Quatertionic Bertrand Curves in the Galilean Space

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Abstract. In this article, we introduce the notion of a spatial quaternionic Bertrand curves in $G^3$ and give some characterizations of such curves. Furthermore, we introduce spatial quaternionic (1,3)--Bertrand curves in $G^4$.

1. Introduction

The geometry of curves in both Euclidean and Minkowski spaces was represented for many years a popular topic in the field of classical differential geometry [4–6, 11]. Global and local properties were studied in several books and new invariants were defined for curves. Two curves which have a common principal normal vector at corresponding points are called Bertrand curves [9, 11].

In four dimensional Euclidean space $E^4$, Bertrand curves are generalized and characterized in [11]. Moreover, in 3−dimensional Galilean space $G^3$, Bertrand curves are defined and characterized in many papers such as [1,12].

K. Bharathi and M. Nagaraj in [3] were studied quaternionic curves in both Euclidean space $E^3$ and Euclidean 4−space $E^4$ and gave the Frenet formulae for quaternionic curves. For other results of quaternionic curves, we refer to the papers [7,8,10,14]. Moreover, O. Kecilioglu and K. Ilarslan [9] were proved that if the bitorsion of a quaternionic curve $\alpha$ is no vanish, then there is no quaternionic curve in the Euclidean 4−space $E^4$ is a Bertrand curve. Therefore, they defined (1,3)− type Bertrand curves for quaternionic curves in the Euclidean 4− space $E^4$ and gave some characterizations for this type in $E^4$ by means of the curvature functions of the curve.

In these regards, we introduce the quaternionic Bertrand curves in the Galilean space $G^3$ and give some characterizations of such curves. Also, we investigate generalized quaternionic Bertrand curves in the Galilean 4− space $G^4$ and deduce general characterizations of these curves.

2. Preliminaries

In this section, we give the basic elements of the theory of quaternions and quaternionic curves. Also, we introduce a brief note about Bertrand curves in the Galilean space.

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A real quaternion $q$ is an expression of the form

$$q = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4$$

where $a_i$, $(1 \leq i \leq 4)$ are real numbers, and $e_i$, $(1 \leq i \leq 4)$ are quaternionic units which satisfy the non-commutative multiplication rules

$$e_i \times e_i = -e_4, \quad (1 \leq i \leq 3)$$
$$e_i \times e_j = e_k = -e_j \times e_i, \quad (1 \leq i, j, k \leq 3)$$

where $(ijk)$ is an even permutation of $(123)$ in the Euclidean space $[8]$. The algebra of the quaternions is denoted by $\mathbb{Q}$ and its natural basis is given by $(e_1, e_2, e_3, e_4)$. A real quaternion can be given by the form $q = S_q + V_q$ where $S_q = a_4$ is scalar part and $V_q = a_1 e_1 + a_2 e_2 + a_3 e_3$ is vector part of $q$. On the other hand, the conjugate of $q = S_q + V_q$ is defined by $\overline{q} = S_q - V_q$.

Let $p$ and $q$ be any two elements of $\mathbb{Q}$. Then the product of $p$ and $q$ is denoted by

$$p \times q = S_p S_q - \langle V_p, V_q \rangle + S_p V_q + S_q V_p + V_p \wedge V_q$$

where we have used the inner and cross products in the Galilean space $G^3$. This defines the symmetric real-valued, non-degenerate, bilinear form as follows:

$$h : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{R}, \quad (p, q) \rightarrow h(p, q) = \frac{1}{2} [p \times q + q \times p]$$

which is called the quaternion inner product. Then the norm of $q$ is given by

$$\|q\|^2 = h(q, q) = \frac{1}{2} [a q \times q + q \times a q] = a_1^2 + a_2^2 + a_3^2 + a_4^2$$

If $\|q\| = 1$, then $q$ is called unit quaternion. $q$ is called a spatial quaternion whenever $q + a q = 0$ and called a temporal quaternion whenever $q - a q = 0$. Then a general quaternion $q$ can be given as $[3]$

$$q = \frac{1}{2} [q + a q] + \frac{1}{2} [q - a q] \quad (1)$$

Now, we consider a quaternionic curve in $E^3$. The Euclidean space $E^3$ is identified with the space of spatial quaternion $[\beta \in \mathbb{Q}/\beta + a \beta = 0]$ in an obvious manner. Let $I = [0, 1]$ be an interval in the real line $R$ and $s \in I$ be the arc-length parameter along the smooth curve

$$\beta : \quad I \subset R \rightarrow \mathbb{Q}$$
$$s \rightarrow \beta(s) = \sum_{i=1}^{3} \beta_i(s)e_i \quad (1 \leq i \leq 3)$$

The tangent vector $\beta'(s) = t(s)$ has unit length $\|t(s)\| = 1$ for all $s$. It follows

$$t' \times at + txat' = 0$$
which implies \( t' \) is orthogonal to \( t \) and \( t' \times at \) is a spatial quaternion. Let \( [t, n, b] \) be the Frenet frame of \( \beta(s) \), then Frenet formulae are given by

\[
\begin{align*}
    t' &= kn \\
    n' &= -kt + \tau b \\
    b' &= -\tau n
\end{align*}
\]

where \( t, n, b \) are the unit tangent, the unit principal normal and the unit binormal vector of a quaternionic curve \( \beta \), respectively. The functions \( k, \tau \) are called the principal curvature and the torsion of \( \beta \), respectively [1].

**Theorem 2.1.** [8, 9] The four-dimensional Euclidean space \( E^4 \) are identified with the space of unique quaternions. Let \( I = [0, 1] \) be a unit interval of the real line \( R \) and

\[
\begin{align*}
    \beta : I \subset R &\rightarrow Q \\
    s &\rightarrow \beta(s) = \sum_{i=1}^{4} \beta_i(s)e_i \quad (1 \leq i \leq 4)
\end{align*}
\]

be a smooth curve in \( E^4 \) with non zero curvatures \( \{K, k, r - K\} \) and \( (T(s), N(s), B_1(s), B_2(s)) \) denotes the Frenet frame of the curve. Then the Frenet formulae are given by

\[
\begin{bmatrix}
    T' \\
    N' \\
    B_1' \\
    B_2'
\end{bmatrix} =
\begin{bmatrix}
    0 & K & 0 & 0 \\
    -K & 0 & k & 0 \\
    0 & -k & 0 & r - K \\
    0 & 0 & -(r - K) & 0
\end{bmatrix}
\begin{bmatrix}
    T \\
    N \\
    B_1 \\
    B_2
\end{bmatrix}
\] (2)

where \( K \) is the principal curvature, \( k \) is the torsion and \( (r - K) \) is the bitorsion of \( \beta \).

The notion of Bertrand curves was discovered by J. Bertrand in 1850 and it plays an important role in classical differential geometry, and a lot of mathematicians have studied on the Bertrand curves in different areas [12]. For more about generalized Bertrand curves in the Galilean space and Minkowski space, we refer to [1, 4, 5, 10, 13].

### 3. Quaternionic Bertrand Curves in the Galilean 3-Space

**Definition 3.1.** Let \( \alpha(s) \) and \( \beta(s') \) be two quaternionic curves in \( G^3 \). \( \{T_\alpha(s), N_\alpha(s), B_\alpha(s)\} \) and \( \{T_\beta(s'), N_\beta(s'), B_\beta(s')\} \) are Frenet frames of these curves, respectively. \( \alpha(s) \) and \( \beta(s') \) are called Bertrand curves if there exist a bijection \( \phi : I \subset R \rightarrow I' \), and \( s \rightarrow \phi(s) = s' \), \( \frac{ds'}{ds} \neq 0 \) and the principal normal lines of \( \alpha(s) \) and \( \beta(s') \) are linearly dependent. In other words, if the principal normal lines of \( \alpha(s) \) and \( \beta(s') \) at \( s \in I \) are parallel.

**Theorem 3.2.** Let \( \alpha(s) \) and \( \beta(s') \) be a spatial quaternionic Bertrand curves in \( G^3 \). Then \( d(\alpha(s), \beta(s')) = \text{constant} \) for all \( s \in I \).

**Proof.** From definition (5.1), we can write

\[
\beta(s') = \alpha(s) + \lambda(s)N_\alpha(s)
\] (3)

differentiating with respect to \( s \) and using Frenet equations, we get

\[
T_\beta(s') \frac{ds'}{ds} = T_\alpha(s) + \lambda'(s)N_\alpha(s) + \lambda(s)\tau_\alpha(s)B_\alpha(s)
\]

Let us put \( \frac{ds'}{ds} = \phi'(s) \neq 0 \), then we have

\[
T_\beta(s') = \frac{1}{\phi'(s)} \left[ T_\alpha(s) + \lambda'(s)N_\alpha(s) + \lambda(s)\tau_\alpha(s)B_\alpha(s) \right]
\] (4)
Since \((N_n(s), N_p(s'))\) is a linearly dependent set, we get \(h(N_n(s), N_p(s')) \neq 0\) hence \(\lambda'(s) = 0\) and \(\lambda\) is a constant function on \(I\). □

**Theorem 3.3.** Let \(\alpha(s)\) and \(\beta(s')\) be a spatial quaternionic Bertrand curves in \(G^3\). Then the measure of the angle between the tangent vector field of \(\alpha(s)\) and \(\beta(s')\) is constant.

**Proof.** Let \(\alpha(s)\) and \(\beta(s')\) be a spatial quaternionic Bertrand curves in \(G^3\) with arc length \(s\) and \(s'\), respectively. Define \(T_\alpha(s')\) as follows

\[
T_\beta(s') = \cos \theta(s)T_\alpha(s) + \sin \theta(s)B_\alpha(s)
\]

where \(\theta\) is the angle between \(T_\alpha(s)\) and \(T_\beta(s')\).

Differentiating with respect to \(s\), we obtain

\[
T'_\beta(s')\varphi'(s) = \frac{d\cos \theta(s)}{ds}T_\alpha(s) + \cos \theta(s)T'_\alpha(s) + \frac{d\sin \theta(s)}{ds}B_\alpha(s) + \sin \theta(s)B'_\alpha(s)
\]

Since \([T_\alpha(s), B_\alpha(s), N_\alpha(s)]\) is the Frenet frame on \(G^3\) along \(\alpha(s)\) and \(\beta(s')\) is a Bertrand mate of \(\alpha(s)\), so

\[
\begin{align*}
    h(N_n(s), T_\alpha(s)) &= h(N_n(s), B_\alpha(s)) = h(T_\alpha(s), B_\alpha(s)) = 0 \\
    h(T_\alpha(s), T_\alpha(s)) &= h(B_\alpha(s), B_\alpha(s)) = h(N_n(s), N_\alpha(s)) = 1
\end{align*}
\]

and hence, we obtain

\[
\frac{d\sin \theta(s)}{ds} = 0 \Rightarrow \sin \theta(s) = \text{constant}
\]

\[
\frac{d\cos \theta(s)}{ds} = 0 \Rightarrow \cos \theta(s) = \text{constant}
\]

Therefore, we can deduce that the angle \(\theta\) is constant. □

**Theorem 3.4.** Let \(\alpha(s)\) be a spatial quaternionic curve in \(G^3\) with arc length \(s\). \(\alpha(s)\) is a spatial quaternionic Bertrand curve if and only if \(\alpha(s)\) has a constant torsion.

**Proof.** Let \(\alpha(s)\) and \(\beta(s')\) be a spatial quaternionic Bertrand curves in \(G^3\). From theorem (3.2), we obtain

\[
\varphi'(s)T_\beta(s') = T_\alpha(s) + \lambda_\alpha(s)B_\alpha(s)
\]

Also, from theorem (3.3), we get

\[
\varphi'(s) = \frac{1}{\cos \theta(s)} \text{ and } \varphi'(s)\sin \theta(s) = \lambda_\alpha(s).\]

Therefore, \(\tan \theta(s) = \lambda_\alpha(s)\) which implies that \(\lambda_\alpha(s) = \frac{\tan \theta(s)}{\lambda}\). Since \(\theta\) is constant we obtain that \(\lambda_\alpha\) is constant. Conversely, assume that \(\lambda_\alpha(s)\) is constant and define \(\beta(s') = \alpha(s) + \lambda(s)N_\alpha(s)\). So, we have

\[
\varphi'(s)T_\beta(s') = T_\alpha(s) + \lambda_\alpha(s)B_\alpha(s)
\]

which implies that

\[
(\varphi'(s))^2 = 1 + \lambda^2_\alpha
\]

Differentiating with respect to \(s\), we get

\[
(\varphi'(s))^2 K_\beta(s')N_\beta(s') = \left(K_\alpha(s) - \lambda^2_\alpha(s)\right)N_\alpha(s)
\]

and therefore,

\[
K_\beta(s')N_\beta(s') = \frac{K_\alpha(s) - \lambda^2_\alpha(s)}{1 + \lambda^2_\alpha(s)}N_\alpha(s)
\]

which means that \(N_\beta(s')\) and \(N_\alpha(s)\) are linearly dependent, and according to definition (3.1), it determines that \(\alpha(s)\) and \(\beta(s')\) are spatial quaternionic Bertrand pair. □
Corollary 3.5. Let $\alpha(s)$ be a spatial quaternionic curve in $G^3$ with arc-length parameter $s$. $\alpha(s)$ is a spatial quaternionic Bertrand curve if $K_\alpha(s') = LK_\alpha(s) + M$ where $L$ and $M$ are constants.

Theorem 3.6. Let $\alpha(s), \beta(s')$ be a spatial quaternionic Bertrand pair in $G^3$. Then $\frac{\tau_\alpha(s)}{\tau_\beta(s')}$ is constant provided that $\tau_\beta(s') \neq 0$.

Proof. If we take $\alpha(s)$ instead of $\beta(s')$, then we can write equation (5) in the form:

$$\alpha(s) = \beta(s') - \lambda N_\beta(s')$$

Differentiating with respect to $s$ and using Frenet equations, we obtain

$$T_\alpha(s) = \varphi'(s)T_\beta(s') - \lambda \varphi'(s) \tau_\beta(s')B_\beta(s')$$

which implies that

$$1 = (\varphi'(s))^2 - \lambda^2 \varphi'(s)^2 \tau_\beta^2(s')$$

Now, from equation (7), we can get

$$\tau_\alpha^2(s) = \tau_\beta^2(s') \left(1 + \lambda^2 \tau_\alpha^2(s)\right)$$

which gives

$$\frac{\tau_\alpha(s)}{\tau_\beta(s')} = \sqrt{1 + \lambda^2 \tau_\alpha^2(s)}$$

Therefore, $\frac{\tau_\alpha(s)}{\tau_\beta(s')}$ is constant. □

4. Quaternionic Bertrand curves in the Galilean 4–Space

Special Bertrand curves in 4D Galilean space was introduced by Oztekin [13]. In this section we introduce the spatial quaternionic Bertrand curves in $G^4$. Let $\alpha(s)$ be a quaternionic curve in the Galilean space $G^4$, if $K_{3a}(s) \neq 0$, then there is no quaternionic curve in $G^4$ is a Bertrand curve. Therefore, we can introduce the following definition.

Definition 4.1. Let $\alpha(s)$ and $\beta(s')$ be two spatial quaternionic curves in $G^4$, $\{T_\alpha(s), N_\alpha(s), B_{1a}(s), B_{2a}(s)\}$ and $\{T_\beta(s'), N_\beta(s'), B_{1b}(s'), B_{2b}(s')\}$ are Frenet frames of these curves, respectively. $\alpha(s)$ and $\beta(s')$ are called spatial quaternionic $(1, 3)$–Bertrand curves if there exist a bijection

$$\phi : I \to I'$$

$$s \to \phi(s) = s', \frac{ds'}{ds} \neq 0$$

and the plane spanned by $N_{\alpha}(s), B_{2a}(s)$ at each point $\alpha(s)$ of the curve $\alpha$ coincides with the plane spanned by $N_\beta(s'), B_{2b}(s')$ at corresponding point $\beta(s') = \beta(\phi(s))$ of the curve $\beta$.

Theorem 4.2. Let $\alpha(s)$ be a spatial quaternionic curve in $G^4$ with curvature functions $K_{1a}(s), K_{2a}(s), K_{3a}(s)$ and $K_{3a}(s) \neq 0$. Then $\alpha(s)$ is a $(1, 3)$–Bertrand curve if there exist constant real numbers $\rho, \sigma, \zeta$ and $\delta$ satisfying

1. $\rho K_{2a}(s) - \sigma K_{3a}(s) \neq 0$,
2. $\zeta (\rho K_{2a}(s) - \sigma K_{3a}(s)) = 1$,
3. $\delta K_{3a}(s) = \zeta K_{1a}(s) - K_{2a}(s)$
Proof. We assume that $\alpha(s)$ is a $(1,3)-$ Bertrand curve parametrized by arc length $s$. The $(1,3)-$ Bertrand mate $\beta(s)$ is given by

$$\beta(s') = \beta(\phi(s)) = \alpha(s) + \rho(s)N_\alpha(s) + \sigma(s)B_{2\alpha}(s)$$

for all $s \in I$, where $\rho(s)$ and $\sigma(s)$ are $C^\infty-$ functions on $I$ and $s'$ is the arc length parameter of $\beta$.

Differentiating equation (11) with respect to $s$ and using the Frenet equations, we obtain

$$\phi'(s)T_\beta(s') = T_\alpha(s) + \rho'(s)N_\alpha(s) + [\rho(s)K_{2\alpha}(s) - \sigma(s)K_{3\alpha}(s)]B_{1\alpha}(s) + \sigma'(s)B_{2\alpha}(s)$$

for all $s \in I$.

Since the plane spanned by $N_\alpha(s)$ and $B_{2\alpha}(s)$ coincides with the plane spanned by $N_\beta(s')$ and $B_{2\beta}(s')$, we can put

$$N_\beta(s') = \cos \theta(s)N_\alpha(s) + \sin \theta(s)B_{2\alpha}(s)$$

and

$$B_{2\beta}(s') = (-\sin \theta(s))N_\alpha(s) + \cos \theta(s)B_{2\alpha}(s)$$

and we notice that $\sin \theta(s) \neq 0$ for all $s \in I$, also we obtain

$$\rho'(s) \cos \theta(s) + \sigma'(s) \sin \theta(s) = 0$$

$$-\rho'(s) \sin \theta(s) + \sigma'(s) \cos \theta(s) = 0$$

from these equations, we get $\sigma'(s) = 0$ and $\rho'(s) = 0$, hence $\rho(s)$ and $\sigma(s)$ are constant functions on $I$ with values $\rho$ and $\sigma$, respectively.

Thus, for all $s \in I$, equation (11) can be rewritten in the form

$$\beta(s') = \beta(\phi(s)) = \alpha(s) + \rho N_\alpha(s) + \sigma B_{2\alpha}(s)$$

Differentiating with respect to $s$ and using the Frenet equations, we obtain

$$\phi'(s)T_\beta(s') = T_\alpha(s) + [\rho K_{2\alpha}(s) - \sigma K_{3\alpha}(s)]B_{1\alpha}(s)$$

and

$$\left(\phi'(s)\right)^2 = 1 + (\rho K_{2\alpha}(s) - \sigma K_{3\alpha}(s))^2 \neq 0$$

for all $s \in I$.

Therefore, we can put

$$T_\beta(s') = (\cos \psi(s))T_\alpha(s) + (\sin \psi(s))B_{1\alpha}(s)$$

and we obtain $\cos \psi(s) = \frac{1}{\phi'(s)}$ and $\sin \psi(s) = \frac{\rho K_{2\alpha}(s) - \sigma K_{3\alpha}(s)}{\phi'(s)}$ where $\psi(s)$ is a $C^\infty-$ functions on $I$.

Differentiating equation (16) and using Frenet equations, we get

$$\phi'(s)K_{1\beta}(s')N_\beta(s') = \left[\cos \psi(s)K_{1\alpha}(s) - \sin \psi(s)K_{2\alpha}(s)\right]N_\alpha(s) + \frac{d \cos \psi(s)}{ds}T_\alpha(s)$$

$$+ \frac{d \sin \psi(s)}{ds}B_{1\alpha}(s) + \sin \psi(s)K_{3\alpha}(s)B_{2\alpha}(s)$$

Since $N_\beta(s') = N_\beta(\phi(s))$ is expressed by a linear combination of $N_\alpha(s)$ and $B_{2\alpha}(s)$, it holds that

$$\frac{d \cos \psi(s)}{ds}T_\alpha(s) = 0 \Rightarrow \frac{d \cos \psi(s)}{ds} = 0$$

$$\frac{d \sin \psi(s)}{ds}B_{1\alpha}(s) = 0 \Rightarrow \frac{d \sin \psi(s)}{ds} = 0$$
i.e., \( \psi(s) \) is a constant function on \( I \) with value \( \psi \), and hence
\[
T_\beta(s^*) = T_\alpha(s) \cos \psi + B_{1\alpha}(s) \sin \psi
\]
which implies that
\[
\sin \psi = (\rho K_{2\alpha}(s) - \sigma K_{3\alpha}(s)) \cos \psi
\]
for all \( s \in I \).

If \( \sin \psi = 0 \), then it holds \( \cos \psi = \mp 1 \) and hence equation (17) becomes
\[
T_\beta(s^*) = \mp T_\alpha(s)
\]
Differentiating with respect to \( s \) and using the Frenet equations, we obtain
\[
\phi'(s)K_{1\beta}(s')N_\beta(s') = \mp K_{1\alpha}(s)N_\alpha(s)
\]
and therefore, we obtain the first relation.

The fact \( \sin \psi \neq 0 \) and equation (18) imply that
\[
\zeta(\rho K_{2\alpha}(s) - \sigma K_{3\alpha}(s)) = 1
\]
where \( \zeta = \frac{\cos \psi}{\sin \psi} \) is a constant number and hence we obtain the second relation.

Differentiating equation (17) with respect to \( s \) and using the Frenet equations, we obtain
\[
\left(\phi'(s)K_{1\beta}(s')\right)^2 = \left[\cos \psi K_{1\alpha}(s) - \sin \psi K_{2\alpha}(s)\right]^2 + \sin^2 \psi K_{3\alpha}(s)
\]
and from equations (22) and (23), we deduce
\[
\left(\phi'(s)K_{1\beta}(s')\right)^2 = \frac{1}{c^2 + 1} \left[\left(\zeta K_{1\alpha}(s) - K_{2\alpha}(s)\right)^2 + (K_{3\alpha}(s))^2\right]
\]
for all \( s \in I \).

From equation (15) and the second relation, we get
\[
\left(\phi'(s)\right)^2 = (c^2 + 1)(\lambda K_{2\alpha}(s) - \mu K_{3\alpha}(s))^2
\]
Thus, we obtain
\[
\left(\phi'(s)K_{1\beta}(s')\right)^2 = \frac{1}{c^2 + 1} \left[\left(\zeta K_{1\alpha}(s) - K_{2\alpha}(s)\right)^2 + (K_{3\alpha}(s))^2\right]
\]
From equations (22) and (23) and the second relation, we can put
\[ N_0(s') = N_0(\phi(s)) = \cos \eta(s)N_a(s) + \sin \eta(s)B_{2a}(s) \] (25)
where
\[ \cos \eta(s) = \frac{(\rho K_{2a}(s) - \alpha K_{3a}(s)) (\zeta K_{1a}(s) - K_{2a}(s))}{K_{1b}(\phi(s)) (\phi'(s))^2} \] (26)
and
\[ \sin \eta(s) = \frac{K_{3a}(s) (\rho K_{2a}(s) - \alpha K_{3a}(s))}{K_{1b}(\phi(s)) (\phi'(s))^2} \] (27)
for all \( s \in I \). Here, \( \eta \) is a \( C^\infty \) function on \( I \).

Differentiating equation (25) with respect to \( s \) and using Frenet equations, we obtain
\[ \frac{d \cos \eta(s)}{ds} = 0, \quad \frac{d \sin \eta(s)}{ds} = 0 \]
that is, \( \eta \) is a constant function on \( I \) with value \( \eta_0 \). From equations (26) and (27), we obtain
\[ K_{3a}(s) \frac{\cos \eta_0}{\sin \eta_0} = (\zeta K_{1a}(s) - K_{2a}(s)) \]
Let \( \delta = \frac{\cos \eta_0}{\sin \eta_0} \) be a constant number, we then get
\[ \delta K_{3a}(s) = \zeta K_{1a}(s) - K_{2a}(s) \]
which is the third relation.

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