Pseudoanalytic Extension on $F(p, p - 2, s)$ Spaces and Applications

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Abstract. In this paper, we generalized the main results in [9]. As an applications, we give a characterization of the closure of $F(p, p - 2, s)$ spaces in Lipschitz-type spaces $A_ω$ by pseudoanalytic extension.

1. Introduction

Let $D$ be the unit disk in the complex plane $C$ and $H(D)$ be the class of functions analytic in $D$. For $0 < p < ∞$, $H^p$ denotes the Hardy space, which consisting of all functions $f ∈ H(D)$ satisfied (see [13])

$$∥f∥_p^H = \sup_{0 < r < 1} \frac{1}{2π} \int_0^{2π} |f(re^iθ)|^p dθ < ∞.$$  

As usual, $H^∞$ is the set of bounded analytic functions in $D$ and $A$ denotes the disc algebra.

Let $0 < p < ∞$, $−2 < q < ∞$ and $s ≥ 0$. The $F(p, q, s)$ ([27]) space is the set of all $f ∈ H(D)$ such that

$$\sup_{a ∈ D} \int_D |f′(z)|^p (1 − |z|^2)^q (1 − |φ_a(z)|^2)^s dA(z) < ∞,$$

where $φ_a(z) = \frac{z - a}{1 - az}$ and $dA(z) = \frac{1}{π} dxdy$. When $q = p − 2$, $F(p, p − 2, s)$ is Möbius invariant Besov-type spaces. When $0 < s < 1$, $F(2, 0, s) = Q_s$ ([24, 25]); If $s = 1$, $F(2, 0, 1) = BMOA$, the space of analytic functions in the Hardy space $H^1(D)$ whose boundary functions have bounded mean oscillation. When $s > 1$, $F(2, 0, s) = B$ (the Bloch space).

Let $ω : [0, ∞) → R$ be a right-continuous with $ω(0) = 0$. If $ω$ is increasing and $\frac{ω(t)}{t}$ is nonincreasing for $t > 0$, there exists constant $C(ω)$ such that

$$\int_0^∞ \frac{ω(t)}{t} dt + \delta \int_0^∞ \frac{ω(t)}{t^2} dt ≤ C(ω) · ω(δ),$$

then we say that $ω$ is a regular majorant, where $0 < δ < 1$.
Given a regular majorant \( \omega \) and a compact set \( E \subset \mathbb{C} \), the Lipschitz-type spaces \( \Lambda_\omega(E) \) consists of those functions \( f : E \to \mathbb{C} \), such that
\[
\|f\|_{\Lambda_\omega} = \sup \left\{ \frac{|f(z) - f(w)|}{\omega(|z - w|)} : z, w \in E, z \neq w \right\} < \infty.
\]

In this paper, we shall be concerned with the space \( \mathcal{A}_\omega = \mathcal{A} \cap \Lambda_\omega(\mathbb{D}) \). When \( \omega(t) = t^\alpha, 0 < \alpha < 1 \), it give the classical Lipschitz space \( \Lambda_\omega \). For more informations on \( \mathcal{A}_\omega \), we refer to [4] and the paper referinthere.

Pseudoanalytic extension, as explained in [10], an analytic function in \( \mathbb{D} \) can be extended to \( \mathbb{D}_e = \{ z : |z| > 1 \} \) as a \( C^1 \) function whose Cauchy-Riemann \( \partial \)-derivative becomes appropriately small. There are many applications for pseudoanalytic extension, for example: \( K \)-property ([9]); inner-outer factorization ([10]); Bernstein-type inequality related to kernel of \( H^p \) spaces ([6]) and so on.

In this paper, we generalize the main results in [9] to \( F(p, p - 2, s) \) spaces. Moreover, we also give an application on our result to studying the closure of \( F(p, p - 2, s) \) spaces in Lipschitz-type spaces \( \mathcal{A}_\omega \) (denoted by \( C_{\mathcal{A}_\omega}(\mathcal{A}_\omega \cap F(p, p - 2, s)) \)) by pseudoanalytic extension.

In this paper, the symbol \( f \approx g \) means that \( f \leq g \leq f \). We say that \( f \leq g \) if there exists a constant \( C \) such that \( f \leq Cg \).

2. Auxiliary results

If \( Q \) is a measurable subset of \( \mathbb{C} \) and \( Q \) varies over all discs in \( \mathbb{C} \), \( |Q| \) will denote the measure (area) of \( Q \). Let \( \omega \) be a positive measurable function on \( \mathbb{C} \). We say that \( \omega \) is an \( A_t \)-weight \((t > 1) \) if (see [22])
\[
\sup_Q \left( \frac{1}{|Q|} \int_Q \omega(z) dA(z) \right) \left( \frac{1}{|Q|} \int_Q \frac{1}{\omega(z)} dA(z) \right)^{t-1} < \infty.
\]

**Remark 1.** Let \( t > 1 \) and \( \omega \) be an \( A_t \)-weight and \( T \) be a Calderon-Zygmund operator. It is well know that (see [22])
\[
\int_{\mathbb{C}} |T f(z)|^t \omega(z) dA(z) \leq \int_{\mathbb{C}} |f(z)|^t \omega(z) dA(z), \text{ for all } f \in L^1(\omega).
\]

Here \( L^1(\omega) \) denote the space of functions \( f \in L^1 \) which satisfy
\[
\int_{\mathbb{C}} |f(z)|^t \omega(z) dA(z) < \infty.
\]

The following lemma generalized [9, Proposition 1].

**Lemma 1.** Suppose that \( 1 < p < \infty, 0 < s < 1, p + s > 2, z \in \mathbb{C} \) and \( a \in \mathbb{D} \). Then \( |1 - |z|^2|^{p-2} \left| \frac{1}{|\omega(z)|^2} - 1 \right|^s \) is an \( A_p \)-weight.

**Proof.** Since
\[
|1 - |z|^2|^{p-2} \left| \frac{1}{|\omega(z)|^2} - 1 \right|^s = \left( 1 - |a|^2 \right)^s |z|^2 - 1 |z - a|^{2s}.
\]

Let
\[
M_a(z) = \frac{\left( 1 - |a|^2 \right)^s |z|^2 - 1 |z - a|^{2s}}{2^s |z - a|^{2s}}
\]
and
\[
N_a(z) = \frac{|z|^2 - 1 |z - a|^{2s}}{2^s |z - a|^{2s}}.
\]
It is easily to see that $M_r(z)$ is an $A_p$-weight if and only if $N_r(z)$ is an $A_p$-weight. Now, we adopt and modify the method in [9, Proposition 1]. Suppose that $N_r(z) = J(z)K_0(z)$, where

$$J(z) = \|z|^{2-s-1}\|^{p-2+s}, \, \, K_0(z) = \frac{1}{|z-a|^{2s}}, \, \, K_0(z) = \frac{1}{|z|^{2s}}.$$  

From [22, page 218], we known that $K_0(z)$ is an $A_t$-weight ($t > 1$). Since $K_0(z)$ are translates of $K_0(z)$, we have $K_0(z)$ is also an $A_t$-weight, that is,

$$\sup_{Q} \left( \frac{1}{|Q|} \int_{Q} K_0(z)dA(z) \right) \left( \frac{1}{|Q|} \int_{Q} \frac{1}{K_0^p(z)} dA(z) \right)^{\frac{1}{t-1}} < \infty.$$  

Let $r \in (1, \frac{p-1}{p-2+s})$ and $Q$ be any disc. Let $\frac{1}{r} + \frac{1}{r'} = 1$. Then, for any $a \in D$, we have

$$\left(\frac{1}{|Q|} \int_{Q} N_a(z) dA(z) \right) \left(\frac{1}{|Q|} \int_{Q} \frac{1}{N_a^p(z)} dA(z) \right)^{\frac{1}{p-1}} = \left(\frac{1}{|Q|} \int_{Q} J(z) K_a(z) dA(z) \right) \left(\frac{1}{|Q|} \int_{Q} \frac{1}{J(z) K_a^p(z)} dA(z) \right)^{\frac{1}{p-1}} \leq \left(\sup_{z \in Q} J(z) \right) \times \left(\frac{1}{|Q|} \int_{Q} K_a(z) dA(z) \right) \times \left(\frac{1}{|Q|} \int_{Q} \frac{1}{K_a(z)} dA(z) \right)^{\frac{1}{p-1}}.$$

By direct calculation (or see [9, page 484]), we obtain

$$\left(\sup_{z \in Q} J(z) \right) \times \left(\frac{1}{|Q|} \int_{Q} \frac{1}{J(z)} dA(z) \right)^{\frac{1}{p-1}} < \infty.$$

Thus,

$$\left(\frac{1}{|Q|} \int_{Q} N_a(z) dA(z) \right) \left(\frac{1}{|Q|} \int_{Q} \frac{1}{N_a^p(z)} dA(z) \right)^{\frac{1}{p-1}} \leq \left(\frac{1}{|Q|} \int_{Q} K_a(z) dA(z) \right) \times \left(\frac{1}{|Q|} \int_{Q} \frac{1}{K_a(z)} dA(z) \right)^{\frac{1}{p-1}}.$$

If $2-s < p \leq 2$, it easily to see that $\frac{p}{p-1} > 1$. If $p > 2$, noted that $r \in (1, \frac{p-1}{p-2+s})$ and $\frac{1}{r} + \frac{1}{r'} = 1$, we can also deduce that $\frac{p}{p-1} > 1$. Let $t = \frac{p-1+s}{p} > 1$. Combined with $(*),$ we have

$$\left(\frac{1}{|Q|} \int_{Q} K_a(z) dA(z) \right) \times \left(\frac{1}{|Q|} \int_{Q} \frac{1}{K_a(z)} dA(z) \right)^{\frac{1}{p-1}} < \infty.$$
Therefore,
\[
\left( \frac{1}{|Q|} \int_Q N_a(z) \ dA(z) \right) \left( \frac{1}{|Q|} \int_Q \frac{1}{N_{a^*}(z)} \ dA(z) \right)^{r-1} < \infty,
\]
for any \(a \in \mathbb{D}\). The proof is completed. \(\square\)

3. Pseudoanalytic extension on \(F(p, p - 2, s)\)

Now, let us consider the pseudoanalytic extension on \(F(p, p - 2, s)\).

**Theorem 1.** Suppose that \(p > 1, 0 < s < 1, p + s > 2\) and \(f \in \bigcap_{0<q<\infty} H^q\). Then the following are equivalent:

1. \(f \in F(p, p - 2, s)\);

2. \[\sup_{a \in \mathbb{D}} \int_D |f'(z)|^p (1 - |z|^2)^{p-2s} \left( \frac{1}{|\partial_A(z)|^2} - 1 \right)^q \ dA(z) < \infty;\]

3. There exists a function \(F \in C^1(\mathbb{D}_r)\) satisfying
   \[F(z) = O(1), \quad \text{as } z \to \infty, \quad (a)\]
   \[\lim_{r \to 1} F(re^{i\theta}) = f(e^{i\theta}), \ a.e \ and \ in \ L^q([\pi, \pi]) \ \text{for all } q \in [1, \infty), \quad (b)\]
   \[\sup_{a \in \mathbb{D}} \int_{\mathbb{D}_r} |\partial F(z)|^p (|z|^2 - 1)^{p-2s} \left( |\partial_A(z)|^2 - 1 \right)^q \ dA(z) < \infty. \quad (c)\]

**Proof.** (1) \(\Leftrightarrow\) (2). Since \(F(p, p - 2, s)\) space is Möbius invariant, we only need to prove that (the case \(a = 0\))

\[
\int_D |f'(z)|^p (1 - |z|^2)^{p-2s} \ dA(z) = \int_D |f'(z)|^p (1 - |z|^2)^{p-2s} \left( \frac{1}{|z|^2} - 1 \right)^q \ dA(z) - \int_D |f'(z)|^p \left( \frac{1}{|z|^2} - 1 \right)^q \ dA(z).
\]

On the one hand, it is obvious that
\[
\int_D |f'(z)|^p (1 - |z|^2)^{p-2s} \ dA(z) \leq \int_D |f'(z)|^p \left( \frac{1}{|z|^2} - 1 \right)^q \ dA(z).
\]

On the other hand, let
\[
M_p(r, f')^p = \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^p \ d\theta.
\]

Bearing in mind that \(M_p(r, f')^p\) is an increasing function of \(r\), we have
\[
\int_D |f'(z)|^p \left( \frac{1}{|z|^2} - 1 \right)^q \ dA(z) = \int_0^1 M_p(r, f')^p (1 - r^2)^{p-2s} r^{1-2s} dr \leq M_p(\frac{1}{2}, f')^p \int_0^1 (1 - r^2)^{p-2s} r^{1-2s} dr + 4^s \int_0^1 M_p(r, f')^p (1 - r^2)^{p-2s} r dr \leq (C(p, s) + 4^s) \int_0^1 M_p(r, f')^p (1 - r^2)^{p-2s} r dr \leq \int_D |f'(z)|^p (1 - |z|^2)^{p-2s} \ dA(z),
\]
where
\[
C(p, s) = \frac{\int_1^2 (1 - r^2)p^{-2s+1-2s} dr}{\int_1^2 (1 - r^2)p^{-2s} rdr} < \infty.
\]

We get the desired result.

(1) \(\Rightarrow\) (3). Suppose \(f \in F(p, p-2, s)\), let \(z^* = \frac{1}{z}\) and
\[
F(z) = f(z^*) , \quad z \in D_r.
\]

Hence, \(F \in C^1(D_n)\) and satisfies (a) and (b). Let \(a \in D\). Using the fact that \(\| \partial F(z) \| = \| f'(z^*) \| z^*^2\) and combining with (1) \(\Leftrightarrow\) (2), we deduce that
\[
\int_{D_r} |\partial F(z)|(\|z\| - 1)^{p-2} (|\varphi(z)| - 1)^s dA(z)
\]
\[
= \int_D |f'(w)|(1 - |w|)^{p-2} (|\varphi(w)| - 1)^s dA(w)
\]
\[
= \int_D |f'(w)|(1 - |w|)^{p-2} \left( \frac{1}{|\varphi(w)|^2} - 1 \right)^s dA(w) < \infty.
\]

(3) \(\Rightarrow\) (1). Let \(z \in D\) and \(R > 1\). Using Cauchy-Green formula we obtain
\[
f(z) = \frac{1}{2\pi i} \int_{|w-z|=R} \frac{g(w)}{w-z} dw - \frac{1}{\pi} \int_{1<|w|<R} \frac{\overline{\partial g(w)}}{w-z} dA(w).
\]

Notice the fact that
\[
\int_{|w|=R} \frac{g(w)}{(w-z)^2} dw \to 0 , \quad \text{as} \quad R \to \infty.
\]

We deduce
\[
f'(z) = -\frac{1}{\pi} \int_{D_r} \frac{\overline{\partial g(w)}}{(w-z)^2} dA(w).
\]

Let \(G\) be defined by
\[
G(z) = \begin{cases} \overline{\partial g(z)}, & z \in D_r, \\ 0, & z \in D. \end{cases}
\]

Let \(T\) denote the Calderón-Zygmund operator defined by
\[
Tg(z) = p.v. \int_C \frac{g(w)}{(w-z)^2} dA(w).
\]

It is not hard to see that
\[
f'(z) = -\frac{1}{\pi} (TG)(z) , \quad z \in D.
\]

Hence, using the boundedness of Calderón-Zygmund operators (see Remark 1) and Lemma 1, we deduce
that
\[
\int_{D} |f'(z)|^p (1 - |z|^2)^{p-2} \frac{1}{|\varphi_a(z)|^p} - 1 \, dA(z) \\
\leq \frac{1}{\pi} \int_{D} |(TG)(z)|^p (1 - |z|^2)^{p-2} \frac{1}{|\varphi_a(z)|^p} - 1 \, dA(z) \\
\leq \int_{\mathcal{C}} |G(z)|^p (1 - |z|^2)^{p-2} \frac{1}{|\varphi_a(z)|^p} - 1 \, dA(z) \\
\leq \int_{D} |\tilde{g}(z)|^p (|z|^2 - 1)^{p-2} (|\varphi_a(z)|^2 - 1)^{p} dA(z) < \infty.
\]

The proof is completed. \( \square \)

**Remark 2.** Such function \( F \) is said to be a pseudonalitical extension of \( f \), clearly it is not uniquelly determined by \( f \).

**Remark 3.** \( T \) is also known as Ahlfors - Beoruling operator, which appears in discussions related to different topics in complex analysis, like Beltrami equation.

Given a function \( v \in L^\infty(\partial D) \), the associated Toeplitz operator \( T_v \) is defined by
\[
(T_v f)(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{v(\xi) f(\xi)}{\xi - z} \, d\xi, \quad f \in H^1, z \in D.
\]

Recall that a subspace \( X \) of \( H^1 \) is said to have the K-property if \( T_v(X) \subset X \) for any \( v \in H^\infty \).

**Corollary 1.** Let \( p > 1, 0 < s < 1 \) and \( p + s > 2 \). The \( F,p - 2,s \) has the K-property.

**Proof.** The proof is similar to [9, Theorem 2]. For completeness, we give the proof. Suppose that \( f \in F(p,p - 2,s) \), \( h \in H^\infty \). We need to show that
\[
g_1 =: T_v f \in F(p,p - 2,s).
\]

Since, by definition of Toeplitz operator, \( g_1 \) is the orthogonal projection of \( \tilde{h} \) onto \( H^2 \), then, we have
\[
\tilde{h} = g_1 + \tilde{g}_2,
\]

where \( \tilde{g}_2 \in H^2_0 \). Therefore, we obtain that
\[
g_1 = \tilde{h} - \tilde{g}_2 \quad a.e. \ on \ \partial D.
\]

From Theorem 1, we know that there is a function \( F \in C^1(\mathbb{D}) \) satisfying (a), (b) and (c). We let
\[
H(z) =: h(z^2), \quad G_2 =: g_2(z^2), \quad G_1(z) =: F(z)H(z) - G_2(z) \quad z \in \mathbb{D}.
\]

Hence, using the fact that
\[
F|_{\partial D} = f, \quad H|_{\partial D} = \tilde{h}, \quad G_2|_{\partial D} = \tilde{g}_2,
\]

we get
\[
G_1|_{\partial D} = g_1.
\]
Since $H$ and $G_2$ are holomorphic in $\Omega$, we obtain
\[ \partial H = 0, \quad \partial G_2 = 0. \]
Thus, we have $\partial G_1 = H \cdot \partial F$ on $\Omega$. Furthermore,
\[ |\partial G_1| \leq ||H||_{C^0} ||\partial F||. \]
It is clear that $G_1$ is $C^1$-smooth in $\Omega$ and bounded at $\infty$. Using the fact of above and Theorem 1, we easy to get (a), (b) and (c) hold true with $G_1$ and $g_1$ in place of $F$ and $f$. The proof is completed. \[ \square \]

4. Closure of $F(p, p - 2, s)$ spaces in $A_\omega$

Let us recall the following result.

**Lemma 2.** [4, Lemma 7] Let $\omega$ be a regular majorant. Suppose that $f \in A$. Then $f \in \Lambda_\omega$ if and only if there exists a bounded function $g \in C^1(\Omega)$ satisfying
\[ \lim_{r \to 1^-} g(re^{i\theta}) = f(e^{i\theta}); \]
\[ \sup_{z \in \Omega} |(\sqrt{|z|^2 - 1}) \partial g(z)| < \infty. \]

Moreover,
\[ ||f||_{\Lambda_\omega} \approx \inf_{g} \sup_{z \in \Omega} |(\sqrt{|z|^2 - 1}) |\partial g(z)|. \]

**Lemma 3.** Let $\omega$ be a regular majorant. Then
\[ \int_{\Omega} \frac{\omega(|w|^2 - 1)}{(|w|^2 - 1)|w - z|^2} dA(w) \leq \frac{\omega(1 - |z|^2)}{(1 - |w|^2)}, \quad z \in \Omega. \]

**Proof.** Making change of variable $w = \frac{1}{r^2}, v \in \Omega$, we have
\[ \int_{\Omega} \frac{\omega(|w|^2 - 1)}{(|w|^2 - 1)|w - z|^2} dA(w) = \int_{\Omega} \frac{\omega(|v|^2 - 1)}{(1 - |v|^2)(1 - |v|^2)|v|^2} \frac{1}{|v|^2} dA(v) \]
\[ = \int_{\Omega} \frac{\omega(|v|^2 - 1)}{(1 - |v|^2)(1 - |v|^2) } dA(v) \]
\[ \leq \int_{0}^{1} \frac{\omega(1 - r^2)}{(1 - r^2)(1 - r^2|z|^2)} r dr. \]

Let $t = \frac{1 - r^2}{r^2}$. Then $r^2 = \frac{1}{1 + t}$ and $r dr = \frac{dt}{2(1 + t)^2}$. We obtain
\[ \int_{0}^{1} \frac{\omega(1 - r^2)}{(1 - r^2)(1 - r^2|z|^2)} r dr \]
\[ \leq \int_{0}^{\infty} \frac{\omega(t)}{t(t + (1 - |z|^2))} dt \]
\[ = \int_{0}^{1} \frac{\omega(t)}{t(t + (1 - |z|^2))} dt + \int_{1}^{\infty} \frac{\omega(t)}{t(t + (1 - |z|^2))} dt. \]
Note that
\[ \int_0^\infty \frac{\omega(t)}{t} \, dt + \delta \int_0^\infty \frac{\omega(t)}{t^2} \, dt \leq C(\omega) \cdot \omega(\delta). \]

We have
\[ \int_0^\infty \frac{\omega(t)}{t} \, dt \leq \omega(\delta) \]
and
\[ \delta \int_0^\infty \frac{\omega(t)}{t^2} \, dt \leq \omega(\delta). \]
Thus,
\[ \int_0^1 \frac{\omega(t)}{t} \, dt \leq \frac{1}{(1 - |z|^2)} \int_0^1 \frac{\omega(t)}{t} \, dt \leq \frac{\omega(1 - |z|^2)}{1 - |z|^2} \]
and
\[ \int_1^{1-|z|^2} \frac{\omega(t)}{t} \, dt \leq \frac{1}{2|z|^2} \int_1^{1-|z|^2} \frac{\omega(t)}{t} \, dt \leq \frac{\omega(1 - |z|^2)}{1 - |z|^2}. \]
That is
\[ \int \omega(|w|^2 - 1) \, dA(w) \leq \frac{\omega(1 - |z|^2)}{1 - |z|^2}. \]

The proof is completed. \( \square \)

**Theorem 2.** Let \( p > 1, 0 < s < 1, p + s > 2 \) and \( \omega \) be a regular majorant. If \( f \in \mathcal{A}_\omega \), then the following statements are equivalent.

(i) \( f \in C_{\mathcal{A}_\omega}(\mathcal{A}_\omega \cap F(p, p - 2, s)) \).

(ii) For any \( \epsilon > 0 \),
\[ \int_{\Omega_\epsilon(F)} \frac{\omega^p(|z|^2 - 1)}{\omega(|z|^2 - 1)^2} \, dA(z) < \infty, \]
where \( \Omega_\epsilon(F) = \{ z \in \mathbb{D}_r : \frac{|z|^2 - 1}{\omega(|z|^2 - 1)^2} |\partial F(z)| \geq \epsilon \} \) and \( F \) is pseudoanalytic extension of \( f \).

**Proof.** (i) \( \Rightarrow \) (ii). Suppose that \( f \in C_{\mathcal{A}_\omega}(\mathcal{A}_\omega \cap F(p, p - 2, s)) \subseteq \mathcal{A}_\omega \). Then for any \( \epsilon > 0 \), there exist a function \( g \in \mathcal{A}_\omega \cap F(p, p - 2, s) \), such that
\[ \| f - g \|_{\mathcal{A}_\omega} \leq \frac{\epsilon}{2}. \]
From Lemma 2, there exist functions \( F, G \in C^1(\mathbb{D}_r) \), such that
\[ \frac{(|z|^2 - 1)}{\omega(|z|^2 - 1)^2} |\partial F - \partial G| \leq \| f - g \|_{\mathcal{A}_\omega} \leq \frac{\epsilon}{2}. \]
Here \( F, G \) are its pseudoanalytic extension of \( f \) and \( g \), respectively. Since
\[ \frac{(|z|^2 - 1)}{\omega(|z|^2 - 1)^2} |\partial F| \leq \frac{(|z|^2 - 1)}{\omega(|z|^2 - 1)^2} |\partial F - \partial G| + \frac{(|z|^2 - 1)}{\omega(|z|^2 - 1)^2} |\partial G|, \]
we have \( \Omega_\epsilon(F) \subseteq \Omega_\frac{\epsilon}{2}(G) \). By Theorem 1, we can deduce that
\[ \int_{\partial G(z)} |\partial F - \partial G| (|z|^2 - 1)^{p-2} (q_{p}(z))^2 - 1)^{p} \, dA(z) \]
\[ \leq \frac{2^p}{\epsilon^2} \int_{\Omega_\frac{\epsilon}{2}(G)} |\partial G(z)| |\partial F - \partial G| (|z|^2 - 1)^{p-2} (q_{p}(z))^2 - 1)^{p} \, dA(z) \]
\[ \leq \frac{2^p}{\epsilon^2} \int_{\mathbb{D}_r} |\partial G(z)| (|z|^2 - 1)^{p-2} (q_{p}(z))^2 - 1)^{p} \, dA(z) < \infty. \]
(ii) $\Rightarrow$ (i). Let $f \in \mathcal{A}_w$. Using Cauchy-Green formula we obtain
\[
f(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{F(w)}{w-z} \, dw - \frac{1}{\pi} \int_{1<|w|<R} \frac{\overline{F}(w)}{w-z} \, dA(w).
\]
Noting the fact that $\int_{|w|=R} \frac{F(w)}{(w-z)^2} \, dw \to 0$, as $R \to \infty$, we obtain
\[
f'(z) = -\frac{1}{\pi} \int_{\Delta_1} \frac{\overline{F}(w)}{(w-z)^2} \, dA(w).
\]
Let
\[
f_1'(z) = -\frac{1}{\pi} \int_{\Omega, (F)} \frac{\overline{F}(w)}{(w-z)^2} \, dA(w)
\]
and
\[
f_2'(z) = -\frac{1}{\pi} \int_{\Delta_1 \backslash \Omega, (F)} \frac{\overline{F}(w)}{(w-z)^2} \, dA(w).
\]
Hence, $f'(z) = f_1'(z) + f_2'(z)$. By Lemma 3,
\[
\frac{(1-|z|^2)}{\omega(1-|z|^2)} |f'(z) - f_1'(z)| = \frac{(1-|z|^2)}{\omega(1-|z|^2)} |f_2'(z)| 
\leq \frac{(1-|z|^2)}{\omega(1-|z|^2)} \int_{\Delta_1 \backslash \Omega, (F)} \frac{|\overline{F}(w)|}{|w-z|^2} \, dA(w)
\leq \frac{(1-|z|^2)}{\omega(1-|z|^2)} \int_{\Delta_1 \backslash \Omega, (F)} \frac{|\overline{F}(w)|}{|w-z|^2} \, dA(w)
\leq \varepsilon \frac{(1-|z|^2)}{\omega(1-|z|^2)} \int_{\Delta_1 \backslash \Omega, (F)} \frac{|w|^2 - 1}{|w|^2 - 1} |w-z|^2 \, dA(w)
\leq \varepsilon \frac{(1-|z|^2)}{\omega(1-|z|^2)} \int_{\Delta_1} \frac{|w|^2 - 1}{|w|^2 - 1} |w-z|^2 \, dA(w)
\leq \varepsilon \frac{(1-|z|^2)}{\omega(1-|z|^2)} \frac{\omega(1-|z|^2)}{(1-|z|^2)} = \varepsilon,
\]
which implies that $f_1 \in \mathcal{A}_w$. Now, we are going to prove that $f_1 \in F(p, p-2, s)$.
Let
\[
G(z) = \begin{cases} \overline{F}(z), & z \in \Omega, (F), \\ 0, & z \in \mathbb{C} \setminus \Omega, (F), \end{cases}
\]
and
\[
Tg(z) = p.v. \int_C \frac{g(w)}{w-z} \, dA(w).
\]
It is easy to see that $f_1'(z) = -\frac{1}{\pi}(TG)(z)$, $z \in \mathbb{D}$. Hence, using the boundedness of the operator $T$ and Lemma
1, we obtain
\[
\int_{\mathbb{D}} |f_1'(z)|^p (1 - |z|^2)^{p-2} (1 - |\varphi_\alpha(z)|^2)^y dA(z) \\
\leq \frac{1}{\pi} \int_{\mathbb{D}} |(TG)(z)|^p (1 - |z|^2)^{p-2} (1 - |\varphi_\alpha(z)|^2)^y dA(z) \\
\leq \int_{\mathbb{C}} |(TG)(z)|^p (1 - |z|^2)^{p-2} \left( \frac{1}{|\varphi_\alpha(z)|^2} - 1 \right)^y dA(z) \\
\leq \int_{\mathbb{C}} |G(z)|^p (1 - |z|^2)^{p-2} \left( \frac{1}{|\varphi_\alpha(z)|^2} - 1 \right)^y dA(z) \\
\leq \int_{\mathbb{C}} \omega^p (|1 - |z|^2|) \left( \frac{1}{|\varphi_\alpha(z)|^2} - 1 \right)^y dA(z) \\
\leq \int_{\mathbb{C}} \omega^p (|z|^2 - 1) \left( |\varphi_\alpha(z)|^2 - 1 \right)^y dA(z) < \infty.
\]

The proof is completed. \(\square\)

**Corollary 2.** Let \(0 < s < 1, p + s > 2\) and \(\omega\) be a regular majorant. The \(C_{\mathcal{A}_\omega}(\mathcal{A}_\omega \cap F(p, p - 2, s))\) has the \(K\)-property.

**Proof.** Let \(f \in C_{\mathcal{A}_\omega}(\mathcal{A}_\omega \cap F(p, p - 2, s)), \varphi \in H^\infty\) and \(g\) be the orthogonal projection of \(f \varphi\) onto \(H^2\). Then \(f \varphi = g + j\), where \(j \in H^2_D\). By pseudoanalytic extension, similar to Corollary 1, there are functions \(G, F, \Phi, J\) on \(D_c\) with
\[
G = g, \quad F = f, \quad \Phi = \varphi, \quad J = j, \quad \text{on } \partial D,
\]
such that \(F \Phi = G + J\). Thus,
\[
|\partial G(z)| \leq ||\varphi||_\infty |\partial F(z)|, \quad z \in D_c.
\]

Combined with Theorem 2, we have
\[
\int_{\mathbb{C}} \omega^p (|z|^2 - 1) \left( |\varphi_\alpha(z)|^2 - 1 \right)^y dA(z) \\
\leq \int_{\mathbb{C}} \omega^p (|z|^2 - 1) \left( |\varphi_\alpha(z)|^2 - 1 \right)^y dA(z) < \infty.
\]

That is \(g \in C_{\mathcal{A}_\omega}(\mathcal{A}_\omega \cap F(p, p - 2, s))\). The proof is completed. \(\square\)

**References**