming in complex spaces, J. Math. Anal. Appl. 164(2) (1992) 399-416.
[9] R.N. Kaul, B Gupta, Multi-objective programming in complex space, Z. Angew. Math. Mech. 61(11) (1981) 599-601.
[10] A. Kumar, PK Garg, Mixed saddle point and its equivalence with an efficient solution under generalized (V, ρ)-invexity, Appl. Math. 6 (2015) 1630-1637.
[11] H.L. Lai, T.Y. Huang, Complex analysis methods related an optimization problem with complex variables, Eur. J. Pure Appl. Math. 3(6) (2010) 989-1005.
[12] M.C. Maciel, S.A. Santos, G.N. Sottosanto, On the Fritz John saddle point problem for differentiable multiobjective optimization, OPSEARCH. DOI: 10.1007/s12597-016-0253-x, 53 (2016) 917–933.
[13] S.K. Mishra, Y. Singh, R.U. Verma, Saddle point criteria in nonsmooth semi-infinite minimax fractional problems, J. Syst. Sci . Complex. 31(2)(2018) 446-462.
[14] Y. Singh, Y. Pandey, S.K. Mishra, Saddle point optimality criteria for mathematical programming problems with equilibrium constraints, Oper. Res. Lett. 45(3) (2017) 254-258.
[15] E.A. Youness, M.E. Elbrolosy, Extension to necessary optimality conditions in complex programming, J. Appl. Math. Comput. 154(1) (2004) 229-237.
[16] E.A. Youness, M.E. Elbrolosy, Extension of proper efficiency in complex space, Math. Meth. Appl. Sci. DOI: 10.1002/mma.4715. 41 (2018) 5792-5800.