



Abelian Theorems for Laplace and Mehler-Fock Transforms of Generalized Functions

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Abstract. The main purpose of this article is to exhibit Abelian theorems for the Laplace and the Mehler-Fock transforms of general order over distributions of compact support and over certain spaces of generalized functions.

1. Introduction and preliminaries

Abelian theorems occur in the context when one has an operator (usually linear with some continuity properties) between function spaces, $T : X \rightarrow Y$, and one deduces some properties (usually asymptotic) of $T(f)$ from properties (usually asymptotic) of f .

We emphasize that these results establish the fact that, knowing the distribution (or generalized function) the transformation behavior of this distribution (or generalized function) can be inferred.

Abelian theorems for distributional transforms were first established by Zemanian in [14] (see also [4] for Abelian theorems on certain index transforms of generalized functions).

In this article we study Abelian theorems for the Laplace and Mehler-Fock transforms of general order over the space of distributions of compact support and over certain spaces of generalized functions.

The conventional one-side Laplace transform of a suitable complex-valued function f defined in $(0, \infty)$ is given by

$$\int_0^{\infty} f(t)e^{-st} dt, \quad s \in \mathbb{C}. \quad (1)$$

On the other hand, the Mehler-Fock transform of general order $\mu \in \mathbb{C}$ of a suitable complex-valued function f defined in $(1, \infty)$ by

$$\int_1^{\infty} f(t)P_{-\frac{1}{2}+i\tau}^{-\mu}(t)dt, \quad \tau > 0, \quad (2)$$

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where P_ν^μ is the associated Legendre function of the first kind (see [1, Chapter 3] for details) given (in terms of the Gauss hypergeometric function ${}_2F_1$) by

$$\begin{aligned}
 P_\nu^\mu(z) &= P_{-\nu-1}^\mu(z) \\
 &= \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right).
 \end{aligned}
 \tag{3}$$

The following results are also summarized in [2], [3] and [11].

We next recall the following integral representation from [1, (7), p. 156]

$$\begin{aligned}
 P_{-\frac{1}{2}+i\tau}^{-\mu}(t) &= \frac{(t^2-1)^{\mu/2}}{2^\mu \sqrt{\pi} \Gamma(\mu+\frac{1}{2})} \int_0^\pi (t+(t^2-1)^{1/2} \cos u)^{-\mu-1/2+i\tau} (\sin u)^{2\mu} du,
 \end{aligned}
 \tag{4}$$

which is valid for

$$t > 1; \tau > 0; \Re\mu > -\frac{1}{2}.$$

Now, observing that

$$t + (t^2 - 1)^{1/2} \cos u > 0 \quad (t > 1; u \in [0, \pi]),$$

it follows from (4) that

$$\begin{aligned}
 &\left| P_{-\frac{1}{2}+i\tau}^{-\mu}(t) \right| \\
 &\leq \frac{(t^2-1)^{\Re\mu/2}}{2^{\Re\mu} \sqrt{\pi} \left| \Gamma\left(\frac{1}{2}+\mu\right) \right|} \int_0^\pi (t+(t^2-1)^{1/2} \cos u)^{-\frac{1}{2}-\Re\mu} (\sin u)^{2\Re\mu} du \\
 &= \frac{\Gamma(\frac{1}{2}+\Re\mu)}{\left| \Gamma(\frac{1}{2}+\mu) \right|} P_{-\frac{1}{2}}^{-\Re\mu}(t).
 \end{aligned}
 \tag{5}$$

Moreover, from [7, (12.08), p. 171]), for $\Re\mu > -1/2$

$$P_{-\frac{1}{2}}^{-\mu}(t) \sim \frac{(t-1)^{\mu/2}}{2^{\mu/2} \Gamma(\mu+1)} \quad (t \rightarrow 1). \tag{6}$$

Also, from [7, (12.20), p. 172], for $\Re\mu > -1/2$, we have

$$P_{-\frac{1}{2}}^{-\Re\mu}(t) \sim \frac{1}{\Gamma(\mu+\frac{1}{2})} \left(\frac{2}{\pi t}\right)^{1/2} \ln t \quad (t \rightarrow \infty). \tag{7}$$

In this point we recall that for any open set $\Omega \subset \mathbb{R}$, the space $\mathcal{E}(\Omega)$ is defined as the vector space of all infinitely differentiable complex-valued functions ϕ defined in Ω . This space equipped with the locally convex topology arising from the family of seminorms

$$\rho_{K,k}(\phi) = \max_{t \in K} |D_t^k \phi(t)| \tag{8}$$

for all compact sets $K \subset \Omega$, all $k \in \mathbb{N} \cup \{0\}$ and with D_t^k denoting the k th derivative with respect to the variable t , becomes a Fréchet space. As usual, we denote by $\mathcal{E}'(\Omega)$ the dual space of $\mathcal{E}(\Omega)$. This $\mathcal{E}'(\Omega)$ agrees with the space of distributions on Ω of compact support.

Related studies concerning Abelian theorems for Mehler-Fock transforms have been done in [10].

Our analysis could be extended to the operators considered in [9], which involve the Kontorovich-Lebedev transform (see [3], Section 2 and 3), and to the transforms considered in [12] and [13].

2. Abelian theorems for the distributional Laplace transform

The one-side Laplace transform of a distribution of compact support f on $(0, \infty)$ is defined by

$$F(s) = \langle f(t), e^{-st} \rangle, \quad s \in \mathbb{C}. \tag{9}$$

In this section we establish Abelian theorems for the transform (9). To do this we prove a previous result.

Lemma 2.1. *Let f be in $\mathcal{E}'((0, \infty))$, and let F be defined by (9). Then there exist $M > 0$ and a nonnegative integer p , all depending on f , such that for $\Re s \geq a$, $a \in \mathbb{R}$, one has*

$$|F(s)| \leq M \max_{0 \leq k \leq p} |s|^k. \tag{10}$$

Proof. From [6, Proposition 2, p. 97] there exist $C > 0$, a compact set $K \subset (0, \infty)$ and a nonnegative integer p , all depending on f , such that

$$|\langle f, \phi \rangle| \leq C \max_{0 \leq k \leq p} \max_{t \in K} |D^k \phi(t)|. \tag{11}$$

Now,

$$|F(s)| = \left| \langle f(t), e^{-st} \rangle \right| \leq C \max_{0 \leq k \leq p} \max_{t \in K} |s^k e^{-st}| \leq M \max_{0 \leq k \leq p} |s|^k, \quad \Re s \geq a,$$

for certain $M > 0$, since t ranges on the compact set $K \subset (0, \infty)$. \square

The smallest integer p which verifies the inequality (11) is defined as the order of the distribution f (cf. [8, Théorème XXIV, p. 88]).

Now we establish an Abelian theorem for the distributional Laplace transform (9).

Theorem 2.2. *(Abelian theorem) Let f be a member of $\mathcal{E}'((0, \infty))$ of order $r \in \mathbb{N} \cup \{0\}$, and let F be given by (9). Then for $\Re s \geq a$, $a \in \mathbb{R}$, and any $\gamma > 0$ one has*

$$\lim_{|s| \rightarrow +\infty} \{|s|^{-r-\gamma} F(s)\} = 0.$$

Proof. From Lemma 2.1 one has

$$|F(s)| \leq M \max_{0 \leq k \leq r} |s|^k, \quad \Re s \geq a,$$

for some $M > 0$, from which the conclusion follows. \square

Now, if f is a locally integrable function on $(0, \infty)$ and f has compact support on $(0, \infty)$, then f gives rise to a regular member T_f in $\mathcal{E}'((0, \infty))$ of order $r = 0$ by means of

$$\langle T_f, \phi \rangle = \int_0^\infty f(t)\phi(t)dt, \quad \forall \phi \in \mathcal{E}((0, \infty)).$$

In fact, taking into account that

$$\begin{aligned} |\langle T_f, \phi \rangle| &= \left| \int_0^\infty f(t)\phi(t)dt \right| \leq \max_{t \in \text{supp}(f)} |\phi(t)| \int_{\text{supp}(f)} |f(t)| dt \\ &= \rho_{\text{supp}(f),0}(\phi) \cdot \int_{\text{supp}(f)} |f(t)| dt, \end{aligned}$$

where $\text{supp}(f)$ represents the support of the function f and $\rho_{\text{supp}(f),0}$ is given by (8), it follows that $T_f \in \mathcal{E}'((0, \infty))$ and its order is $r = 0$.

So, we have

$$F(s) = \langle T_f(t), e^{-st} \rangle = \int_0^\infty f(t)e^{-st} dt, \quad s \in \mathbb{C}. \tag{12}$$

In this sense we get

Corollary 2.3. *Let f be a locally integrable function on $(0, \infty)$ such that f has compact support on $(0, \infty)$, and let F be given by (12). Then for $\Re s \geq a$, $a \in \mathbb{R}$, and any $\gamma > 0$ one has*

$$\lim_{|s| \rightarrow +\infty} \{|s|^{-\gamma} F(s)\} = 0.$$

3. Abelian theorems for the Laplace transform of generalized functions

Zemanian [15, p. 90] introduced the vector space $\mathcal{L}_{+,a}$ of the all infinitely differentiable complex-valued functions ϕ defined on $(0, \infty)$ such that

$$\lambda_{k,a}(\phi) = \sup_{t \in (0, \infty)} |e^{at} D^k \phi(t)| < \infty, \quad k = 0, 1, 2, \dots \tag{13}$$

where $a \in \mathbb{R}$.

The space $\mathcal{L}_{+,a}$, $a \in \mathbb{R}$, equipped with the topology arising from the family of seminorms $\{\lambda_{k,a}\}$ is a Fréchet space.

As usual, we denote by $\mathcal{L}'_{+,a}$ the dual space of $\mathcal{L}_{+,a}$.

Observe that e^{-st} , $t \in (0, \infty)$, is a member of $\mathcal{L}_{+,a}$ for $\Re s \geq a$.

For $f \in \mathcal{L}'_{+,a}$, $a \in \mathbb{R}$, the generalized Laplace transform is defined by means of

$$F(s) = \langle f(t), e^{-st} \rangle, \quad \Re s \geq a. \tag{14}$$

Set $f \in \mathcal{L}'_{+,a}$. From [6, Proposition 2, p. 97], there exist $C > 0$ and a nonnegative integer p , all depending on f , such that

$$|\langle f, \phi \rangle| \leq C \max_{0 \leq k \leq p} \sup_{t \in (0, \infty)} |e^{at} D^k \phi(t)| \tag{15}$$

for all $\phi \in \mathcal{L}_{+,a}$.

In this section we establish Abelian theorems for the transform (14). To do this we prove a previous result.

Lemma 3.1. *Let f be in $\mathcal{L}'_{+,a}$, $a \in \mathbb{R}$, and let F be defined by (14). Then there exist an $M > 0$ and a nonnegative integer p , all depending on f , such that*

$$|F(s)| \leq M \max_{0 \leq k \leq p} |s|^k, \quad \Re s \geq a. \tag{16}$$

Proof. From [6, Proposition 2, p. 97] one has

$$|F(s)| = \left| \langle f(t), e^{-st} \rangle \right| \leq C \max_{0 \leq k \leq p} \sup_{t \in (0, \infty)} |e^{at} s^k e^{-st}|.$$

Now, taking into account the fact that $\Re s \geq a$, it follows that

$$|F(s)| \leq M \max_{0 \leq k \leq p} |s|^k, \quad \Re s \geq a,$$

for certain $M > 0$. \square

The smallest integer p which verifies the inequality (15) is called the order of the generalized function f .

The next theorem establishes an Abelian theorem for the Laplace transform of generalized functions in $\mathcal{L}'_{+,a}$.

Theorem 3.2. (Abelian theorem) *Let f be a member of $\mathcal{L}'_{+,a}$, $a \in \mathbb{R}$, of order $r \in \mathbb{N} \cup \{0\}$, and let F be given by (9). Then for $\Re s \geq a$ and any $\gamma > 0$ one has*

$$\lim_{|s| \rightarrow +\infty} \{|s|^{-r-\gamma} F(s)\} = 0.$$

Proof. From Lemma 3.1 one has

$$|F(s)| \leq M \max_{0 \leq k \leq r} |s|^k, \quad \Re s \geq a,$$

for some $M > 0$, from which the conclusion follows. \square

Moreover, a locally integrable function f defined on $(0, \infty)$, such that $e^{-at} f(t)$, $a \in \mathbb{R}$, is Lebesgue integrable on $(0, \infty)$, gives rise to a regular generalized function T_f in $\mathcal{L}'_{+,a}$ of order $r = 0$ through

$$\langle T_f, \phi \rangle = \int_0^\infty f(t) \phi(t) dt, \quad \forall \phi \in \mathcal{L}_{+,a}.$$

In fact, taking into account that

$$\begin{aligned} |\langle T_f, \phi \rangle| &= \left| \int_0^\infty f(t) \phi(t) dt \right| \\ &= \left| \int_0^\infty e^{-at} f(t) e^{at} \phi(t) dt \right| \leq \sup_{t \in (0, \infty)} |e^{at} \phi(t)| \int_0^\infty |e^{-at} f(t)| dt \\ &= \lambda_{0,a}(\phi) \cdot \int_0^\infty e^{-at} |f(t)| dt, \end{aligned}$$

it follows that $T_f \in \mathcal{L}'_{+,a}$ and its order is $r = 0$.

Thus, we have

$$F(s) = \langle T_f(t), e^{-st} \rangle = \int_0^\infty f(t) e^{-st} dt, \quad \Re s \geq a. \tag{17}$$

Corollary 3.3. *Let f be a locally integrable function defined in $(0, \infty)$ such that $e^{-at} f(t)$ is Lebesgue integrable on $(0, \infty)$, $a \in \mathbb{R}$, and let F be given by (17). Then for $\Re s \geq a$ and any $\gamma > 0$ one has*

$$\lim_{|s| \rightarrow +\infty} \{|s|^{-\gamma} F(s)\} = 0.$$

Remark 3.4. *Observe that Corollary 3.3 includes Corollary 2.3.*

4. Abelian theorems for the Mehler–Fock transform of general order of generalized functions

In [5], Hayek and the first author studied the Mehler–Fock transform of general order over certain spaces of generalized functions with the associated Legendre function $P_{-\frac{1}{2}+i\tau}^{-\mu}$ as kernel, $\mu \in \mathbb{C}$ and $\tau > 0$.

Here we consider the space of testing functions $\mathcal{M}_{a,\mu}$, $a \in \mathbb{R}$ and $\mu \in \mathbb{C}$, which is defined as the vector space of all infinitely differentiable complex-valued functions ϕ defined on $(1, \infty)$ such that

$$\gamma_{k,a,\mu}(\phi) = \sup_{t \in (1, \infty)} |(t-1)^a A_t^k \phi(t)| < \infty$$

for every nonnegative integer k , where A_t is the differential operator given by

$$A_t = (t^2 - 1)^{-\frac{\mu}{2}} D_t (t^2 - 1)^{\mu+1} D_t (t^2 - 1)^{-\frac{\mu}{2}}. \tag{18}$$

The space $\mathcal{M}_{a,\mu}$ is equipped with the topology arising from the family of seminorms $\{\gamma_{k,a,\mu}\}$ is a Fréchet space.

Observe that from (5), (6) and (7), the function $P_{-\frac{1}{2}+i\tau}^{-\mu}(t)$, $t \in (1, \infty)$, is a member of $\mathcal{M}_{a,\mu}$ for $\Re\mu > -\frac{1}{2}$, $-\frac{\Re\mu}{2} \leq a < \frac{1}{2}$ and all $\tau > 0$.

This space $\mathcal{M}_{a,\mu}$ differs from the space $M_{a,\mu}$ considered in [3] and [5] in terms of the function $(t - 1)^a$ instead of t^a from the definition of the seminorms $\gamma_{k,a,\mu}$. This fact allows us to obtain a new Abelian theorem concerning this transform. Specifically, we replace the condition $\Re\mu \geq 0$, $a < \frac{1}{2}$ in [3, Theorem 5.2] with

$$\Re\mu > -\frac{1}{2}, -\frac{\Re\mu}{2} \leq a < \frac{1}{2}.$$

As usual, we denote by $\mathcal{M}'_{a,\mu}$ the dual space of $\mathcal{M}_{a,\mu}$.

So, for $f \in \mathcal{M}'_{a,\mu}$, $\Re\mu > -\frac{1}{2}$ and $-\frac{\Re\mu}{2} \leq a < \frac{1}{2}$, the generalized Mehler–Fock transform is defined by

$$F(\tau) = \langle f(t), P_{-\frac{1}{2}+i\tau}^{-\mu}(t) \rangle, \quad \tau > 0. \tag{19}$$

From [6, Proposition 2, p. 97], one has that for all $f \in \mathcal{M}'_{a,\mu}$, there exist a $C > 0$ and a nonnegative integer p , all depending on f , such that

$$|\langle f, \phi \rangle| \leq C \max_{0 \leq k \leq p} \gamma_{k,a,\mu}(\phi) = C \max_{0 \leq k \leq p} \sup_{t \in (1, \infty)} |(t - 1)^a A_t^k \phi(t)| \tag{20}$$

for all $\phi \in \mathcal{M}_{a,\mu}$.

In this section we establish Abelian theorems for the transform (19). To do this we prove a previous result.

Lemma 4.1. *Set $\Re\mu > -\frac{1}{2}$ and $-\frac{\Re\mu}{2} \leq a < \frac{1}{2}$. Let f be in $\mathcal{M}'_{a,\mu}$ and let F be defined by (19). Then there exist $M > 0$ and a nonnegative integer p , all depending on f , such that*

$$|F(\tau)| \leq M \max_{0 \leq k \leq p} \left[\left(|\mu| + \frac{1}{2} \right)^2 + \tau^2 \right]^k, \quad \forall \tau > 0. \tag{21}$$

Proof. From (20) and (5) one has

$$\begin{aligned} |F(\tau)| &= \left| \langle f(t), P_{-\frac{1}{2}+i\tau}^{-\mu}(t) \rangle \right| \leq C \max_{0 \leq k \leq p} \sup_{t \in (1, \infty)} \left| (t - 1)^a A_t^k P_{-\frac{1}{2}+i\tau}^{-\mu}(t) \right| \\ &\leq C \max_{0 \leq k \leq p} \sup_{t \in (1, \infty)} \left[\left(\mu + \frac{1}{2} \right)^2 + \tau^2 \right]^k (t - 1)^a \frac{\Gamma(\frac{1}{2} + \Re\mu)}{\Gamma(1 + \mu)} P_{-\frac{1}{2}}^{-\Re\mu}(t). \end{aligned}$$

From (6) and (7), and taking into account the fact that $\Re\mu > -\frac{1}{2}$ and $-\frac{\Re\mu}{2} \leq a < \frac{1}{2}$ it follows that

$$|F(\tau)| \leq M \max_{0 \leq k \leq p} \left[\left(|\mu| + \frac{1}{2} \right)^2 + \tau^2 \right]^k, \quad \forall \tau > 0$$

for certain $M > 0$. \square

The smallest integer p which verifies the inequality (20) is called the order of the generalized function f .

The next theorem establishes an Abelian theorem for the Mehler–Fock transform of generalized functions in $\mathcal{M}'_{a,\mu}$.

Theorem 4.2. (Abelian theorem) Set $\Re\mu > -\frac{1}{2}$ and $-\frac{\Re\mu}{2} \leq a < \frac{1}{2}$. Let f be a member of $\mathcal{M}'_{a,\mu}$ of order $r \in \mathbb{N} \cup \{0\}$, and let F be given by (19). Then

(i) for any $\gamma > 0$ one has

$$\lim_{\tau \rightarrow 0^+} \{\tau^\gamma F(\tau)\} = 0,$$

(ii) for any $\gamma > 0$ one has

$$\lim_{\tau \rightarrow +\infty} \{\tau^{-2r-\gamma} F(\tau)\} = 0.$$

Proof. From Lemma 4.1 one has

$$|F(\tau)| \leq M \max_{0 \leq k \leq r} \left[\left(|\mu| + \frac{1}{2} \right)^2 + \tau^2 \right]^k, \quad \forall \tau > 0,$$

for some $M > 0$, from which the conclusion follows. \square

Otherwise, a locally integrable function f defined on $(1, \infty)$ such that $(t - 1)^{-a} f(t)$ is Lebesgue integrable on $(1, \infty)$, gives rise to a regular generalized function T_f in $\mathcal{M}'_{a,\mu}$ of order $r = 0$ through

$$\langle T_f, \phi \rangle = \int_1^\infty f(t)\phi(t)dt, \quad \forall \phi \in \mathcal{M}_{a,\mu}.$$

In fact, taking into account that

$$\begin{aligned} |\langle T_f, \phi \rangle| &= \left| \int_1^\infty f(t)\phi(t)dt \right| \\ &= \left| \int_1^\infty (t - 1)^{-a} f(t)(t - 1)^a \phi(t)dt \right| \leq \sup_{t \in (1, \infty)} |(t - 1)^a \phi(t)| \int_1^\infty |(t - 1)^{-a} f(t)| dt \\ &= \gamma_{0,a,\mu}(\phi) \cdot \int_1^\infty (t - 1)^{-a} |f(t)| dt, \end{aligned}$$

it follows that $T_f \in \mathcal{M}'_{a,\mu}$ and its order is $r = 0$.

In this case,

$$F(\tau) = \langle T_f(t), P_{-\frac{1}{2}+i\tau}^{-\mu}(t) \rangle = \int_1^\infty f(t)P_{-\frac{1}{2}+i\tau}^{-\mu}(t)dt, \quad \tau > 0, \tag{22}$$

for $\Re\mu > -\frac{1}{2}$ and $-\frac{\Re\mu}{2} \leq a < \frac{1}{2}$.

Corollary 4.3. Set $\Re\mu > -\frac{1}{2}$ and $-\frac{\Re\mu}{2} \leq a < \frac{1}{2}$. Let f be a locally integrable function defined on $(1, \infty)$ such that $(t - 1)^{-a} f(t)$ is Lebesgue integrable on $(1, \infty)$, and F is given by (22). Then

(i) for any $\gamma > 0$ one has

$$\lim_{\tau \rightarrow 0^+} \{\tau^\gamma F(\tau)\} = 0,$$

(ii) for any $\gamma > 0$ one has

$$\lim_{\tau \rightarrow +\infty} \{\tau^{-\gamma} F(\tau)\} = 0.$$

Remark 4.4. Observe that, for $\Re\mu > -\frac{1}{2}$ and $-\frac{\Re\mu}{2} \leq a < \frac{1}{2}$, Corollary 4.3 includes Corollary 4.5 of [3].

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