Constraction of a Core Regular Double $MS$-Algebra

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Abstract. In this paper, we introduce and characterize a core regular double $MS$-algebra. A construction of a core regular double $MS$-algebra $M^{[2]}$ via a de Morgan algebra $M$ is given. A one to one correspondence between the class of de Morgan algebras and the class of core regular double $MS$-algebras is obtained. According to such construction we investigate many properties of a core regular double $MS$-algebra deal with subalgebras, homomorphisms, atoms and dual atoms. A description of an atomic core regular double $MS$-algebra is established. Also, we discuss some properties of a complete core regular double $MS$-algebra.

1. Introduction

De Morgan Stone algebra (briefly $MS$-algebra) is introduced by T.S. Blyth and J.C. Varlet [8] as a common properties of a de Morgan algebra and a Stone algebra. T.S. Blyth and J.C. Varlet [9] described the lattice of all subclasses of the class $MS$ of all $MS$-algebras which contains twenty subclasses, for examples, the class $S$ of all Stone algebras and the class $M$ of all de Morgan algebras. Also, T.S. Blyth and J.C. Varlet [10] presented the class $DMS$ of all double $MS$-algebras which containing the class $DS$ of all double Stone algebras. J.C. Varlet [18] studied a regular variety of type $(2,2,1,1,0,0)$. T. Katriňák [16] presented a construction of a regular double Stone algebra from a suitable Boolean algebra $B$ and a filter $F$ of $B$. S.D. Comer [14] proved the existence and uniqueness of perfect extensions of a regular double stone algebra using Katriňák’s construction [16]. Recently, A. Badawy [2] introduced and characterized the class of double $MS$-algebras satisfying the generalized complement property (briefly $DMS^{gr}$-algebras) which includes the class of double $MS$-algebras satisfying the complement property presented by L. Congwen [13]. Also, A. Badawy [2] gave a construction of $DMS^{gr}$-algebras generalizing the construction due to T. Katriňák [11] for regular double Stone algebras. Many important properties of $MS$-algebras and double $MS$-algebras deal with homomorphisms, subalgebras, filters and congruences are studied in [3-7].

In this paper, we introduce and characterize a subclass of the class of double $MS^{gr}$-algebras which is called core regular double $MS$-algebras. In fact the class $CRDMS$ of all core regular double $MS$-algebras includes the class $CRDS$ of all core regular double Stone algebras due to R. Kumar et al. [17]. A construction of a core regular double $MS$-algebra from a suitable de Morgan algebra is obtained. Also, we construct
a core regular double Stone algebra from a suitable Boolean algebra. We observe that there is a one to one correspondence between the class $\mathbf{M}$ of all de Morgan algebras and the class $\text{CRDMS}$. We give many applications of such construction. Characterizations of homomorphisms and subalgebras of core regular double $\text{MS}$-algebras are obtained. We describe atoms and dual atoms of a core regular double $\text{MS}$-algebra by using this construction. A description of atomic core regular double $\text{MS}$-algebras is given. We observe that the completeness of a core regular double $\text{MS}$-algebra $L$ depends on only the completeness of its skeleton $L^\circ$, in particular the last two applications of our construction are to discuss complete homomorphisms and complete subalgebras of core regular double $\text{MS}$-algebras.

2. Preliminaries

In this section, we recall certain definitions and important results. We refer the reader to the references [5], [7], [8], [9], [10], [11], [12] and [15] as a guide references.

Definition 2.1. [15] An algebra $(L; \wedge, \vee)$ is said to be a lattice if for every $a, b, c \in L$, it satisfies the following properties:
1. $a \wedge a = a, a \vee a = a$ (Idempotency),
2. $a \wedge b = b \wedge a, a \vee b = b \vee a$ (Commutativity),
3. $(a \wedge (b \wedge c)) = (a \wedge b) \wedge (a \wedge c)$, $(a \vee (b \vee c)) = (a \vee b) \vee (a \vee c)$ (Associativity),
4. $(a \wedge (b \vee c)) = (a \wedge b) \vee (a \wedge c)$, $(a \vee (b \wedge c)) = (a \vee b) \wedge (a \vee c)$ (Absorption).

If a lattice $L$ has a greatest element (denoted by 1) and a smallest element (denoted by 0), then $L$ is said to be a bounded lattice.

Definition 2.2. [15] A lattice $L$ is called distributive if it satisfies either of the following equivalent distributive laws:
1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, for all $a, b, c \in L$.

Definition 2.3. [11] A de Morgan algebra is an algebra $(L; \lor, \land, ^{-}, 0, 1)$ of type $(2,2,1,0,0)$ where $(L; \lor, \land, 0, 1)$ is a bounded distributive lattice and the unary operation $^-$ of pseudocomplementation has the properties that $x \land a = 0 \iff x \leq a^-$ and $x^+ \lor x^+ = 1$.

Definition 2.4. [12] A Stone algebra is a universal algebra $(L; \lor, \land, ^{-}, 0, 1)$ of type $(2,2,1,0,0)$, where $(L; \lor, \land, 0, 1)$ is a bounded distributive lattice and the unary operation $^*$ of pseudocomplementation has the properties that $x \land a = 0 \iff x \leq a^*$ and $x^+ \lor x^* = 1$.

Definition 2.5. [16] A dual Stone algebra is a universal algebra $(L; \lor, \land, ^{+}, 0, 1)$ of type $(2,2,1,0,0)$, where $(L; \lor, \land, 0, 1)$ is a bounded distributive lattice and the unary operation $^*$ of dual pseudocomplementation has the properties that $x \lor a = 1 \iff x \geq a^*$ and $x^+ \land x^+ = 0$.

Definition 2.6. [16] A double Stone algebra is an algebra $(L; ^{+}, ^{-})$ such that $(L; ^{-})$ is a Stone algebra, $(L; ^{+})$ is a dual Stone algebra and for every $x \in L$, $x^{++} = x^+, x^{++} = x^{++}$.

Definition 2.7. [8] An $\text{MS}$-algebra is an algebra $(L; \lor, \land, ^{-}, 0, 1)$ of type $(2,2,1,0,0)$ where $(L; \lor, \land, 0, 1)$ is a bounded distributive lattice and the unary operation $^-$ satisfies:
$$x \leq x^{\circ}, (x \land y)^{-} = x^{\circ} \lor y^{\circ}, 1^{\circ} = 0.$$
\[ x \geq x^+, (x \wedge y)^+ = x^+ \lor y^+, 0^+ = 1. \]

**Definition 2.9.** [10] A double MS-algebra is an algebra \((L; \vee, *, +, 0, 1)\) such that \((L; \vee^*, *)\) is an MS-algebra, \((L; +)\) is a dual MS-algebra, and the unary operations \(*, +\) are linked by the identities \(x^* = x^+\) and \(x^+ = x^*\), for all \(x \in L\).

The class DS of all double Stone algebras is a subclass of the class DMS of all double MS-algebras and is characterized by the identities \(x \wedge x^* = 0\) and \(x \lor x^* = 1\).

Throughout this paper, we adopt the following rules of computation in a double MS-algebra \((L; \vee, \wedge, \vee^*, *, +, 0, 1)\) (see [8] and [10]).

**Theorem 2.10.** For any two elements \(a, b\) of a double MS-algebra \(L\), we have

1. \(0^\wedge = 0\) and \(1^\wedge = 1\)
2. \(a \leq b \Rightarrow b^+ \leq a^+\)
3. \(a^\wedge = a^\vee\)
4. \(a^\wedge = a^\vee\)
5. \((a \vee b)^+ = a^\wedge \land b^\wedge\)
6. \((a \vee b)^+ = a^\vee \lor b^\vee\)
7. \((a \land b)^+ = a^\vee \land b^\vee\)

**Theorem 2.11.** [9] Let \((L; \vee, \wedge, \vee^*, *, +, 0, 1)\) be a double MS-algebra. Then

1. \(L^\wedge = \{x \in L : x = x^\wedge\} = \{x \in L : x = x^+\} = \{x \in L : x = x^*\}\) is a de Morgan subalgebra of \(L\),
2. \(L^\vee = \{x \in L^\wedge : x \leq x^{\wedge}\}\) is an increasing subset (dual order ideal) of \(L\),
3. \(L^ \vee \wedge = \{a \land a^\wedge : a \in L\} = L^\wedge \cap L^\vee\).

**Definition 2.12.** [15] Let \(L = (L; \vee, \wedge, \vee^*, *, +, 0, 1)\) and \(L_1 = (L_1; \vee, \wedge, 0, 1)\) be bounded lattices. A mapping \(f : L \rightarrow L_1\) is called a \([0, 1]\)-lattice homomorphism if \(f(0) = 0, f(1) = 1, f(x \lor y) = f(x) \lor f(y)\) and \(f(x \land y) = f(x) \land f(y)\) for all \(x, y \in L\). A \([0, 1]\)-lattice homomorphism is called an isomorphism if \(f\) is a bijective mapping, in this case, we call \(L\) and \(L_1\) are isomorphic lattices and write \(L \cong L_1\).

### 3. Core regular double MS-algebras

In this section, we introduce the concept of core regular double MS-algebras that includes the class of core regular double Stone algebras.

**Definition 3.1.** [2] A double MS-algebra \((L; \vee, \wedge, *, +, 0, 1)\) is said to be a regular double MS-algebra (or simply RDMS-algebra) if for any \(x, y \in L\), \(x^\wedge = y^\wedge\) and \(x^+ = y^+\) imply \(x = y\).

A relation \(\Phi^*_x\) defined by \((x, y) \in \Phi^*_x \Leftrightarrow x^\wedge = y^\wedge\) and \(x^+ = y^+\) is a congruence relation on a double MS-algebra \(L\).

A characterization of regular double MS-algebra in terms of the congruence \(\Phi^*_x\) is given in the following.

**Theorem 3.2.** Let \(L\) be a double MS-algebra. Then \(L\) is regular if and only if \(\Phi^*_x = \omega\), where \(\omega = \{(x, x) : x \in L\}\).

**Proof.** Let \(L\) be a regular double MS-algebra. Let \((x, y) \in \Phi^*_x\). Then \(x^\wedge = y^\wedge\) and \(x^+ = y^+\) and hence by regularity of \(L\), we get \(x = y\). Therefore \(\Phi^*_x = \omega\). Conversely, let \(\Phi^*_x = \omega\). Let \(x^\wedge = y^\wedge\) and \(x^+ = y^+\). Then \((x, y) \in \omega\). So, \(x = y\) and \(L\) is regular.  

**Definition 3.3.** [1] Let \(L\) be an MS-algebra. An element \(d \in L\) is called a dense element of \(L\) if \(d^\wedge = 0\), the set of all dense elements of \(L\) is denoted by \(D(L)\).
Definition 3.4. Let $L$ be a dual MS-algebra. An element $d \in L$ is called a dual dense element of $L$ if $d^+ = 1$, the set of all dual dense elements of $L$ is denoted by $D(L)$.

Lemma 3.5. Let $L$ be a double MS-algebra. Then $D(L)$ is a filter of $L$ and $D(L)$ is an ideal of $L$.

Proof. It is observed that $D(L)$ is a filter of $L$ (see [1]). Let $x, y \in D(L)$. Then $x^+ = y^+ = 1$. So by Theorem 2.10(5d), $(x \lor y)^+ = x^+ \land y^+ = 1$. Hence $x \lor y \in D(L)$. Now, let $z \leq x \in D(L)$ and $z \in L$. Then by Theorem 2.10(2d), $z^+ \geq x^+ = 1$. This means that $z \in D(L)$. Therefore $D(L)$ is an ideal of $L$.

Definition 3.6. Let $L$ be a double MS-algebra. The set $K(L) = D(L) \cap D(L)$ is called the core of $L$.

Definition 3.7. A core regular double MS-algebra (briefly CRDMS-algebras) is a regular double MS-algebra with non empty core, that is, $K(L) \neq \emptyset$.

Lemma 3.8. Let $L$ be a CRDMS-algebra. Then $|K(L)| = 1$.

Proof. Let $k_1, k_2 \in K(L)$. Then $k_1^o = k_2^o = 0$ and $k_1^+ = k_2^+ = 1$. Hence by regularity of $L$, $k_1 = k_2$. Therefore $K(L)$ has a unique element and hence $|K(L)| = 1$.

We will denote the core element by $k$. The core element $k$ will play an important role throughout the rest of this paper.

Example 3.9.

(1) Every core regular double Stone algebra is a core regular double MS-algebra.

(2) Consider the bounded distributive lattice $L$ in Figure 1. Define unary operations $\circ,^+$ on $L$ by

$$k^o = x^o = y^o = z^o = 0, d^o = b^o = b, c^o = a^o = a, 1^o = 0$$

and

$$k^+ = c^+ = d^+ = 0^+ = 1, y^+ = b^+ = b, x^+ = a^+ = a, 1^+ = 0.$$  

It is observed that $(L^o,^+)$ is a regular double MS-algebra. We have $D(L) = \{k, x, y, 1\}$, $D(L) = \{0, c, d, k\}$ and $K(L) = \{k\}$. Then $L$ represents a CRDMS-algebra. Since $c^o \wedge c \neq 0$ and $c^+ \lor c \neq 1$ then $L$ is not a double Stone algebra. This example deduce that $\text{CRDS} \subsetneq \text{CRDMS}$.

(3) Consider the bounded distributive lattice $L$ in Figure 1. Define unary operations $\circ,^+$ on $L$ by
Example 3.12. Consider \( L \) the following.

\[
x^0 = 1^0 = 0, k^0 = y^0 = c, d^0 = b^0 = a, a^0 = b, c^0 = y, 1^0 = 0
\]

and

\[
d^+ = 0^+ = 1, k^+ = c^+ = y, x^+ = a^+ = b, b^+ = a, y^+ = c, 1^+ = 0.
\]

Clearly \((L^0,+^0)\) is a regular double MS-algebra. We have \( D(L) = \{x, 1\}, D^+(L) = \{0, d\} \) and \( K(L) = \phi \). Then \( L \) is not a core regular double MS-algebra.

Definition 3.10. [2] A double MS-algebra \( L \) is called a double MS-algebra satisfying the generalized complement property (or briefly DMS\( ^{\phi} \)-algebra) if

1. \( L \) is a regular double MS-algebra,
2. Given \( a, b \in L^\circ \) and a filter \( F \) of \( L^\circ \) containing \( L^{\text{cov}} \) such that \( a \leq b \) and \( a \lor b^\circ \in F \), then there exists an element \( x \in L \) such that \( x^+ = a \) and \( x^{\circ} = b \).

Lemma 3.11. Every CRDMS-algebra with core element \( k \) is a DMS\( ^{\phi} \)-algebra.

Proof. We can choose \( F = L^\circ \). Let \( a, b \in L^\circ \) be such that \( a \leq b \). Clearly \( a \lor b^\circ \in F \) as \( F = L^\circ \). Set \( x = (a \lor k) \land b \). Then \( x^+ = a \) and \( x^{\circ} = b \). Then condition (ii) of Definition 3.9 holds. Then \( L \) is a DMS\( ^{\phi} \)-algebra. \( \square \)

Now we illustrate an example to show that the converse of the above Lemma is not true, that is, the class CRDMS of all core regular double MS-algebras is a proper subclass of the class of DMS\( ^{\phi} \) of all DMS\( ^{\phi} \)-algebras.

Example 3.12. Consider \( L = \{0 < c < a < d < 1\} \) be a five element chain and \( a = a^0 = d^0 = d^+ = 1^0 = 0, 0^+ = c^+ = 1 \). It is clear that \((L^0,+^0)\) is a regular double MS-algebra, \( L^{\circ} = \{0, a, 1\} \) and \( L^{\text{cov}} = \{a, 1\} \). A filter \( F = \{a, 1\} \) of \( L^\circ \) contains \( L^{\text{cov}} \). It is observed that \((L^0,+^0)\) is a DMS\( ^{\phi} \)-algebra. Since \( D(L) = \{1, d\} \) and \( D^+(L) = \{0, c\} \) then \( K(L) = D(L) \cap D^+(L) = \phi \). Then \( L \) is not a CRDMS-algebra.

4. The construction

The construction of a core regular double MS-algebra from a suitable de Morgan algebra is given in the following.

Theorem 4.1. (Construction Theorem)

Let \((M; \lor, \land, 0, 1)\) be a de Morgan algebra. Then

\[
M^{[2]} = \{(a, b) \in M \times M : a \leq b\}
\]

is a core regular double CRDMS-algebra with core element \((0, 1)\), whenever

\[
\begin{align*}
(a, b) \lor (c, d) &= (a \lor c, b \lor d), \\
(a, b) \land (c, d) &= (a \land c, b \land d), \\
(a, b)^+ &= (a, a), \\
(a, b)^0 &= (b, b) \\
0_{M^{[2]}} &= (0, 0) \\
1_{M^{[2]}} &= (1, 1).
\end{align*}
\]

Moreover, \( M \) is isomorphic to \( D(M^{[2]}) \) as well as \( D(M^{[2]}) \) as lattices.
Proof. T.S. Blyth and J.c. Varlet [10] observed that \( M^{[2]} = (M^{[2]}; \lor, \land, ^+, (0, 0), (1, 1)) \) is a double MS-algebra. Let \((a, b)\)\(^+\) = \((c, d)\)\(^+\) and \((a, b)^* = (c, d)^*\). Then \((\bar{b}, \bar{b}) = (d, d)\) and \((\bar{a}, \bar{a}) = (c, c)\) imply \(a = c\) and \(b = d\). Thus \((a, b) = (c, d)\). Therefore \(M^{[2]}\) is a regular double MS-algebra. By Theorem 3.5, \(D(M^{[2]})\) is a filter of \(M^{[2]}\) and \(\overline{D(M^{[2]})}\) is an ideal of \(M^{[2]}\). We observe that

\[
D(M^{[2]}) = \{(x, y) \in M^{[2]} : (x, y)^+ = (0, 0)\} = \{(x, y) \in M^{[2]} : (\bar{y}, \bar{y}) = (0, 0)\} = \{(x, y) \in M^{[2]} : \bar{y} = 0\} = \{(x, y) \in M^{[2]} : y = 1\} = \{(x, 1) \in M^{[2]} : x \in M\},
\]

and

\[
\overline{D(M^{[2]})} = \{(x, y) \in M^{[2]} : (x, y)^+ = (1, 1)\} = \{(x, y) \in M^{[2]} : (\bar{x}, \bar{x}) = (1, 1)\} = \{(x, y) \in M^{[2]} : \bar{x} = 1\} = \{(x, y) \in M^{[2]} : x = 0\} = \{(0, y) \in M^{[2]} : y \in M\}.
\]

Now, we prove that the element \((0, 1)\) is the core element of \(M^{[2]}\). Since \((0, 1)^+ = (0, 0)\), then \((0, 1) \in D(L)\). We claim that \(D(L)\) is a principal filter of \(M^{[2]}\) generated by \((0, 1)\). Let \((x, 1)\) be any element of \(D(L)\). Then \(x \geq 0\) implies \((x, 1) \geq (0, 1)\). Therefore \((0, 1)\) is the smallest element of \(D(L)\) and \(D(L) = [(0, 1)]\). Similarly, we can prove that \(\overline{D(L)}\) is a principal ideal of \(M^{[2]}\) generated by \((0, 1)\). Thus \(D(L) = [(0, 1)]\). Consequently, the core of \(M^{[2]}\) is \(K(M^{[2]}) = D(M^{[2]}) \cap \overline{D(M^{[2]})} = [(0, 1)] \cap [(0, 1)] = [(0, 1)]\). To prove that the lattices \(M\) and \(D(M^{[2]})\) are isomorphic, define a map \(f : M \to D(M^{[2]})\) by \(f(a) = (a, 1)\). Clearly \(f(0) = (0, 1)\) and \(f(1) = (1, 1)\). For every \(a, b \in M\), we have

\[
f(a \land b) = (a \land b, 1) = (a, 1) \land (b, 1) = f(a) \land f(b).
\]

Also, \(f(a \lor b) = f(a) \lor f(b)\). Therefore \(f\) is a \((0, 1)\)-lattice homomorphism. Obviously \(f\) is a bijective map. Therefore \(f\) is an isomorphism and \(M \cong D(M^{[2]})\). Similarly, we can deduce that \(M \cong \overline{D(M^{[2]})}\) under the lattice isomorphism \(a \mapsto (0, a)\). Therefore \(D(M^{[2]})\) and \(\overline{D(M^{[2]})}\) are also isomorphic lattices.

We illustrate the above construction on the following example.

**Example 4.2.** Let \(M\) be the four-element de Morgan algebra (see Fig. 2).

![Diagram](M.png)

**Figure 2:** \(M\) is a de Morgan algebra.

Using the construction Theorem (4.1), we obtain a core regular double MS-algebra \(M^{[2]}\) in figure 3.

Where \(^\circ\) and \(^*\) are given as follows:
Proof. Let $B$ be a Boolean subalgebra of $B$. If $B$ is a homomorphism if $f$ is a mapping $f : M 	o B$. Theorem 2.11(1).

Then clearly a map $a \mapsto (a, a)$ is an isomorphism of $M$ onto $(M^{[2]})^\circ$. Consequently, $M \cong (M^{[2]})^\circ$. □

For a core regular double Stone algebra, we have.

Corollary 4.5. If $B = (B; \lor, \land, \neg, 0, 1)$ is a Boolean algebra, then $B^{[2]}$ is a core regular double Stone algebra and $(B^{[2]})^\circ$ is a Boolean subalgebra of $B^{[2]}$, where $\neg$ is a unary operation of complementation on $B$.

Proof. For any element $x$ of a Boolean algebra $B$, we have the facts $x \lor x' = 1$ and $x \land x' = 0$. Since each Boolean algebra is a de Morgan algebra, then according to the above Theorem 4.1, $B^{[2]} = \{(a, b) : a \leq b\}$ is a core regular double Stone algebra with core element $(0, 1)$. We prove that $(a, b) \land (a, b)^\circ = (0, 0)$ and $(a, b) \lor (a, b)^\circ = (1, 1)$ for all $(a, b) \in B^{[2]}$.

$$(a, b) \land (a, b)^\circ = (a, b) \lor (b', b') = (a \land b', b \land b')$$

$$= (a \land b', 0) \in B^{[2]} \text{ as } b \land b' = 0$$

$$= (0, 0) \text{ as } a \land b' \leq 0 \Rightarrow a \land b' = 0$$

$$(a, b) \lor (a, b)^\circ = (a, b) \lor (a', a') = (a' \lor a', b \lor a')$$

$$= (1, b \lor a') \in B^{[2]} \text{ as } a \lor a' = 1$$

$$= (1, 1) \text{ as } 1 \leq b \lor a' \Rightarrow b \lor a' = 1.$$
Therefor $B^{[2]}$ is a core double Stone algebra. By Theorem 2.11(1), $(B^{[2]})^\omega$ is a de Morgan subalgebra of $B^{[2]}$. From corollary 4.2, $(B^{[2]})^\omega = \{(a, a) : a \in B\}$. Since $(a, a) \lor (a, a)^\circ = (1, 1)$ and $(a, a) \land (a, a)^\circ = (0, 0)$ for all $(a, a) \in (B^{[2]})^\omega$, then $(B^{[2]})^\omega$ is a Boolean subalgebra of $B^{[2]}$. 

**Definition 4.6.** A mapping $f : L \rightarrow L_1$ of a CRDMS-algebra $L$ with core element $k$ into a CRDMS-algebra $L_1$ with core element $k_1$ is called a homomorphism if

1. $f(k) = k_1$, $f(x^\circ) = (f(x))^\circ$ and $f(x^+) = (f(x))^+$. A bijective homomorphism of CRDMS-algebras is called isomorphism.

**Theorem 4.7.** A CRDMS-algebra $L$ with core element $k$ is isomorphic to $L^{\omega[2]}$.

**Proof.** Since $L^{\omega}$ is a de Morgan algebra, then by Theorem 4.1, $L^{\omega[2]} = \{(a, b) \in L^{\omega} \times L^{\omega} : a \leq b\}$ is a CRDMS-algebra with core element $(0, 1)$. Define $\phi : L \rightarrow L^{\omega[2]}$ by $\phi(x) = (x^+, x^\omega)$. Since $x^+ \leq x^\omega$, then $\phi(x) \in L^{\omega[2]}$. To prove that $\phi$ is an injective map, let $\phi(x) = \phi(y)$. Then $(x^+, x^\omega) = (y^+, y^\omega)$. Hence $x^+ = y^+$ and $x^\omega = y^\omega$. Then by Theorem 2.10(3), we have $x^+ = x^++ = y^++ = y^+$ and $x^\omega = x^\omega = y^\omega = y^\omega$. By regularity of $L$, $x = y$. Now, we prove that $\phi$ is surjective. For all $(a, b) \in L^{\omega[2]}$, we have $a \leq b$ and $a, b \in L^{\omega}$. Set $d = (a \lor k) \land b$. Using (6),(6),(7),(7) of Theorem 2.10, and $k^+ = 1, k^\omega = 0$, we have

$$d^+ = ((a \lor k) \land b)^+ = (a^+ \lor k^+) \land b^+ = (a \lor 0) \land b = a \land b = a,$$

and

$$d^\omega = ((a \lor k) \land b)^\omega = (a^\omega \lor k^\omega) \land b^\omega = (a \lor 1) \land b = 1 \land b = b.$$

Thus $\phi(d) = (d^+, d^\omega) = (a, b)$. Therefore $\phi$ is a bijective mapping. Clearly, $\phi(0) = (0, 0)$, $\phi(1) = (1, 1)$ and $\phi(k) = (0, 1)$. For all $x, y \in L$, we get

$$\phi(x \land y) = ((x \land y)^+, (x \land y)^\omega)$$

$$= (x^+ \land y^+, x^\omega \land y^\omega) \text{ by Theorem 2.10(7), (7)}$$

$$= (x^+, x^\omega) \land (y^+, y^\omega)$$

$$= \phi(x) \land \phi(y),$$

$$\phi(x \lor y) = ((x \lor y)^+, (x \lor y)^\omega)$$

$$= (x^+ \lor y^+, x^\omega \lor y^\omega) \text{ by Theorem 2.10(6), (6)}$$

$$= (x^+, x^\omega) \lor (y^+, y^\omega)$$

$$= \phi(x) \lor \phi(y).$$

Therefore $\phi$ is a $[0, 1]$-lattice homomorphism. Now, for all $x \in L$ we have

$$\phi(x^+) = (x^{++}, x^{\omega\omega})$$

$$= (x^{++}, x^{++}) \text{ as } x^{++} = x^+$$

$$= (x^+, x^{\omega\omega})^+$$

$$= \phi(x)^+,$$

$$\phi(x^\omega) = (x^{\omega\omega}, x^{\omega\omega})$$

$$= (x^{\omega\omega}, x^{\omega\omega}) \text{ as } x^{\omega\omega} = x^\omega$$

$$= (x^+, x^{\omega\omega})^+$$

$$= \phi(x)^\omega.$$

Then $\phi$ preserves $+$ and $\circ$. Consequently, $\phi$ is an isomorphism of a CRDMS-algebra $L$ onto a CRDMS-algebra $L^{\omega[2]}$. So $L \cong L^{\omega[2]}$. 

From the above discussion, we immediately get the following important result.
Theorem 4.8. There is a one to one correspondence between the class of core regular double MS-algebras and the class of de Morgan algebras.

Now, we give another useful characterization of a core regular double MS-algebra.

Theorem 4.9. Let $L$ be a RDMS-algebra. Then the following statements are equivalent.

(i) $L$ has core element,

(ii) For $a, b \in L^\infty$ and $a \leq b$, there exists an element $x \in L$ such that $x^{+} = a$ and $x^{\infty} = b$.

Proof. (i) $\Rightarrow$ (ii): Let $L$ has core element $k$. Let $a \leq b, a, b \in L^\infty$. Set $x = (a \lor k) \land b$. It is clear that $x^{+} = a$ and $x^{\infty} = b$. Then condition (ii) holds.

(ii) $\Rightarrow$ (i): Let $L$ be a regular double MS-algebra satisfying the condition (ii). Then by Theorem 4.1, $L^{[2]} = \{(a, b) \in L^\infty \times L^\infty : a \leq b\}$ is a core regular double MS-algebra with core element $(0, 1)$. Define a map $\phi : L \to L^{[2]}$ by $\phi(x) = (x^{+}, x^{\infty})$. In the proof of Theorem 4.7, we show that $\phi$ is an injective mapping of $L$ into $L^{[2]}$. Now we show that $\phi$ is a surjective mapping using (ii). Let $(a, b) \in L^{[2]}$. Then $a \leq b$ and $a, b \in L^\infty$. By (ii) there exists $x \in L$ such that $x^{+} = a$ and $x^{\infty} = b$. Then $\phi(x) = (x^{+}, x^{\infty}) = (a, b)$. Therefore $\phi$ is a bijective mapping of $L$ onto $L^{[2]}$. We claim that the inverse image of the core element $(0, 1)$ of $L^{[2]}$ is the core element of $L$. Suppose that $d = \phi^{-1}(0, 1)$. Then $\phi(d) = (0, 1)$ implies $(d^{+}, d^{\infty}) = (0, 1)$. Thus $d^{+} = 0$ and $d^{\infty} = 1$. It follows that $d^{+} = 1$ and $d^{\infty} = 0$. This deduce that $d$ is the core element of $L$. \qed

Now, for any de Morgan algebra $M = (M; \lor, \land^\ast, 0, 1)$ and any filter $F$ of $M$ containing $M^\prime$, the author proved in [2] that $(L; \lor, \land^\ast, 0, 1, (1, 1))$ forms a $DMS^\infty$-algebra, where

$$L = (M, F) = \{(a, b) : a \leq b, a \lor b \in \bar{F}\}$$

and the operations $\lor, \land^\ast$ and $+$ are given as in Theorem 4.1.

The following result gives the necessary and sufficient condition for a DMS$^\infty$-algebra $L = (M, F)$ to become a core regular double MS-algebra.

Theorem 4.10. A DMS$^\infty$-algebra $L = (M, F)$ is a CRDMS-algebra iff $F = M$.

Proof. Let $F = M$. Then $L = (M, M) = M^{[2]}$. Thus by Theorem 4.1, $L = M^{[2]}$ is a core regular double MS-algebra with core element $(0, 1)$. Conversely, Let $L = (M, F)$ is a core regular double MS-algebra with core element $(a, b)$. Then $(a, b) \in D(L) \cap \bar{D}(L)$ and $a \lor b \in \bar{F}$. Hence $(a, b)^{+} = (0, 0)$ and $(a, b)^{\infty} = (1, 1)$. It follows that $(\bar{b}, \bar{b}) = (0, 0)$ and $(\bar{a}, \bar{a}) = (1, 1)$, respectively. Then $\overline{b} = 0$ and $\overline{a} = 1$ implies $b = 1$ and $a = 0$, respectively. Then $(a, b) = (0, 1)$ and hence 0 = 0 $\lor \overline{1} = a \lor \overline{b} \in \bar{F}$. Therefore $F = M$. \qed

5. Applications of the construction Theorem

We start this section with subalgebras of a CRDMS-algebra.

Definition 5.1. A bounded sublattice $H$ of a CRDMS-algebra $L$ with core element $k$ is said to be a subalgebra of $L$ if

1. $x^{+}, x^{\infty} \in H$ for all $x \in H$,
2. $k \in H$.

It is observed that $\{0, k, 1\}$ is the smallest subalgebra of any CRDMS-algebra $L$.

The subalgebras of a CRDMS-algebra $L$ in example 3.9(2) are $\{0, k, 1\}, \{0, c, a, k, x, 1\}, \{0, d, b, k, y, 1\}$ and $L$.

Theorem 5.2. There is one to one correspondence between the set of all subalgebras of a de Morgan algebra $M$ and the set of all subalgebras of a CRDMS-algebra $M^{[2]}$. 

Proof. Let $M_1$ be a subalgebra of $M$. We prove that a set $M_1^{[2]} = \{(a, b) \in M_1 \times M_1 : a \leq b\}$ is a subalgebra of $M^{[2]}$. Since $0, 1 \in M_1$, then $(0, 0), (1, 1)$ and $(0, 1)$ belong to $M_1^{[2]}$. For every $(a, b), (c, d) \in M_1^{[2]}$: Then $a, b, c, d \in M_1$ and hence $a \lor c, b \lor d, a \land c, b \land d \in M_1$. Thus
\[
(a, b) \lor (c, d) = (a \lor c, b \lor d) \in M_1^{[2]} \quad \text{as} \quad a \leq b \lor d,
\]
\[
(a, b) \land (c, d) = (a \land c, b \land d) \in M_1^{[2]} \quad \text{as} \quad a \leq b \land d.
\]
Therefore $M_1^{[2]}$ is a bounded sublattice of $M^{[2]}$. Let $(a, b) \in M_1^{[2]}$. Then $a, b \in M_1$ and hence $\bar{a}, \bar{b} \in M_1$ (as $M_1$ is a subalgebra of $M$). Thus
\[
(a, b)^+ = (\bar{a}, \bar{a}) \in M_1^{[2]},
\]
\[
(a, b)^* = (\bar{b}, \bar{b}) \in M_1^{[2]}.
\]
The core element $(0, 1)$ of $M^{[2]}$ belongs to $M_1^{[2]}$. Therefore $M_1^{[2]}$ is a subalgebra of $M^{[2]}$.

Conversely, let $L_1$ be a subalgebra of $M^{[2]}$. Consider a subset $M_1$ of $M$ as follows:
\[
M_1 = \{a \in M : (a, a) \in L_1\}.
\]
We claim that $M_1$ is a subalgebra of $M$. Since $(0, 0), (1, 1) \in L_1$, then $0, 1 \in M_1$. Let $x, y \in M_1$. Hence $(x, x), (y, y) \in L_1$. Now
\[
(x, x) \land (y, y) = (x \land y, x \land y) \in L_1 \Rightarrow x \land y \in M_1,
\]
\[
(x, x) \lor (y, y) = (x \lor y, x \lor y) \in L_1 \Rightarrow x \lor y \in M_1,
\]
\[
(x, x)^\circ = (x,x) \in L_1 \Rightarrow x \in M_1.
\]
Therefore $M_1$ is a subalgebra of a de Morgan algebra $M$. \(\Box\)

A clarification of the correspondence between subalgebras of a de Morgan algebra $M$ and a CRDMS-algebra $M^{[2]}$ is provided in the following example.

Example 5.3. Consider a de Morgan algebra $M$ and a CRDMS-algebra $M^{[2]}$ in example 4.2. We observe that the subalgebras $M_1 = [0, 1], M_2 = [0, a, 1], M_3 = [0, b, 1], M_4 = M$ of a de Morgan algebra $M$ are corresponding to the subalgebras $M_2^{[2]} = \{(0, 0), (0, 1), (1, 1)\}$,
\[
M_1^{[2]} = \{(0, 0), (0, a), (0, 1), (a, a), (a, 1), (1, 1)\},
\]
$M_2^{[2]} = \{(0, 0), (0, b), (b, b)(0, 1), (b, 1), (1, 1)\}$,
\[
M_4^{[2]} = M^{[2]} \quad \text{of a CRDMS-algebra} \quad M^{[2]}{\text{, respectively.}}
\]

Definition 5.4. A subalgebra $L_1$ of a CRDMS-algebra $L$ is said to be a Stone subalgebra if $x^\circ \lor x^{**} = 1$ and $x^+ \land x^{++} = 0$ for all $x \in L_1$.

Corollary 5.5. There is one to one correspondence between the set of all Boolean subalgebras of a de Morgan algebra $M$ and the set of all Stone subalgebras of the CRDMS-algebra $M^{[2]}$.

Proof. Let $M_1$ be a Boolean subalgebra of a de Morgan algebra $M$. Then $x \land \bar{x} = 0$ and $x \lor \bar{x} = 1$ for all $x \in M_1$. Theorem 5.2 shows that $M_1^{[2]}$ is a subalgebra of $M^{[2]}$. We need to prove that the Stone identities hold in $M_1^{[2]}$. For all $(x, y) \in M_1^{[2]}$, we get
\[
(x, y)^+ \land (x, y)^{++} = (\bar{x}, \bar{x}) \land (x, x) = (\bar{x} \land x, \bar{x} \land x) = (0, 0)
\]
\[
(x, y)^\circ \lor (x, y)^{***} = (\bar{y}, \bar{y}) \lor (y, y) = (\bar{y} \lor y, \bar{y} \lor y) = (1, 1).
\]

Conversely, let $L_1$ be a Stone subalgebra of $M^{[2]}$. Then by Theorem 5.2, $M_1 = \{a \in M : (a, a) \in L_1\}$ is a subalgebra of a de Morgan algebra $M$. To prove $M_1$ is a Boolean subalgebra of $M$, we have to show that $a \lor \bar{a} = 1$ and $a \land \bar{a} = 0$ for $a \in M_1$. Let $a \in M_1$. Then $(a, a) \in L_1$. Since $L_1$ is a Stone subalgebra of $M^{[2]}$ then $(1, 1) = (a, a)^\circ \lor (a, a)^{**} = (\bar{a} \lor a, \bar{a} \lor a)$. Therefore $a \lor \bar{a} = 1$. Also, $(0, 0) = (a, a)^\circ \land (a, a)^{***} = (\bar{a} \land a, \bar{a} \land a)$ implies that $\bar{a} \land a = 0$. \(\Box\)
It is known that the center \( Z(M) = \{ x \in M : x \lor x = 1 \} \) of a de Morgan algebra \( M \) forms a Boolean subalgebra of \( M \).

**Corollary 5.6.** \((Z(M))^{2} \) is the greatest Stone subalgebra of \( M^{[2]} \).

**Example 5.7.** Consider a de Morgan algebra \( M \) and a CRDMD-algebra \( M^{[2]} \) in example 4.2. The center \( Z(M) = \{ 0, 1 \} \) of \( M \) correspond to the greatest Stone subalgebra \( M^{[2]} = \{ (0, 0), (0, 1), (1, 1) \} \) of a CRDMS-algebra \( M^{[2]} \).

Let \( h : L \rightarrow L_{1} \) be a homomorphism of a CRDMS-algebra \( L \) into a CRDMS-algebra \( L_{1} \). We will denote by \( h_{L^\circ}, h_{D(L)} \) and \( h_{D(L)} \) to the restrictions of \( h \) on \( L^\circ, D(L) \) and \( D(L) \), respectively. It is easy to show the following.

**Lemma 5.8.** Let \( h : L \rightarrow L_{1} \) be a homomorphism of a CRDMS-algebra \( L \) into a CRDMS-algebra \( L_{1} \). Then

1. \( h_{L^\circ} \) is a homomorphism of a de Morgan algebras \( L^\circ \) into a de Morgan algebra \( L_{1}^\circ \).
2. \( h_{D(L)} \) is a \([0, 1]\)-lattice homomorphism of a lattice \( D(L) \) into a lattice \( D(L_{1}) \).
3. \( h_{D(L)} \) is a \([0, 1]\)-lattice homomorphism of a lattice \( D(L) \) into a lattice \( D(L_{1}) \).

**Theorem 5.9.** Let \( M \) and \( M_{1} \) be de Morgan algebras. If \( f : M \rightarrow M_{1} \) is a homomorphism, then a map \( h : M^{[2]} \rightarrow M_{1}^{[2]} \) defined by \( h(a, b) = (f(a), f(b)) \) is a homomorphism of a CRDMS-algebra \( M^{[2]} \) into a CRDMS-algebra \( M_{1}^{[2]} \). Conversely, if \( h : M^{[2]} \rightarrow M_{1}^{[2]} \) is a homomorphism of CRDMS-algebras, then \( f : M \rightarrow M_{1} \) defined by \( f(a) = b \Leftrightarrow h_{M^{[2]}}(a, a) = (b, b) \) for all \( a \in M \) is homomorphism of de Morgan algebras.

Proof. Let \( f : M \rightarrow M_{1} \) be a homomorphism between de Morgan algebras \( M \) and \( M_{1} \). It is ready seen that a map \( h : M^{[2]} \rightarrow M_{1}^{[2]} \) defined by \( h(a, b) = (f(a), f(b)) \) is a homomorphism of a DMS-algebra \( M^{[2]} \) into a DMS-algebra \( M_{1}^{[2]} \). Since \( h(0, 1) = (f(0), f(1)) = (0, 1) \), then \( h \) is a homomorphism of CRDMS-algebra \( M^{[2]} \) into a CRDMS-algebra \( M_{1}^{[2]} \).

Conversely, let \( h : M^{[2]} \rightarrow M_{1}^{[2]} \) be a homomorphism of \( M^{[2]} \) into \( M_{1}^{[2]} \). Define a map \( f : M \rightarrow M_{1} \) as follows:

\[
 f(a) = b \Leftrightarrow h_{M^{[2]}}(a, a) = h(a, a) = (b, b) \text{ for all } a \in M. 
\]

Using Lemma 5.8(1), \( h(a, a) = (b, b) \in M_{1}^{[2]} \). Then \( f(a) = b \in M_{1} \) for all \( a \in M \). Since \( h(0, 0) = (0, 0) \) and \( h(1, 1) = (1, 1) \), then \( f(0) = 0 \) and \( f(1) = 1 \), respectively. For all \( x, y \in M \), by Lemma 5.8(1), we have \( h(x, x) = (x_{1}, x_{1}) \) and \( h(y, y) = (y_{1}, y_{1}) \). Then \( f(x) = x_{1} \) and \( f(y) = y_{1} \). Now,

\[
 h(x \land y, x \land y) = h((x, x) \land (y, y)) = h(x, x) \land h(y, y) = (x_{1}, x_{1}) \land (y_{1}, y_{1}) = (x_{1} \land y_{1}, x_{1} \land y_{1}) .
\]

Then \( f(x \land y) = x_{1} \land y_{1} = f(x) \land f(y) \). Using similar way, we get \( f(x \lor y) = f(x) \lor f(y) \). Since \( h((x, x)') = (h(x, x))' \), then \( h(x, x)' = (x_{1}, x_{1})' = (x_{1}, x_{1}) \). Hence \( f(x)' = x_{1} = f(x) \). Therefore \( f \) is a homomorphism of de Morgan algebra \( M \) into a de Morgan algebra \( M_{1} \).

**Definition 5.10.** [10] An element \( a \) of a lattice \( L \) with \( 0 \) is said to be an atom of \( L \) if \( a \neq 0 \) and for any \( x \in L, x \leq a \), then either \( x = 0 \) or \( x = a \). Dually, an element \( d \) of a lattice \( L \) with \( 1 \) is said to be a coatom (dual atom) of \( L \) if \( d \neq 1 \) and for any \( x \in L, d \leq x \), then either \( x = 1 \) or \( x = d \). Let \( At(L) \) be the set of all atoms of \( L \). A lattice \( L \) with zero element is said to be atomic if for every nonzero element \( x \) of \( L \), there exists an atom \( a \) of \( L \) such that \( a \leq x \).

Now, we obtain many properties of atoms and coatoms of CRDMS-algebras that should be useful for further discussion.

**Lemma 5.11.** For a CRDMS-algebra \( M^{[2]} \), we have
Theorem 5.13. The following.

Example 5.15. Consider a de Morgan algebra \(M\) and a CRDMD-algebra \(M\). In the following example, we clarify the properties of atoms and coatoms of \(M\).

Proof. (1). Suppose that \(x = (a, b) \in M^{[2]}\) is an atom of \(M^{[2]}\) if and only if \(0 < b, a < 0\) and \(y = (b, a, b) \in M^{[2]}\). Thus \(y < x\), which contradicts with the fact that \(x\) is an atom of \(M^{[2]}\). Hence \(b\) is an atom of \(M\). Now, since \(a \leq b\) and \(b\) is an atom of \(M\), we have \(a = 0\) or \(a = 1\). If \(a = b\) then \((0, 0) < (a, b)\), which contradicts with that \((a, b)\) is an atom of \(M^{[2]}\). Then \(a = 0\). Conversely, let \(b\) be an atom of \(M\) and \(b = 1\). Then we have to show that \(x = (0, b)\) is an atom of \(M^{[2]}\). Let \(y = (c, d)\) is an element of \(M^{[2]}\) such that \(y \leq x\). Then \(c = 0\) and \(d \leq b\). Since \(b\) is an atom of \(M\), then \(d = 0\) and \(y = (0, 0)\). Therefore \(x = (0, b)\) is an atom of \(M^{[2]}\) as claimed.

(2) By duality of (1). \(\square\)

Corollary 5.12. (1) \(b\) is an atom of \(M\) if and only if \((0, b)\) is an atom of \(M^{[2]}\),
(2) \(b\) is a coatom of \(M\) if and only if \((b, 1)\) is a coatom of \(M^{[2]}\),
(3) there is a one to one correspondence between the set of all atoms (coatoms) of \(M\) and the set of all atoms (coatoms) of \(M^{[2]}\).

Theorem 5.13. Let \(M\) be a de Morgan algebra and \(a \in M\). Then
(1) \((0, a)\) is an atom of \(M^{[2]}\) implies \((a, 1)\) is a coatom of \(M^{[2]}\),
(2) \((a, 1)\) is a coatom of \(M^{[2]}\) implies \((0, a)\) is an atom of \(M^{[2]}\),
(3) there is a one to one correspondence between the set of all atoms of \(M^{[2]}\) and the set of all coatoms of \(M^{[2]}\).

Proof. (1). Let \((0, a)\) is an atom of \(M^{[2]}\). Then by Corollary 5.12(1), \(a\) is an atom of \(M\). Clearly \((a, 1) \in M^{[2]}\). Let \((x, y) \geq (a, 1)\) for some \((x, y) \in M^{[2]}\). Then \(x \geq a\) and \(y = 1\) implies \(x \leq a\) and \(y = 1\). Since \(a\) is an atom of \(M\), then \(x = 0\) or \(x = a\). It follows that \(x = 1\), \(y = 1\) or \(x = a\), \(y = 1\). Hence \((x, y) = (1, 1)\) or \((x, y) = (a, 1)\). \(\therefore (a, 1)\) is a coatom of \(M^{[2]}\).

The proof of (2) is similar to that of (1) and the proof of (3) follows (1) and (2). \(\square\)

Theorem 5.14. A de Morgan algebra \(M\) is atomic if and only if \(M^{[2]}\) is atomic.

Proof. Let \(M\) be an atomic de Morgan algebra. Let \((a, b)\) is a nonzero element of \(M^{[2]}\). Then \(a \leq b\). Hence \(a = 0\) or \(a \neq 0\) but \(b \neq 0\). If \(a = 0\), then there exist atom of \(M\) say \(c\) such that \(c \leq b\). Then by Corollary 5.10(1), \((0, c)\) is an atom of \(M^{[2]}\) and \((0, c) \leq (0, b) = (a, b)\). If \(a \neq 0\) then there exists an atom of \(M\) say \(x\) such that \(x \leq a\). Hence \((0, x)\) is an atom of \(M^{[2]}\) with \((0, x) \leq (a, a) \leq (a, b)\). Therefore \(M^{[2]}\) is an atomic core regular double MS-algebra. Conversely, let \(M^{[2]}\) is atomic. Let \(0 \neq a \in M\). Then \((a, a)\) is a nonzero element of \(M^{[2]}\). Thus there exists an atom of \(M^{[2]}\) say \((0, y)\) with \((0, y) \leq (a, a)\). Consequently \(y\) is an atom of \(M\) with \(y \leq a\). Therefore \(M\) is atomic. \(\square\)

In the following example, we clarify the properties of atoms and coatoms of \(M\) and \(M^{[2]}\).

Example 5.15. Consider a de Morgan algebra \(M\) and a CRDMD-algebra \(M^{[2]}\) in example 4.2. We observe the following.
(1) \(\text{At}(M) = \{a, b\}\) and \(\text{At}(M^{[2]}) = \{(0, a), (0, b)\}\), where \(a, b\) are corresponding to \((0, a), (0, b)\), respectively.
(2) \([a, b]\) and \(\{(a, 1), (b, 1)\}\) are the sets of coatoms of \(M\) and \(\text{At}(M^{[2]})\), respectively. Also, \(a, b\) are corresponding to \((a, 1), (b, 1)\), respectively.
(3) The atoms \((0, a), (0, b)\) of \(M^{[2]}\) are corresponding to the coatoms \((a, 1), (b, 1)\) of \(M^{[2]}\), respectively.
(4) It is ready seen that \(M\) is an atomic de Morgan algebra and \(M^{[2]}\) is an atomic CRDMS-algebra.

Definition 5.16. [7] A lattice \(L\) is called complete if \(\text{inf}_L H\) and \(\text{sup}_L H\) exist for each \(\phi \neq H \subseteq L\).
A CRDMS-algebra $L$ is called complete if considered as a lattice it is complete.

Let $H = \{x_i = (a_i, b_i) : i \in I\} \subseteq \mathcal{M}^{[2]}$. We can write $\sup_L H = \bigvee_{i \in I} x_i$ and $\inf_L H = \bigwedge_{i \in I} x_i$.

**Theorem 5.17.** If $M$ is a complete de Morgan algebra, then $\mathcal{M}^{[2]}$ is complete CRDMS-algebra.

**Proof.** Let $\phi \neq H \subseteq \mathcal{M}^{[2]}$. Consider $H = \{(a_i, b_i) \in \mathcal{M}^{[2]}, i \in I\}$. Since $M$ is complete, then $\bigvee_{i \in I} a_i$ and $\bigvee_{i \in I} b_i$ exist. Hence $a_i \leq \bigvee_{i \in I} a_i$ and $b_i \leq \bigvee_{i \in I} b_i$. So, $(a_i, b_i) \leq (\bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i)$ and hence $(\bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i)$ is an upper bound of $H$. Let $(x, y)$ be an upper bound of $H$. Then $(a_i, b_i) \leq (x, y)$ implies $a_i \leq x$ and $b_i \leq y$. Therefore $\bigvee a_i \leq x$ and $\bigvee b_i \leq y$ and $(\bigvee a_i, \bigvee b_i) \leq (x, y)$. Then $(\bigvee a_i, \bigvee b_i) = \sup H$. Similarly, we can show that $(\bigwedge a_i, \bigwedge b_i) = \inf H$. Then $\mathcal{M}^{[2]}$ is complete. \qed

**Theorem 5.18.** Let $\mathcal{M}^{[2]}$ be a complete CRDMS-algebra. Then

1. $(M^{[2]})^\circ \subseteq \mathcal{M}^{[2]}$ is complete,
2. $M$ is complete.

**Proof.** (1). Let $\phi \neq H \subseteq (M^{[2]})^\circ$. Since $\mathcal{M}^{[2]}$ is complete and $H \subseteq \mathcal{M}^{[2]}$, then $\sup H$ and $\inf H$ exist in $\mathcal{M}^{[2]}$. Assume that $(a, b) = \sup_{M^{[2]}} H$ and $(c, d) = \inf_{M^{[2]}} H$. We prove that $(b, b) = \sup_{(M^{[2]})^\circ} H$. Since $(a, b) = \sup_{M^{[2]}} H$, then $(h, h) \leq (a, b)$ for all $h \in H$. Thus $(h, h) = (h, h)^+ \leq (a, b)^+ = (a, a)$ and hence $(a, a)$ is an upper bound of $H$. Since $(a, b) = \sup_{M^{[2]}} H$, then $(a, b) \leq (a, a)$ implies $b \leq a$. But $a \leq b$ as $(a, b) \in \mathcal{M}^{[2]}$. Therefore $a = b$ and $(a, b) = (b, b) \in (M^{[2]})^\circ$ and $(b, b) = \sup_{(M^{[2]})^\circ} H$. Similarly, we can show that $\inf H \in (M^{[2]})^\circ = (d, d)$. Therefor $(M^{[2]})^\circ$ is complete de Morgan algebra.

(2) Let $\phi \neq C \subseteq M$. Since $M$ isomorphic to $(M^{[2]})^\circ$ (see Corollary 4.4) then $\mathring{C} = \{(c, c) : c \in C\} \subseteq (M^{[2]})^\circ$ corresponds to $C$. By (1), $(M^{[2]})^\circ$ is complete and $\mathring{C} \subseteq (M^{[2]})^\circ$ then $\sup_{(M^{[2]})^\circ} \mathring{C}$ and $\inf_{(M^{[2]})^\circ} \mathring{C}$ exist. Assume $(x, x) = \sup_{(M^{[2]})^\circ} \mathring{C}$ and $(y, y) = \inf_{(M^{[2]})^\circ} \mathring{C}$. Then $(c, c) \leq (x, x)$ for all $(c, c) \in \mathring{C}$ implies $c \leq x$ for all $c \in C$. Thus $x$ is an upper bound of $C$. Let $y$ be an upper bound of $C$. Then $c \leq y$ for all $c \in C$ implies $(c, c) \leq (y, y)$ for all $(c, c) \in C$. Hence $(y, y)$ is an upper bound of $\mathring{C}$. Then $(x, x) \leq (y, y)$ as $(x, x) = \sup_{(M^{[2]})^\circ} \mathring{C}$. Therefore $x \leq y$ and $x = \sup_M C$. Using a similar way, we get $y = \inf_M C$. Then $M$ is complete. \qed

Combining Theorem 5.17 and Theorem 5.18(2), we have

**Theorem 5.19.** A de Morgan algebra $M$ is complete if and only if $\mathcal{M}^{[2]}$ is a complete CRDMS-algebra.

Now, we give two examples of complete atomic CRDMS-algebras, the first one is finite and the second one is infinite.

**Example 5.20.**

1. Consider a CRDMS-algebra $L$ in example 3.9(2). We have $\text{At}(L) = \{c, d\}$. It is clear that $L$ is a finite complete atomic CRDMS-algebra.

2. Let $M = [0] \oplus [0, 1] \oplus 1$ be an infinite chain, where $[0, 1]$ is a real closed interval and $\oplus$ stands for the ordinal sum. Then $(M; \lor, \land, 0, 1)$ forms a bounded distributive lattice, where $x \lor y = \max(x, y)$, $x \land y = \min(x, y)$, $x, y \in [0, 1]$ and $0, 1$ are the smallest and the greatest elements of $M$, respectively. Define a negation $\sim$ on $M$ by $\sim x = 1 - x$ for all $x \in [0, 1]$, $\sim 0 = 1$ and $\sim 1 = 0$. Since $\text{At}(M) = \{0\}$, then $M$ is atomic. As $\sup_M H$ and $\inf_M H$ exist, for $\phi \neq H \subseteq M$, then $M$ is complete. Therefore $M$ is a complete de Morgan algebra. Using the construction Theorem, we obtain
the core regular double MS-algebra \( M^{[2]} \), where
\[
M^{[2]} = \{(\emptyset, \emptyset), (\emptyset, 0), \ldots, (\emptyset, 1/2), \ldots, (\emptyset, \bar{T}),
(0, 0), \ldots, (0, 1/2), \ldots, (0, 1), (0, \bar{T}), \ldots \}
\]

(1). Since Definition 5.22. Therefore \( M = M \).

(4)
(3)
(2)
(1)
(4) The proof is similar to that of (3).
(3) Since \( \bigvee_{i \in I} x_i \geq x \), then \( (\bigvee_{i \in I} X_i)^* \leq x^* \). Hence \( (\bigvee_{i \in I} X_i)^* \) is a lower bound of \( x^* \). Let \( y \) be a lower bound of \( x^* \). Then \( y \leq x^* \) implies \( y^* \geq x^* \). Then \( y^* \) is an upper bound of \( x \) and this gives \( \bigvee_{i \in I} x_i \leq y^* \). Therefore \( (\bigvee_{i \in I} X_i)^* \geq y^* \geq y \). Then we deduce that \( (\bigvee_{i \in I} X_i)^* \) is the greatest lower bound of \( x^* \) and \( (\bigvee_{i \in I} X_i)^* = \bigwedge_{i \in I} x_i^* \).
(4) The proof is similar to that of (3). \( \square \)

**Definition 5.22.** A subalgebra \( L_1 \) of a complete CRDMS-algebra \( L \) is called a complete subalgebra of \( L \) if \( \inf_{L_1} H \in L_1 \) and \( \sup_{L_1} H \in L_1 \) for every subset \( H \) of \( L_1 \).

**Theorem 5.23.** Let \( M^{[2]} \) be a subalgebra of a complete CRDMS-algebra \( M^{[2]} \). Then \( M^{[2]} \) is complete subalgebra of \( M^{[2]} \) if and only if \( M^{[2]} \) is a complete subalgebra of \( M^{[2]} \).

**Proof.** Let \( M^{[2]} \) be a complete subalgebra of \( M^{[2]} \). Let \( \phi \neq H \subseteq M_1 \). Consider the subset \( \hat{H} = \{x_i = (a_i, a_i) : a_i \in H, i \in I\} \) of \( M^{[2]} \) corresponding to \( H \). Since \( M^{[2]} \) is complete, then by Lemma 5.21(1), (2), we have
\[
\sup_{M} \hat{H} = \bigvee_{i \in I} x_i = (\bigvee_{i \in I} a_i, \bigvee_{i \in I} a_i) \in M^{[2]},
\]
\[
\inf_{M} \hat{H} = \bigwedge_{i \in I} x_i = (\bigwedge_{i \in I} a_i, \bigwedge_{i \in I} a_i) \in M^{[2]}.
\]
Hence $\bigvee_{i \in I} a_i \in M_1$ and $\bigwedge_{i \in I} a_i \in M_1$. Then $M_1$ is complete complete subalgebra of $M$. Conversely, let $M_1$ is a complete subalgebra of a complete de Morgan algebra $M$. Let $\phi \neq H \subseteq M_1^{[2]}$. Then

$H = \{x_i = (a_i, b_i) \in M_1^{[2]} : i \in I\} \subseteq M_1^{[2]}$.

Since $M_1$ is complete subalgebra of $M$, then we have

$\bigvee_{i \in I} a_i \in M_1$ and $\bigvee_{i \in I} b_i \in M_1$.

Also, $\bigwedge_{i \in I} a_i \in M_1$ and $\bigwedge_{i \in I} b_i \in M_1$. Then by Lemma 5.21,(1),(2), respectively, we get

$\bigvee_{M^{[2]}_1} H = \bigvee_{i \in I} x_i = \left(\bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i\right) \in M_1^{[2]}$, 

$\bigwedge_{M^{[2]}_1} H = \bigwedge_{i \in I} x_i = \left(\bigwedge_{i \in I} a_i, \bigwedge_{i \in I} b_i\right) \in M_1^{[2]}$.

Then $M_1^{[2]}$ is a complete subalgebra of a CRDMS-algebra $M^{[2]}$.

**Definition 5.24.** [6] A lattice homomorphism $h : L \to L_1$ of a complete lattice $L$ into a complete lattice $L_1$ is called complete if

$h(\inf_L H) = \inf_{L_1} h(H)$ and $h(\sup_L H) = \sup_{L_1} h(H)$ for each $\phi \neq H \subseteq L$.

A homomorphism $h : L \to L_1$ of a complete CRDMS-algebra $L$ into a complete CRDMS-algebra $L_1$ is called complete if it is complete as a lattice homomorphism.

**Theorem 5.25.** Let $M$ and $M_1$ be complete de Morgan algebras. If $f : M \to M_1$ is a complete homomorphism, then $h : M^{[2]} \to M_1^{[2]}$ defined by $h(a, b) = (f(a), f(b))$ is a complete homomorphism of $M^{[2]}$ into $M_1^{[2]}$. Conversely, if $g : M^{[2]} \to M_1^{[2]}$ is a complete homomorphism, then $f : M \to M_1$ defined by $f(a) = b \Leftrightarrow g(a, a) = (b, b)$ is a complete homomorphism of de Morgan algebras.

**Proof.** Let $f : M \to M_1$ is a complete homomorphism. Then by Theorem 5.9, $h : M^{[2]} \to M_1^{[2]}$ defined by $h(a, b) = (f(a), f(b))$ is a homomorphism of CRDMS-algebras $M^{[2]}$ and $M_1^{[2]}$. We prove that $\sup_{M^{[2]}_1} h(H) = h(\sup_{M^{[2]}_1} H)$ for $\phi \neq H \subseteq M^{[2]}$. Consider $H = \{x_i = (a_i, b_i) \in M^{[2]} : i \in I\}$ for $\phi \neq H \subseteq M^{[2]}$. Using Lemma 5.21(1), we get $\sup_{M^{[2]}_1} H = \bigvee_{i \in I} x_i = \bigvee_{i \in I} (a_i, b_i) = \left(\bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i\right)$. Thus

$h(\sup H) = h\left(\bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i\right)$

$= \left(\bigvee_{i \in I} f(a_i), \bigvee_{i \in I} f(b_i)\right)$

$= \bigvee_{i \in I} (f(a_i), f(b_i))$

$= \bigvee_{i \in I} h(a_i, b_i)$

$= \sup_{M^{[2]}_1} h(H)$.

Using Lemma 5.21(2), we can get $\inf_{M^{[2]}_1} = h(\inf_{M^{[2]}_1} H)$. Therefore $h$ is complete. Conversely, let $g : M^{[2]} \to M_1^{[2]}$ is a complete homomorphism of a CRDMS-algebra $M^{[2]}$ into $M_1^{[2]}$. Then by Theorem 5.9, a mapping $f : M \to M_1$ defined by $f(a) = b \Leftrightarrow g(a, a) = (b, b)$ is a homomorphism of $M$ into $M_1$. We have to show
that \( f \) is complete. Let \( \phi \neq H = \{ a_i : i \in I \} \subseteq M \), we prove that \( f(\inf_M H) = \inf_M f(H) \). Consider a subset \( \hat{H} = \{ x_i = (a_i, a_i) : a_i \in H, i \in I \} \) of \( M^{[2]} \) corresponding to \( H \). Since \( M \) and \( M^{[2]} \) are complete, then by Lemma 5.21(2), we get

\[
\inf_{M^{[2]}} \hat{H} = \bigwedge_{i \in I} x_i = (\bigwedge_M a_i, \bigwedge_M a_i)
\]

Let \( g(a_i, a_i) = (b_i, b_i) \). Then by definition of \( f \), we have \( f(a_i) = b_i \). Since \( g \) is complete, then \( g(\inf_{M^{[2]}} \hat{H}) = \inf_{M^{[2]}} g(\hat{H}) \). Now

\[
\begin{align*}
g(\inf_{M^{[2]}} \hat{H}) &= g(\bigwedge_{i \in I} (a_i, a_i)) = g(\bigwedge_{i \in I} a_i, \bigwedge_{i \in I} a_i), \\
\inf_{M^{[2]}} g(\hat{H}) &= \bigwedge_{i \in I} g(a_i, a_i) = \bigwedge_{i \in I} (b_i, b_i) = (\bigwedge_{i \in I} b_i, \bigwedge_{i \in I} b_i).
\end{align*}
\]

Then \( g(\bigwedge_{i \in I} a_i, \bigwedge_{i \in I} a_i) = (\bigwedge_{i \in I} b_i, \bigwedge_{i \in I} b_i) \) implies \( f(\inf_{M^{[2]}} H) = f(\bigwedge_{i \in I} a_i) = \bigwedge_{i \in I} b_i = \inf_M f(H) \). Similarly, we can show that \( f(\sup_M H) = \sup_{M^{[2]}} f(\hat{H}) \). Then \( f \) is complete. \( \square \)

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**References**


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