Filomat 34:10 (2020), 3349–3360
https://doi.org/10.2298/FIL2010349N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Novel Inequalities Involving a Function's Fractional Integrals in Relation to Another Function through Generalized Quasiconvex Mappings

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Abstract. In this paper, we establish new inequalities of the Hermite–Hadamard, midpoint and trapezoid types for functions whose first derivatives in absolute value are η-quasiconvex by means of generalized fractional integral operators with respect to another function $\omega : [\alpha, \beta] \to (0, \infty)$. Our theorems reduce to results involving the Riemann–Liouville fractional integral operators if ω is the identity map, and results involving the Hadamard operators if $\omega(x) = \ln x$. More inequalities can be deduced by choosing different bifunctions for η. To the best of our knowledge, the results obtained herein are new and we hope that they will stimulate further interest in this direction.

1. Introduction

In the theory of convex analysis, the standard Hermite–Hadamard inequality, christened after the french mathematicians, Charles Hermite and Jacques S. Hadamard, stipulates the following two-sided estimate of the mean value of a continuous convex function $h : [\alpha, \beta] \to \mathbb{R}$:

$$h\left(\frac{\alpha+\beta}{2}\right) \le \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} h(r)\,dr \le \frac{h(\alpha)+h(\beta)}{2}. \tag{1}$$

The above inequality has generated loads of papers in this direction. In 2013, Sarikaya et al. [17] extended (1) via the Riemann–Liouville fractional integral operators. Specifically, they proved:

Theorem 1.1. *Let $\epsilon > 0$ and $h : [\alpha, \beta] \to \mathbb{R}$ be a positive function with $0 \le \alpha \le \beta$ and $h \in L([\alpha, \beta])$. If h is a convex function on $[\alpha, \beta]$, then the following inequalities hold:*

$$h\left(\frac{\alpha+\beta}{2}\right) \le \frac{\Gamma(\epsilon+1)}{2(\beta-\alpha)^{\epsilon}}\left[\mathbf{I}^{\epsilon}_{\alpha^+}h(\beta) + \mathbf{I}^{\epsilon}_{\beta^-}h(\alpha)\right] \le \frac{h(\alpha)+h(\beta)}{2},$$

2010 *Mathematics Subject Classification*. 26A51, 26D15, 26E60, 41A55
Keywords. Hermite–Hadamard inequality, convex functions, quasiconvex functions, Riemann–Liouville operators, Hadamard fractional operators.
Received: 02 November 2019; Accepted: 17 January 2020
Communicated by Miodrag Spalević
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where the Riemann–Liouville fractional integrals, $\mathbf{I}^{\epsilon}_{\alpha^+}$ and $\mathbf{I}^{\epsilon}_{\beta^-}$, are defined as thus:

$$\mathbf{I}^{\epsilon}_{\alpha^+}h(x) = \frac{1}{\Gamma(\epsilon)}\int_a^x (x-r)^{\epsilon-1}h(r)\,dr$$

and

$$\mathbf{I}^{\epsilon}_{\beta^-}h(x) = \frac{1}{\Gamma(\epsilon)}\int_x^\beta (r-x)^{\epsilon-1}h(r)\,dr.$$

Here $\Gamma(\epsilon)$ is the Gamma function defined by $\Gamma(\epsilon) = \int_0^\infty e^{-x}x^{\epsilon-1}\,dx$.

More papers in this sense can be found in [6, 13]. We now recall the definition of functions integrated with respect to another function in the fractional sense:

Definition 1.2 ([10]). *Let $\omega : [\alpha,\beta] \to \mathbb{R}$ be an increasing and positive monotone function on $(\alpha,\beta]$ having a continuous derivative $\omega'(x)$ on (α,β). The left- and right-sided fractional integral of h with respect to the function ω on $[\alpha,\beta]$ of order $\epsilon > 0$ are defined respectively by:*

$$\mathfrak{I}^{\epsilon}_{\alpha^+;\omega}h(x) = \frac{1}{\Gamma(\epsilon)}\int_\alpha^x \frac{\omega'(r)}{[\omega(x)-\omega(r)]^{1-\epsilon}}h(r)\,dr, \qquad x > \alpha$$

and

$$\mathfrak{I}^{\epsilon}_{\beta^-;\omega}h(x) = \frac{1}{\Gamma(\epsilon)}\int_x^\beta \frac{\omega'(r)}{[\omega(r)-\omega(x)]^{1-\epsilon}}h(r)\,dr, \qquad x < \beta.$$

Remark 1.3. *In view of the above definition, we make the following observations that will aid the readability of this article.*

1. *If $\omega(x) = x$, then*
$$\mathfrak{I}^{\epsilon}_{\alpha^+;\omega}h(x) = \mathbf{I}^{\epsilon}_{\alpha^+}h(x) \quad \textit{and} \quad \mathfrak{I}^{\epsilon}_{\beta^-;\omega}h(x) = \mathbf{I}^{\epsilon}_{\beta^-}h(x).$$
2. *Let $\omega(x) = \ln x$. Then the fractional operators given in Definition 1.2 become the Hadamard fractional integrals, $\mathbf{J}^{\epsilon}_{\alpha^+}$ and $\mathbf{J}^{\epsilon}_{\beta^-}$, defined as follows:*
$$\mathfrak{I}^{\epsilon}_{\alpha^+;\omega}h(x) = \mathbf{J}^{\epsilon}_{\alpha^+}h(x) = \frac{1}{\Gamma(\epsilon)}\int_a^x \left(\ln\frac{x}{r}\right)^{\epsilon-1}\frac{h(r)}{r}\,dr$$
and
$$\mathfrak{I}^{\epsilon}_{\beta^-;\omega}h(x) = \mathbf{J}^{\epsilon}_{\beta^-}h(x) = \frac{1}{\Gamma(\epsilon)}\int_x^\beta \left(\ln\frac{r}{x}\right)^{\epsilon-1}\frac{h(r)}{r}\,dr.$$

Using the operators given in Definition 1.2, Budak [2] recently established the following inequalities of the Hermite–Hadamard type:

Theorem 1.4 ([2]). *Let $\epsilon > 0$. Suppose $\omega : [\alpha,\beta] \to \mathbb{R}$ is an increasing and positive monotone function on $(\alpha,\beta]$ having a continuous derivative $\omega'(x)$ on (α,β). If h is a convex function on $[\alpha,\beta]$, then we have the following Hermite–Hadamard type inequality for generalized fractional integrals,*

$$h\left(\frac{\alpha+\beta}{2}\right) \le \frac{\Gamma(\epsilon+1)}{2\Lambda^{\epsilon}_{\omega}(1)}\left[\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^-;\omega}H(\alpha) + \mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^+;\omega}H(\beta)\right] \le \frac{h(\alpha)+h(\beta)}{2},$$

where

$$H(x) = h(x) + \tilde{h}(x) \quad \textit{and} \quad \tilde{h}(x) = h(\alpha+\beta-x) \tag{2}$$

and

$$\Lambda^{\epsilon}_{\omega}(\tau) := \left[\omega(\beta) - \omega\left(\frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta\right)\right]^{\epsilon} + \left[\omega\left(\frac{\tau}{2}\beta + \frac{2-\tau}{2}\alpha\right) - \omega(\alpha)\right]^{\epsilon}. \tag{3}$$

In the same paper, Budak established the following midpoint and trapezoid type results:

Theorem 1.5 ([2]). *Let $\epsilon > 0$. Suppose $\omega : [\alpha, \beta] \to \mathbb{R}$ is an increasing and positive monotone function on $(\alpha, \beta]$ having a continuous derivative $\omega'(x)$ on (α, β). If $|h'|$ is a convex function on $[\alpha, \beta]$, then we have the following inequality for generalized fractional integrals:*

$$\left|\frac{\Gamma(\epsilon+1)}{2\Lambda_{\omega}^{\epsilon}(1)}\left[\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-};\omega}H(\alpha)+\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega}H(\beta)\right]-h\left(\frac{\alpha+\beta}{2}\right)\right|$$
$$\leq \frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)}\Big[|h'(\alpha)|+|h'(\beta)|\Big]\int_{0}^{1}\Lambda_{\omega}^{\epsilon}(\tau)\,d\tau.$$

Theorem 1.6 ([2]). *Let $\epsilon > 0$. Suppose $\omega : [\alpha, \beta] \to \mathbb{R}$ is an increasing and positive monotone function on $(\alpha, \beta]$ having a continuous derivative $\omega'(x)$ on (α, β). If $|h'|$ is a convex function on $[\alpha, \beta]$, then we have the following trapezoid type inequality for generalized fractional integrals:*

$$\left|\frac{h(\alpha)+h(\beta)}{2}-\frac{\Gamma(\epsilon+1)}{2\Lambda_{\omega}^{\epsilon}(1)}\left[\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-};\omega}H(\alpha)+\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega}H(\beta)\right]\right|$$
$$\leq \frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)}\Big[|h'(\alpha)|+|h'(\beta)|\Big]\int_{0}^{1}\left|\Lambda_{\omega}^{\epsilon}(1)-\Lambda_{\omega}^{\epsilon}(\tau)\right|d\tau.$$

Next, we recall the notion of quasiconvexity.

Definition 1.7. *A function $h : I \subset \mathbb{R} \to \mathbb{R}$ is called quasiconvex on the interval I if*

$$h(\tau x + (1-\tau)y) \leq \max\{h(x), h(y)\}$$

for all $x, y \in I$ and $\tau \in [0, 1]$.

Recently, Gordji et al. [5] further generalized the class of quasiconvex functions in the following manner:

Definition 1.8 ([5]). *A function $h : I \subset \mathbb{R} \to \mathbb{R}$ is called η-quasiconvex on I with respect to $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ if*

$$h(\tau x + (1-\tau)y) \leq \max\{h(y), h(y) + \eta(h(x), h(y))\}$$

for all $x, y \in I$ and $\tau \in [0, 1]$.

It is well known that every convex function is quasiconvex but the converse is not necessarily true. For this reason, it is our purpose to, among other things, extend Theorems 1.4, 1.5 and 1.6 to a larger class of functions – the η-quasiconvex functions. If, in particular, we take the bifunction $\eta(x, y) = x - y$, then our results boil down to that of the quasiconvex functions. Results involving the Riemann–Liouville and Hadamard fractional integral operators are deduced as special cases. Since this class of functions is new, it will be of interest to further develop it in this direction. For some recent results involving these generalized fractional integral operators, we invite the interested reader to see the papers [7, 12] and the references cited therein.

This paper is organized as follows: in Section 2, we state and give proofs to our main results in three subsections. In the next section thereafter, a brief conclusion is presented.

2. Main Results

For the sake of convenience, we set the following notations: for any η-quasiconvex function $f : [\alpha, \beta] \to \mathbb{R}$, we denote

$$\mathcal{M}_{\alpha}^{\beta}(f;\eta) := \max\{f(\alpha), f(\alpha) + \eta(f(\beta), f(\alpha))\} \tag{4}$$

and

$$\mathcal{N}_{\alpha}^{\beta}(f;\eta) := \max\{f(\beta), f(\beta) + \eta(f(\alpha), f(\beta))\}. \tag{5}$$

2.1. **Right-sided Inequality of the Hermite–Hadamard type**

Theorem 2.1. *Let $\epsilon > 0$. Suppose $\omega : [\alpha, \beta] \to \mathbb{R}$ is an increasing and positive monotone function on $(\alpha, \beta]$ having a continuous derivative $\omega'(x)$ on (α, β). If h is an η-quasiconvex function on $[\alpha, \beta]$, then we have the succeeding inequality for generalized fractional integrals*

$$\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-};\omega} H(\alpha) + \mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega} H(\beta) \leq \frac{\Lambda^{\epsilon}_{\omega}(1)}{\Gamma(\epsilon+1)} \left[\mathcal{M}^{\beta}_{\alpha}(h;\eta) + \mathcal{N}^{\beta}_{\alpha}(h;\eta)\right],$$

where

$$\Lambda^{\epsilon}_{\omega}(1) = \left(\omega(\beta) - \omega\left(\frac{\alpha+\beta}{2}\right)\right)^{\epsilon} + \left(\omega\left(\frac{\alpha+\beta}{2}\right) - \omega(\alpha)\right)^{\epsilon}, \tag{6}$$

and H, $\mathcal{M}^{\beta}_{\alpha}(h;\eta)$ and $\mathcal{N}^{\beta}_{\alpha}(h;\eta)$ are defined by (2), (4) *and* (5), *respectively.*

Proof. Using the η-quasiconvexity of h, one gets that for all $\tau \in [0, 1]$, the following inequalities hold:

$$h\left(\frac{\tau}{2}\beta + \frac{2-\tau}{2}\alpha\right) \leq \mathcal{M}^{\beta}_{\alpha}(h;\eta) \tag{7}$$

and

$$h\left(\frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta\right) \leq \mathcal{N}^{\beta}_{\alpha}(h;\eta). \tag{8}$$

Adding (7) and (8) gives:

$$h\left(\frac{\tau}{2}\beta + \frac{2-\tau}{2}\alpha\right) + h\left(\frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta\right) \leq \mathcal{M}^{\beta}_{\alpha}(h;\eta) + \mathcal{N}^{\beta}_{\alpha}(h;\eta). \tag{9}$$

Now multiplying both sides of (9) by

$$\frac{\beta-\alpha}{2\Gamma(\epsilon)} \frac{\omega'\left(\frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta\right)}{\left[\omega(\beta) - \omega\left(\frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta\right)\right]^{1-\epsilon}}$$

and integrate the resultant inequality over $[0, 1]$ to give:

$$\begin{aligned} &\frac{\beta-\alpha}{2\Gamma(\epsilon)} \int_0^1 \left[h\left(\frac{\tau}{2}\beta + \frac{2-\tau}{2}\alpha\right) + h\left(\frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta\right)\right] \frac{\omega'\left(\frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta\right)}{\left[\omega(\beta) - \omega\left(\frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta\right)\right]^{1-\epsilon}} d\tau \\ &\leq \frac{\beta-\alpha}{2\Gamma(\epsilon)} \left[\mathcal{M}^{\beta}_{\alpha}(h;\eta) + \mathcal{N}^{\beta}_{\alpha}(h;\eta)\right] \int_0^1 \frac{\omega'\left(\frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta\right)}{\left[\omega(\beta) - \omega\left(\frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta\right)\right]^{1-\epsilon}} d\tau. \end{aligned} \tag{10}$$

If we substitute $x = \frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta$, then $d\tau = -\frac{2}{\beta-\alpha}\, dx$, $\alpha + \beta - x = \frac{\tau}{2}\beta + \frac{2-\tau}{2}\alpha$ and Inequality (10) becomes:

$$\begin{aligned} &\frac{1}{\Gamma(\epsilon)} \int_{\frac{\alpha+\beta}{2}}^{\beta} [h(x) + h(\alpha+\beta-x)] \frac{\omega'(x)}{[\omega(\beta) - \omega(\alpha+\beta-x)]^{1-\epsilon}} dx \\ &\leq \frac{1}{\Gamma(\epsilon)} \left[\mathcal{M}^{\beta}_{\alpha}(h;\eta) + \mathcal{N}^{\beta}_{\alpha}(h;\eta)\right] \int_{\frac{\alpha+\beta}{2}}^{\beta} \frac{\omega'(x)}{[\omega(\beta) - \omega(\alpha+\beta-x)]^{1-\epsilon}} dx. \end{aligned} \tag{11}$$

Employing the fact that

$$\int_{\frac{\alpha+\beta}{2}}^{\beta} \frac{\omega'(x)}{[\omega(\beta)-\omega(\alpha+\beta-x)]^{1-\epsilon}}\,dx = \frac{1}{\epsilon}\left[\omega(\beta)-\omega\left(\frac{\alpha+\beta}{2}\right)\right]^{\epsilon}$$

and Definition 1.2, we obtain from (11) the following inequality

$$\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega} h(\beta) + \mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega} \tilde{h}(\beta) \le \frac{1}{\epsilon\Gamma(\epsilon)}\left[\mathcal{M}_{\alpha}^{\beta}(h;\eta)+\mathcal{N}_{\alpha}^{\beta}(h;\eta)\right]\left[\omega(\beta)-\omega\left(\frac{\alpha+\beta}{2}\right)\right]^{\epsilon}.$$

That is;

$$\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega} H(\beta) \le \frac{1}{\Gamma(\epsilon+1)}\left[\mathcal{M}_{\alpha}^{\beta}(h;\eta)+\mathcal{N}_{\alpha}^{\beta}(h;\eta)\right]\left[\omega(\beta)-\omega\left(\frac{\alpha+\beta}{2}\right)\right]^{\epsilon}. \tag{12}$$

Similarly, one gets by multiplying (9) by

$$\frac{\beta-\alpha}{2\Gamma(\epsilon)}\frac{\omega'\left(\frac{\tau}{2}\beta+\frac{2-\tau}{2}\alpha\right)}{\left[\omega\left(\frac{\tau}{2}\beta+\frac{2-\tau}{2}\alpha\right)-\omega(\alpha)\right]^{1-\epsilon}}$$

and integrating the resultant inequality over $[0,1]$ the succeeding inequality:

$$\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-};\omega} H(\alpha) \le \frac{1}{\Gamma(\epsilon+1)}\left[\mathcal{M}_{\alpha}^{\beta}(h;\eta)+\mathcal{N}_{\alpha}^{\beta}(h;\eta)\right]\left[\omega\left(\frac{\alpha+\beta}{2}\right)-\omega(\alpha)\right]^{\epsilon}. \tag{13}$$

Adding inequalities (12) and (13) amounts to:

$$\begin{aligned}&\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-};\omega} H(\alpha) + \mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega} H(\beta)\\ &\le \frac{1}{\Gamma(\epsilon+1)}\left[\mathcal{M}_{\alpha}^{\beta}(h;\eta)+\mathcal{N}_{\alpha}^{\beta}(h;\eta)\right]\left[\left(\omega(\beta)-\omega\left(\frac{\alpha+\beta}{2}\right)\right)^{\epsilon}+\left(\omega\left(\frac{\alpha+\beta}{2}\right)-\omega(\alpha)\right)^{\epsilon}\right],\end{aligned}$$

from where the desired inequality follows. □

By taking $\eta(x,y) = x - y$, Theorem 2.1 becomes:

Corollary 2.2. *Let $\epsilon > 0$. Suppose $\omega : [\alpha,\beta] \to \mathbb{R}$ is an increasing and positive monotone function on $(\alpha,\beta]$ having a continuous derivative $\omega'(x)$ on (α,β). If h is a positive quasiconvex function on $[\alpha,\beta]$, then we have the succeeding inequality for generalized fractional integrals:*

$$\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-};\omega} H(\alpha) + \mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega} H(\beta) \le \frac{2\Lambda_{\omega}^{\epsilon}(1)}{\Gamma(\epsilon+1)}\max\left\{h(\alpha),h(\beta)\right\}.$$

If $0 < \alpha < \beta$ and $\omega : [\alpha,\beta] \to \mathbb{R}$ is defined by $\omega(x) = k(x) = x$, then (6) becomes

$$\Lambda_{k}^{\epsilon}(1) = \frac{(\beta-\alpha)^{\epsilon}}{2^{\epsilon-1}}, \tag{14}$$

and hence the inequality in Corollary 2.2 reduces to:

$$\mathbf{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-}} H(\alpha) + \mathbf{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+}} H(\beta) \le \frac{1}{\Gamma(\epsilon+1)}\frac{(\beta-\alpha)^{\epsilon}}{2^{\epsilon-2}}\max\left\{h(\alpha),h(\beta)\right\}.$$

Also, by letting $0 < \alpha < \beta$ and $\omega : [\alpha, \beta] \to \mathbb{R}$ be defined by $\omega(x) = \ln x$, then (6) becomes

$$\Lambda_{\ln}^{\epsilon}(1) = \left[\ln \frac{2\beta}{\alpha+\beta}\right]^{\epsilon} + \left[\ln \frac{\alpha+\beta}{2\alpha}\right]^{\epsilon}. \tag{15}$$

Using (15), the inequality in Corollary 2.2 amounts to:

$$\frac{\Gamma(\epsilon+1)}{\left[\ln \frac{2\beta}{\alpha+\beta}\right]^{\epsilon} + \left[\ln \frac{\alpha+\beta}{2\alpha}\right]^{\epsilon}} \left[\mathbf{J}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-}} H(\alpha) + \mathbf{J}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+}} H(\beta)\right] \leq 2 \max\left\{h(\alpha), h(\beta)\right\}.$$

2.2. Midpoint type Inequalities in Generalized form

The following lemma will be needed in the proof of our results in this subsection:

Lemma 2.3 ([2]). *Let $\epsilon > 0$ and let the mapping ω be as in Theorem 2.1. If $h : [\alpha, \beta] \to \mathbb{R}$ be a differentiable mapping on (α, β) with $\alpha < \beta$, then the following equality holds:*

$$\begin{aligned}
&\frac{\Gamma(\epsilon+1)}{2\Lambda_{\omega}^{\epsilon}(1)} \left[\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-};\omega} H(\alpha) + \mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega} H(\beta)\right] - h\left(\frac{\alpha+\beta}{2}\right) \\
&= \frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)} \int_0^1 \Lambda_{\omega}^{\epsilon}(\tau) \left[h'\left(\frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta\right) - h'\left(\frac{\tau}{2}\beta + \frac{2-\tau}{2}\alpha\right)\right] d\tau,
\end{aligned}$$

where the mappings H and $\Lambda_{\omega}^{\epsilon}$ are defined as in (2) and (3), respectively.

Theorem 2.4. *Let $\epsilon > 0$ and let the mappings ω and h be as in Theorem 2.1 and Lemma 2.3, respectively. If $|h'|$ is an η-quasiconvex mapping on $[\alpha, \beta]$, then the following generalized fractional integral inequality holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\epsilon+1)}{2\Lambda_{\omega}^{\epsilon}(1)} \left[\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-};\omega} H(\alpha) + \mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega} H(\beta)\right] - h\left(\frac{\alpha+\beta}{2}\right)\right| \\
&\leq \frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)} \left[\mathcal{M}_{\alpha}^{\beta}(|h'|;\eta) + \mathcal{N}_{\alpha}^{\beta}(|h'|;\eta)\right] \int_0^1 \Lambda_{\omega}^{\epsilon}(\tau)\, d\tau.
\end{aligned}$$

Proof. Given that the function $|h'|$ is η-quasiconvex on $[\alpha, \beta]$ implies that for all $\tau \in [0, 1]$, the following inequalities hold:

$$\left|h'\left(\frac{\tau}{2}\beta + \frac{2-\tau}{2}\alpha\right)\right| \leq \mathcal{M}_{\alpha}^{\beta}(|h'|;\eta) \tag{16}$$

and

$$\left|h'\left(\frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta\right)\right| \leq \mathcal{N}_{\alpha}^{\beta}(|h'|;\eta). \tag{17}$$

Taking absolute values of both sides of Lemma 2.3 and using inequalities (16) and (17), we get

$$\begin{aligned}
&\left|\frac{\Gamma(\epsilon+1)}{2\Lambda_{\omega}^{\epsilon}(1)} \left[\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-};\omega} H(\alpha) + \mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega} H(\beta)\right] - h\left(\frac{\alpha+\beta}{2}\right)\right| \\
&\leq \frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)} \int_0^1 \left|\Lambda_{\omega}^{\epsilon}(\tau)\right| \left|h'\left(\frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta\right) - h'\left(\frac{\tau}{2}\beta + \frac{2-\tau}{2}\alpha\right)\right| d\tau \\
&\leq \frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)} \int_0^1 \left|\Lambda_{\omega}^{\epsilon}(\tau)\right| \left|h'\left(\frac{\tau}{2}\alpha + \frac{2-\tau}{2}\beta\right)\right| d\tau \\
&\quad + \frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)} \int_0^1 \left|\Lambda_{\omega}^{\epsilon}(\tau)\right| \left|h'\left(\frac{\tau}{2}\beta + \frac{2-\tau}{2}\alpha\right)\right| d\tau \\
&\leq \frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)} \left[\mathcal{M}_{\alpha}^{\beta}(|h'|;\eta) + \mathcal{N}_{\alpha}^{\beta}(|h'|;\eta)\right] \int_0^1 \left|\Lambda_{\omega}^{\epsilon}(\tau)\right| d\tau.
\end{aligned}$$

This completes the proof. □

Corollary 2.5. *Let $\epsilon > 0$ and let the mappings ω and h be as in Theorem 2.1 and Lemma 2.3, respectively. If $|h'|$ is a quasiconvex mapping on $[\alpha, \beta]$, then the following generalized fractional integral inequality holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\epsilon+1)}{2\Lambda_{\omega}^{\epsilon}(1)}\left[\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-};\omega}H(\alpha)+\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega}H(\beta)\right]-h\left(\frac{\alpha+\beta}{2}\right)\right|\\ &\leq \frac{\beta-\alpha}{2\Lambda_{\omega}^{\epsilon}(1)}\max\left\{|h'(\alpha)|,|h'(\beta)|\right\}\int_0^1\Lambda_{\omega}^{\epsilon}(\tau)\,d\tau.\end{aligned}$$

Proof. The proof follows by taking $\eta(x, y) = x - y$ and then observing that

$$\mathcal{M}_{\alpha}^{\beta}(|h'|;\eta)=\mathcal{N}_{\alpha}^{\beta}(|h'|;\eta)=\max\left\{|h'(\alpha)|,|h'(\beta)|\right\}.$$

□

Corollary 2.6. *Let $\epsilon > 0$ and let the mapping h be as in Lemma 2.3. If $|h'|$ is a quasiconvex mapping on $[\alpha, \beta]$, then the following Riemann–Liouville fractional integral inequality holds:*

$$\begin{aligned}&\left|\mathbf{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-}}H(\alpha)+\mathbf{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+}}H(\beta)-\frac{(\beta-\alpha)^{\epsilon}}{\Gamma(\epsilon+1)2^{\epsilon-2}}h\left(\frac{\alpha+\beta}{2}\right)\right|\\ &\leq \frac{(\beta-\alpha)^{\epsilon+1}}{2^{\epsilon-1}\Gamma(\epsilon+2)}\max\left\{|h'(\alpha)|,|h'(\beta)|\right\}.\end{aligned}$$

Proof. Let $0 < \alpha < \beta$ and $\omega : [\alpha, \beta] \to \mathbb{R}$ be defined by $\omega(x) = k(x) = x$. From (3), we observe that

$$\Lambda_k^{\epsilon}(\tau):=\frac{(\beta-\alpha)^{\epsilon}}{2^{\epsilon-1}}\tau^{\epsilon},$$

and

$$\int_0^1\left|\Lambda_k^{\epsilon}(\tau)\right|d\tau=\int_0^1\frac{(\beta-\alpha)^{\epsilon}}{2^{\epsilon-1}}\tau^{\epsilon}\,d\tau=\frac{(\beta-\alpha)^{\epsilon}}{(\epsilon+1)2^{\epsilon-1}}.$$

Hence, we get the desired inequality by applying Corollary 2.5. □

Corollary 2.7. *Let $\epsilon > 0$ and let the mapping h be as in Lemma 2.3. If $|h'|$ is a quasiconvex mapping on $[\alpha, \beta]$, then the following Hadamard fractional integral inequality holds:*

$$\begin{aligned}&\left|\mathbf{J}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-}}H(\alpha)+\mathbf{J}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+}}H(\beta)-\frac{2}{\Gamma(\epsilon+1)}\left(\left[\ln\frac{2\beta}{\alpha+\beta}\right]^{\epsilon}+\left[\ln\frac{\alpha+\beta}{2\alpha}\right]^{\epsilon}\right)h\left(\frac{\alpha+\beta}{2}\right)\right|\\ &\leq \frac{\beta-\alpha}{\Gamma(\epsilon+1)}\max\left\{|h'(\alpha)|,|h'(\beta)|\right\}\int_0^1\Lambda_{\ln}^{\epsilon}(\tau)\,d\tau,\end{aligned}$$

where

$$\Lambda_{\ln}^{\epsilon}(\tau)=\left[\ln\left(\frac{2\beta}{\tau\alpha+(2-\tau)\beta}\right)\right]^{\epsilon}+\left[\ln\left(\frac{\tau\beta+(2-\tau)\alpha}{2\alpha}\right)\right]^{\epsilon}. \tag{18}$$

Proof. In this case, let $0 < \alpha < \beta$ and $\omega : [\alpha, \beta] \to \mathbb{R}$ be defined by $\omega(x) = \ln x$. We get the intended result by applying (3) and (15) in Corollary 2.5. □

Theorem 2.8. *Let $\epsilon > 0$ and let the mappings ω and h be as in Theorem 2.1 and Lemma 2.3, respectively. If $|h'|^q$, $q > 1$, is an η-quasiconvex mapping on $[\alpha, \beta]$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following generalized fractional integral inequality holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\epsilon+1)}{2\Lambda_{\omega}^{\epsilon}(1)}\left[\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-};\omega}H(\alpha)+\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega}H(\beta)\right]-h\left(\frac{\alpha+\beta}{2}\right)\right|\\ &\leq \frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)}\left[\int_0^1\left|\Lambda_{\omega}^{\epsilon}(\tau)\right|^p d\tau\right]^{\frac{1}{p}}\left[\left(\mathcal{M}_{\alpha}^{\beta}(|h'|^q;\eta)\right)^{\frac{1}{q}}+\left(\mathcal{N}_{\alpha}^{\beta}(|h'|^q;\eta)\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. Following a similar approach as in the proof of Theorem 2.4 and using the Hölder's inequality, one gets

$$\begin{aligned}
&\left|\frac{\Gamma(\epsilon+1)}{2\Lambda_\omega^\epsilon(1)}\left[\mathfrak{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^-;\omega}H(\alpha)+\mathfrak{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^+;\omega}H(\beta)\right]-h\left(\frac{\alpha+\beta}{2}\right)\right| \\
&\leq \frac{\beta-\alpha}{4\Lambda_\omega^\epsilon(1)}\int_0^1\left|\Lambda_\omega^\epsilon(\tau)\right|\left|h'\left(\frac{\tau}{2}\alpha+\frac{2-\tau}{2}\beta\right)-h'\left(\frac{\tau}{2}\beta+\frac{2-\tau}{2}\alpha\right)\right|d\tau \\
&\leq \frac{\beta-\alpha}{4\Lambda_\omega^\epsilon(1)}\int_0^1\left|\Lambda_\omega^\epsilon(\tau)\right|\left|h'\left(\frac{\tau}{2}\alpha+\frac{2-\tau}{2}\beta\right)\right|d\tau \\
&\quad+\frac{\beta-\alpha}{4\Lambda_\omega^\epsilon(1)}\int_0^1\left|\Lambda_\omega^\epsilon(\tau)\right|\left|h'\left(\frac{\tau}{2}\beta+\frac{2-\tau}{2}\alpha\right)\right|d\tau \\
&\leq \frac{\beta-\alpha}{4\Lambda_\omega^\epsilon(1)}\left[\int_0^1\left|\Lambda_\omega^\epsilon(\tau)\right|^p d\tau\right]^{\frac{1}{p}}\left[\int_0^1\left|h'\left(\frac{\tau}{2}\alpha+\frac{2-\tau}{2}\beta\right)\right|^q d\tau\right]^{\frac{1}{q}} \\
&\quad+\frac{\beta-\alpha}{4\Lambda_\omega^\epsilon(1)}\left[\int_0^1\left|\Lambda_\omega^\epsilon(\tau)\right|^p d\tau\right]^{\frac{1}{p}}\left[\int_0^1\left|h'\left(\frac{\tau}{2}\beta+\frac{2-\tau}{2}\alpha\right)\right|^q d\tau\right]^{\frac{1}{q}} \\
&\leq \frac{\beta-\alpha}{4\Lambda_\omega^\epsilon(1)}\left[\int_0^1\left|\Lambda_\omega^\epsilon(\tau)\right|^p d\tau\right]^{\frac{1}{p}}\left[\left(\mathcal{M}_\alpha^\beta(|h'|^q;\eta)\right)^{\frac{1}{q}}+\left(\mathcal{N}_\alpha^\beta(|h'|^q;\eta)\right)^{\frac{1}{q}}\right].
\end{aligned}$$

The desired inequality is achieved. □

Substituting $\eta(x,y)=x-y$, one gets:

Corollary 2.9. *Let $\epsilon>0$ and let the mappings ω and h be as in Theorem 2.1 and Lemma 2.3, respectively. If $|h'|^q$, $q>1$, is a quasiconvex mapping on $[\alpha,\beta]$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following generalized fractional integral inequality holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\epsilon+1)}{2\Lambda_\omega^\epsilon(1)}\left[\mathfrak{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^-;\omega}H(\alpha)+\mathfrak{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^+;\omega}H(\beta)\right]-h\left(\frac{\alpha+\beta}{2}\right)\right| \\
&\leq \frac{\beta-\alpha}{2\Lambda_\omega^\epsilon(1)}\left[\int_0^1\left|\Lambda_\omega^\epsilon(\tau)\right|^p d\tau\right]^{\frac{1}{p}}\left[\max\left\{|h'(\alpha)|^q,|h'(\beta)|^q\right\}\right]^{\frac{1}{q}}.
\end{aligned}$$

2.3. Trapezoid type Inequalities in Generalized form

The main result, in this subsection, shall be anchored on the following lemma:

Lemma 2.10 ([2]). *Let $\epsilon>0$ and let the mapping ω be as in Theorem 2.1. If $h:[\alpha,\beta]\to\mathbb{R}$ be a differentiable mapping on (α,β) with $\alpha<\beta$, then the following equality holds:*

$$\begin{aligned}
&\frac{h(\alpha)+h(\beta)}{2}-\frac{\Gamma(\epsilon+1)}{2\Lambda_\omega^\epsilon(1)}\left[\mathfrak{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^-;\omega}H(\alpha)+\mathfrak{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^+;\omega}H(\beta)\right] \\
&=\frac{\beta-\alpha}{4\Lambda_\omega^\epsilon(1)}\int_0^1\left(\Lambda_\omega^\epsilon(1)-\Lambda_\omega^\epsilon(\tau)\right)\left[h'\left(\frac{\tau}{2}\alpha+\frac{2-\tau}{2}\beta\right)-h'\left(\frac{\tau}{2}\beta+\frac{2-\tau}{2}\alpha\right)\right]d\tau,
\end{aligned}$$

where the mappings H and Λ_ω^ϵ are the same as in Lemma 2.3.

Theorem 2.11. *Let $\epsilon > 0$ and let the mappings ω and h be as in Theorem 2.1 and Lemma 2.10, respectively. If $|h'|$ is an η-quasiconvex mapping on $[\alpha,\beta]$, then the following generalized fractional integral inequality holds:*

$$\begin{aligned}&\left|\frac{h(\alpha)+h(\beta)}{2}-\frac{\Gamma(\epsilon+1)}{2\Lambda_\omega^\epsilon(1)}\left[\mathfrak{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^-;\omega}H(\alpha)+\mathfrak{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^+;\omega}H(\beta)\right]\right|\\&\le\frac{\beta-\alpha}{4\Lambda_\omega^\epsilon(1)}\left[\mathcal{M}_\alpha^\beta(|h'|;\eta)+\mathcal{N}_\alpha^\beta(|h'|;\eta)\right]\int_0^1\left|\Lambda_\omega^\epsilon(1)-\Lambda_\omega^\epsilon(\tau)\right|d\tau.\end{aligned}$$

Proof. In this case, we employ Lemma 2.10 and take absolute values of both sides to obtain:

$$\begin{aligned}&\left|\frac{h(\alpha)+h(\beta)}{2}-\frac{\Gamma(\epsilon+1)}{2\Lambda_\omega^\epsilon(1)}\left[\mathfrak{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^-;\omega}H(\alpha)+\mathfrak{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^+;\omega}H(\beta)\right]\right|\\&\le\frac{\beta-\alpha}{4\Lambda_\omega^\epsilon(1)}\int_0^1\left|\Lambda_\omega^\epsilon(1)-\Lambda_\omega^\epsilon(\tau)\right|\left|h'\left(\frac{\tau}{2}\alpha+\frac{2-\tau}{2}\beta\right)-h'\left(\frac{\tau}{2}\beta+\frac{2-\tau}{2}\alpha\right)\right|d\tau,\\&\le\frac{\beta-\alpha}{4\Lambda_\omega^\epsilon(1)}\int_0^1\left|\Lambda_\omega^\epsilon(1)-\Lambda_\omega^\epsilon(\tau)\right|\left|h'\left(\frac{\tau}{2}\alpha+\frac{2-\tau}{2}\beta\right)\right|d\tau\\&\quad+\frac{\beta-\alpha}{4\Lambda_\omega^\epsilon(1)}\int_0^1\left|\Lambda_\omega^\epsilon(1)-\Lambda_\omega^\epsilon(\tau)\right|\left|h'\left(\frac{\tau}{2}\beta+\frac{2-\tau}{2}\alpha\right)\right|d\tau\\&\le\frac{\beta-\alpha}{4\Lambda_\omega^\epsilon(1)}\left[\mathcal{M}_\alpha^\beta(|h'|;\eta)+\mathcal{N}_\alpha^\beta(|h'|;\eta)\right]\int_0^1\left|\Lambda_\omega^\epsilon(1)-\Lambda_\omega^\epsilon(\tau)\right|d\tau.\end{aligned}$$

This finishes the proof. □

Putting $\eta(x,y)=x-y$, gives:

Corollary 2.12. *Let $\epsilon > 0$ and let the mappings ω and h be as in Theorem 2.1 and Lemma 2.10, respectively. If $|h'|$ is a quasiconvex mapping on $[\alpha,\beta]$, then the following generalized fractional integral inequality holds:*

$$\begin{aligned}&\left|\frac{h(\alpha)+h(\beta)}{2}-\frac{\Gamma(\epsilon+1)}{2\Lambda_\omega^\epsilon(1)}\left[\mathfrak{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^-;\omega}H(\alpha)+\mathfrak{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^+;\omega}H(\beta)\right]\right|\\&\le\frac{\beta-\alpha}{2\Lambda_\omega^\epsilon(1)}\max\left\{|h'(\alpha)|,|h'(\beta)|\right\}\int_0^1\left|\Lambda_\omega^\epsilon(1)-\Lambda_\omega^\epsilon(\tau)\right|d\tau.\end{aligned}$$

Let $0<\alpha<\beta$ and $\omega:[\alpha,\beta]\to\mathbb{R}$ be defined by $\omega(x)=k(x)=x$. Then

$$\int_0^1\left|\Lambda_\omega^\epsilon(1)-\Lambda_\omega^\epsilon(\tau)\right|d\tau=\frac{(\beta-\alpha)^\epsilon}{2^{\epsilon-1}}\int_0^1(1-\tau^\epsilon)\,d\tau=\frac{(\beta-\alpha)^\epsilon}{2^{\epsilon-1}}\frac{\epsilon}{\epsilon+1}.$$

Hence, the inequality in Corollary 2.12 reduces to the following result involving the fractional Riemann–Liouville operators:

$$\begin{aligned}&\left|\frac{h(\alpha)+h(\beta)}{2}-\frac{2^{\epsilon-2}\Gamma(\epsilon+1)}{(\beta-\alpha)^\epsilon}\left[\mathbf{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^-}H(\alpha)+\mathbf{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^+}H(\beta)\right]\right|\\&\le\frac{(\beta-\alpha)\epsilon}{2(\epsilon+1)}\max\left\{|h'(\alpha)|,|h'(\beta)|\right\}.\end{aligned}$$

If, in the other hand, we let $\omega(x)=\ln x$ with $x\in[\alpha,\beta]\subset(0,\infty)$. Then we deduce from Corollary 2.12 the following Hadamard fractional integral inequality:

$$\left|\frac{h(\alpha)+h(\beta)}{2}-\frac{\Gamma(\epsilon+1)}{2\Lambda_{\ln}^{\epsilon}(1)}\left[\mathbf{J}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-}}H(\alpha)+\mathbf{J}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+}}H(\beta)\right]\right|$$
$$\leq \frac{\beta-\alpha}{2\Lambda_{\ln}^{\epsilon}(1)}\max\left\{|h'(\alpha)|,|h'(\beta)|\right\}\int_0^1\left|\Lambda_{\ln}^{\epsilon}(1)-\Lambda_{\ln}^{\epsilon}(\tau)\right|d\tau,$$

where $\Lambda_{\ln}^{\epsilon}(1)$ and $\Lambda_{\ln}^{\epsilon}(\tau)$ are defined by (15) and (18), respectively.

Theorem 2.13. *Let $\epsilon > 0$ and let the mappings ω and h be as in Theorem 2.1 and Lemma 2.10, respectively. If $|h'|^q$, $q > 1$, is an η-quasiconvex mapping on $[\alpha,\beta]$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following generalized fractional integral inequality holds:*

$$\left|\frac{h(\alpha)+h(\beta)}{2}-\frac{\Gamma(\epsilon+1)}{2\Lambda_{\omega}^{\epsilon}(1)}\left[\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-};\omega}H(\alpha)+\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega}H(\beta)\right]\right|$$
$$\leq \frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)}\left[\int_0^1\left|\Lambda_{\omega}^{\epsilon}(1)-\Lambda_{\omega}^{\epsilon}(\tau)\right|^p d\tau\right]^{\frac{1}{p}}\left[\left(\mathcal{M}_{\alpha}^{\beta}(|h'|^q;\eta)\right)^{\frac{1}{q}}+\left(\mathcal{N}_{\alpha}^{\beta}(|h'|^q;\eta)\right)^{\frac{1}{q}}\right].$$

Proof. Applying Lemma 2.10 together with the Hölder's inequality and the η-quasiconvexity of $|h'|^q$, one gets

$$\left|\frac{h(\alpha)+h(\beta)}{2}-\frac{\Gamma(\epsilon+1)}{2\Lambda_{\omega}^{\epsilon}(1)}\left[\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-};\omega}H(\alpha)+\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega}H(\beta)\right]\right|$$
$$\leq \frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)}\int_0^1\left|\Lambda_{\omega}^{\epsilon}(1)-\Lambda_{\omega}^{\epsilon}(\tau)\right|\left|h'\left(\frac{\tau}{2}\alpha+\frac{2-\tau}{2}\beta\right)-h'\left(\frac{\tau}{2}\beta+\frac{2-\tau}{2}\alpha\right)\right|d\tau$$
$$\leq \frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)}\int_0^1\left|\Lambda_{\omega}^{\epsilon}(1)-\Lambda_{\omega}^{\epsilon}(\tau)\right|\left|h'\left(\frac{\tau}{2}\alpha+\frac{2-\tau}{2}\beta\right)\right|d\tau$$
$$+\frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)}\int_0^1\left|\Lambda_{\omega}^{\epsilon}(1)-\Lambda_{\omega}^{\epsilon}(\tau)\right|\left|h'\left(\frac{\tau}{2}\beta+\frac{2-\tau}{2}\alpha\right)\right|d\tau$$
$$\leq \frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)}\left[\int_0^1\left|\Lambda_{\omega}^{\epsilon}(1)-\Lambda_{\omega}^{\epsilon}(\tau)\right|^p d\tau\right]^{\frac{1}{p}}\left[\int_0^1\left|h'\left(\frac{\tau}{2}\alpha+\frac{2-\tau}{2}\beta\right)\right|^q d\tau\right]^{\frac{1}{q}}$$
$$+\frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)}\left[\int_0^1\left|\Lambda_{\omega}^{\epsilon}(1)-\Lambda_{\omega}^{\epsilon}(\tau)\right|^p d\tau\right]^{\frac{1}{p}}\left[\int_0^1\left|h'\left(\frac{\tau}{2}\beta+\frac{2-\tau}{2}\alpha\right)\right|^q d\tau\right]^{\frac{1}{q}}$$
$$\leq \frac{\beta-\alpha}{4\Lambda_{\omega}^{\epsilon}(1)}\left[\int_0^1\left|\Lambda_{\omega}^{\epsilon}(1)-\Lambda_{\omega}^{\epsilon}(\tau)\right|^p d\tau\right]^{\frac{1}{p}}\left[\left(\mathcal{M}_{\alpha}^{\beta}(|h'|^q;\eta)\right)^{\frac{1}{q}}+\left(\mathcal{N}_{\alpha}^{\beta}(|h'|^q;\eta)\right)^{\frac{1}{q}}\right].$$

The desired inequality is obtained. □

Corollary 2.14. *Let $\epsilon > 0$ and let the mappings ω and h be as in Theorem 2.1 and Lemma 2.10, respectively. If $|h'|^q$, $q > 1$, is an quasiconvex mapping on $[\alpha,\beta]$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following generalized fractional integral inequality holds:*

$$\left|\frac{h(\alpha)+h(\beta)}{2}-\frac{\Gamma(\epsilon+1)}{2\Lambda_{\omega}^{\epsilon}(1)}\left[\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{-};\omega}H(\alpha)+\mathfrak{I}^{\epsilon}_{\left(\frac{\alpha+\beta}{2}\right)^{+};\omega}H(\beta)\right]\right|$$
$$\leq \frac{\beta-\alpha}{2\Lambda_{\omega}^{\epsilon}(1)}\left[\int_0^1\left|\Lambda_{\omega}^{\epsilon}(1)-\Lambda_{\omega}^{\epsilon}(\tau)\right|^p d\tau\right]^{\frac{1}{p}}\left[\max\left\{|h'(\alpha)|^q,|h'(\beta)|^q\right\}\right]^{\frac{1}{q}}.$$

Remark 2.15. *Many interesting inequalities can be deduced from Corollaries 2.9 and 2.14 by choosing different functions for ω. For instance, if we take $\omega(x) = x$ and $\epsilon \in (0,1]$, then*

$$\begin{aligned}\frac{\beta-\alpha}{2\Lambda_\omega^\epsilon(1)}\left[\int_0^1 \left|\Lambda_\omega^\epsilon(1)-\Lambda_\omega^\epsilon(\tau)\right|^p d\tau\right]^{\frac{1}{p}} &= \frac{\beta-\alpha}{2}\left[\int_0^1 |1-\tau^\epsilon|^p\, d\tau\right]^{\frac{1}{p}}\\ &\le \frac{\beta-\alpha}{2}\left[\int_0^1 |1-\tau|^{p\epsilon}\, d\tau\right]^{\frac{1}{p}}\\ &= \frac{\beta-\alpha}{2}\left(\frac{1}{p\epsilon+1}\right)^{\frac{1}{p}}.\end{aligned}$$

Using this, the inequality in Corollary 2.14 becomes the following estimate involving the Riemann–Liouville fractional integral operators:

$$\begin{aligned}&\left|\frac{h(\alpha)+h(\beta)}{2}-\frac{2^{\epsilon-2}\Gamma(\epsilon+1)}{(\beta-\alpha)^\epsilon}\left[\mathbf{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^-}H(\alpha)+\mathbf{I}^\epsilon_{\left(\frac{\alpha+\beta}{2}\right)^+}H(\beta)\right]\right|\\ &\le \frac{\beta-\alpha}{2}\left(\frac{1}{p\epsilon+1}\right)^{\frac{1}{p}}\left[\max\left\{|h'(\alpha)|^q,|h'(\beta)|^q\right\}\right]^{\frac{1}{q}}.\end{aligned}$$

3. Conclusion

In 2016, the class of η-(quasi)convex function was introduced. Some results concerning this class of functions have been published, see [1, 3, 4, 8, 9, 11, 14–16] and the references therein. In this article, fractional integral inequalities of the Hermite–Hadamard, midpoint and trapezoid types are established. Our results reduce to inequalities involving the Riemann–Liouville and Hadamard operators as particular cases. Since this is a new class, we anticipate that this paper will stimulate further interest in this regard.

Acknowledgement

Many thanks to the anonymous referee(s).

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