Filomat 34:11 (2020), 3705–3709
https://doi.org/10.2298/FIL2011705Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Singular Value Inequalities Related to Linear Maps

Junjian Yang[a,b,c,d], Linzhang Lu[a,e], Zhen Chen[a]

[a] *School of Mathematical Sciences, Guizhou Normal University, Guiyang, P. R. China*
[b] *School of Mathematics and Statistics, Hainan Normal University, Haikou, P. R. China*
[c] *Key Laboratory Of Data Science And Intelligence Education, Hainan Normal University, Ministry of Education*
[d] *Key Laboratory of Computational Science and Application of Hainan Province*
[e] *School of Mathematical Sciences, Xiamen University, Xiamen, P. R. China*

Abstract.
If $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ is positive semidefinite, Lin [7] conjectured that

$$2s_j(\Psi(X)) \quad \leq \quad s_j(\Psi(A) + \Psi(B)), \quad j = 1, \ldots, n,$$

and

$$s_j(\Psi(X)) \quad \leq \quad s_j(\Psi(A) \sharp \Psi(B)), \quad j = 1, \ldots, n,$$

where the linear map $\Psi : X \mapsto 2\mathrm{tr}(X)I_n - X$ and $s_j(\cdot)$ means the j-th largest singular value.
In this paper, we reprove that

$$\begin{pmatrix} \Psi(A) & \Psi(X) \\ \Psi(X^*) & \Psi(B) \end{pmatrix}$$

is PPT by using an alternative approach and prove the above singular value inequalities hold for the linear map $\Psi_1 : X \mapsto (2n + 1)\mathrm{tr}(X)I_n - X$.

1. Introduction

We denote by $\mathbb{M}_n(\mathbb{M}_k)$ the set of $n \times n$ block matrices with each block in $\mathbb{M}_k$. By convention, the $n \times n$ identity matrix is denoted by I_n. We use $E_{j,k}$ to denote the $n \times n$ matrix with 1 at the i, k component and zeros elsewhere. A positive semidefinite matrix A will be expressed as $A \geq 0$. Likewise, we write $A > 0$ to refer that A is a positive definite matrix. For any $n \times n$ matrix A, the singular values $s_j(A)$ are nonincreasingly arranged, $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. If A is Hermitian, we also arrange its eigenvalues $\lambda_j(A)$ in nonincreasing order $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. The geometric mean of two positive definite matrices $A, B \in \mathbb{M}_n$, denoted by $A \sharp B$, is the positive definite solution of the Ricatti equation $XB^{-1}X = A$ and

2010 *Mathematics Subject Classification*. 15A45, 15A60.
Keywords. Block matrices, Singular value inequalities, Linear maps.
Received: 24 November 2019; Revised: 20 December 2019; Accepted: 14 May 2020
Communicated by Fuad Kittaneh
Research supported by Hainan Provincial Natural Science Foundation for High-level Talents grant no. 2019RC171, the Ministry of Education of Hainan grant no. Hnky2019ZD-13, the Provincial Key Laboratory, Hainan Normal University grant no. JSKX201904, Hainan Provincial Natural Science Foundation of China grant no. 120MS032.
Email addresses: junjianyang1981@163.com (Junjian Yang), llz@gznu.edu.cn, lzlu@xmu.edu.cn (Linzhang Lu), zchen@gznu.edu.cn (Zhen Chen)

it has the explicit expression $A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$. More details on the matrix geometric mean can be found in [1, Chapter 4].

A matrix $H = [H_{i,j}]_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$ is said to be positive partial transpose (i.e., PPT) if H is positive semidefinite and its partial transpose $H^\tau = [H_{j,i}]_{i,j=1}^n$ is also positive semidefinite.

A linear map $\Phi : \mathbb{M}_k \mapsto \mathbb{M}_m$ is said to be n-positive if for $A = [A_{i,j}]_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$,

$$[A_{i,j}]_{i,j=1}^n \geq 0 \quad \Rightarrow \quad [\Phi(A_{i,j})]_{i,j=1}^n \geq 0. \tag{1}$$

It is said to be completely positive if (1) is true for any positive integer. On the other hand, a linear map $\Phi : \mathbb{M}_k \mapsto \mathbb{M}_m$ is said to be n-copositive if

$$[A_{i,j}]_{i,j=1}^n \geq 0 \quad \Rightarrow \quad [\Phi(A_{j,i})]_{i,j=1}^n \geq 0, \tag{2}$$

and Φ is said to be completely copositive if (2) is true for any positive integer n. Furthermore, Φ is called a completely PPT map if $[\Phi(A_{i,j})]_{i,j=1}^n$ and $[\Phi(A_{j,i})]_{i,j=1}^n$ are both positive semidefinite.

Lin [7] left the following inequalities unsolved:

Conjecture 1.1. Let $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ be positive semidefinite. Then

$$2s_j(\Psi(X)) \quad \leq \quad s_j(\Psi(A) + \Psi(B)), \quad j = 1, \ldots, n, \tag{3}$$

and

$$s_j(\Psi(X)) \quad \leq \quad s_j(\Psi(A)\sharp\Psi(B)), \quad j = 1, \ldots, n, \tag{4}$$

where $\Psi : X \mapsto 2\mathrm{tr}(X)I_n - X$.

Actually, Lin [5] has showed that (3) holds for the linear map $\Phi : X \mapsto \mathrm{tr}(X)I_n + X$. However, (4) has not been proven for the linear map $\Phi : X \mapsto tr(X)I_n + X$.

In this paper, we first prove that

$$\begin{pmatrix} \Psi(A) & \Psi(X) \\ \Psi(X^*) & \Psi(B) \end{pmatrix}$$

is PPT for $\Psi : X \mapsto 2\mathrm{tr}(X)I_n - X$ by a new approach which is different from that in [7, Exampe 3.6] and then show (3) and (4) hold for linear map $\Psi_1 : X \mapsto (2n+1)\mathrm{tr}(X)I_n - X$.

2. Auxilliary results and proofs

Before presenting our results, we start with two lemmas which are useful for our proofs. The first one is standard in matrix analysis.

Lemma 2.1. [1, p. 14] Let $B > 0$. Then the block matrix $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is positive semidefinite if and only if $A \geq XB^{-1}X^*$.

The following result is due to Choi [2, Theorem 2].

Lemma 2.2. Let Φ be a linear map from $\mathbb{M}_n$ to $\mathbb{M}_k$. Then Φ is completely positive if and only if $[\Phi(E_{i,j})]_{i,j=1}^n \geq 0$.

Remark 2.3. It is easy to see from Lemma 2.2 that Φ is completely PPT if and only if $[\Phi(E_{i,j})]_{i,j=1}^n \geq 0$ and $[\Phi(E_{j,i})]_{i,j=1}^n \geq 0$.

Lin proved the following theorem in [7]. Now, we provide an alternative proof for the result.

Theorem 2.4. Let $\Psi : X \mapsto 2\mathrm{tr}(X)I_n - X$ be a linear map from $\mathbb{M}_n$ to $\mathbb{M}_n$ and $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ be positive semidefinite. Then

$$\begin{pmatrix} \Psi(A) & \Psi(X) \\ \Psi(X^*) & \Psi(B) \end{pmatrix}$$

is PPT.

Proof. By Remark 2.3, it suffices to show that

$$\begin{pmatrix} \Psi(E_{11}) & \Psi(E_{12}) \\ \Psi(E_{21}) & \Psi(E_{22}) \end{pmatrix} \geq 0 \text{ and } \begin{pmatrix} \Psi(E_{11}) & \Psi(E_{21}) \\ \Psi(E_{12}) & \Psi(E_{22}) \end{pmatrix} \geq 0.$$

Since

$$\begin{pmatrix} \Psi(E_{11}) & \Psi(E_{12}) \\ \Psi(E_{21}) & \Psi(E_{22}) \end{pmatrix} = \begin{pmatrix} 2I_n - E_{11} & -E_{12} \\ -E_{21} & 2I_n - E_{22} \end{pmatrix},$$

then it is easy to compute

$$\begin{aligned} &2I_n - E_{22} - (-E_{21})(2I_n - E_{11})^{-1}(-E_{12}) \\ &= 2I_n - E_{22} - E_{21}(2I_n - E_{11})^{-1}E_{12} \\ &= 2I_n - E_{22} - E_{22} \\ &= 2I_n - 2E_{22} \geq 0. \end{aligned}$$

Hence by Lemma 2.1,

$$\begin{pmatrix} \Psi(E_{11}) & \Psi(E_{12}) \\ \Psi(E_{21}) & \Psi(E_{22}) \end{pmatrix} \geq 0.$$

In a similar way, we can have

$$\begin{pmatrix} \Psi(E_{11}) & \Psi(E_{21}) \\ \Psi(E_{12}) & \Psi(E_{22}) \end{pmatrix} \geq 0.$$

So Ψ is completely PPT. □

Remark 2.5. Lin [7] proved Theorem 2.4 by using the approach in [3].

Next, we prove a result which is related to [7, Example 3.6.].

Theorem 2.6. Let $\Psi : X \mapsto 2(\mathrm{tr}X)I_n - X$, $\Phi : C \mapsto (\mathrm{tr}X)I_n + X$ be both linear maps from $\mathbb{M}_n$ to $\mathbb{M}_n$ and $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ be positive semidefinite. Then

$$2s_j(\Phi(\Psi(X))) \quad \leq \quad s_j(\Phi(\Psi(A) + \Psi(B))), \quad j = 1, \ldots, n,$$

and

$$s_j(\Phi(\Psi(X))) \quad \leq \quad s_j(\Phi(\Psi(A))\sharp\Phi(\Psi(B))), \quad j = 1, \ldots, n.$$

Proof. By Theorem 2.4, we know that

$$\begin{pmatrix} \Psi(A) & \Psi(X) \\ \Psi(X^*) & \Psi(B) \end{pmatrix}$$

is PPT.
So the inequalities below follow from [5, (1.1)]

$$2s_j(\Phi(\Psi(X))) \quad \leq \quad s_j(\Phi(\Psi(A) + \Psi(B))), \quad j = 1, \ldots, n, \tag{5}$$

which means that

$$\begin{aligned}2s_j(\Phi(\Psi(X))) &= 2s_j(\Psi(X)+\mathrm{tr}(\Psi(X))I_n)\\ &= 2s_j((2n+1)\mathrm{tr}(X)I_n - X)\\ &\le s_j((\Psi(A)+\Psi(B))+\mathrm{tr}(\Psi(A)+\Psi(B))I_n)\\ &= s_j((2n+1)\mathrm{tr}(A+B)I_n-(A+B)),\ \ j=1,\dots,n.\end{aligned}$$

Now setting the linear map $\Psi_1 : X \mapsto (2n+1)\mathrm{tr}(X)I_n - X$ in (5) yields

$$2s_j(\Psi_1(X)) \le s_j(\Psi_1(A)+\Psi_1(B)),\quad j=1,\dots,n.$$

Moreover, by [5, (1.1)] and [4, Lemma 4.2], we have the stronger inequalities below:

$$s_j(\Phi(\Psi(X))) \le s_j(\Phi(\Psi(A)\sharp\Psi(B))),\quad j=1,\dots,n.$$

Compute

$$\begin{aligned}s_j(\Phi(\Psi(X))) &= s_j((2n+1)\mathrm{tr}(X)I_n - X)\\ &= s_j(\Psi_1(X))\\ &\le s_j(\Phi(\Psi(A))\sharp\Phi(\Psi(B)))\\ &= s_j(((2n+1)\mathrm{tr}(A)I_n - A)\sharp((2n+1)\mathrm{tr}(B)I_n - B))\\ &\le s_j(\Psi_1(A)\sharp\Psi_1(B))\\ &= s_j(\Phi(\Psi(A))\sharp\Phi(\Psi(B))),\quad j=1,\dots,n.\end{aligned}$$

□

Finally, we show that the inequality (3) holds for $n=2$.

Theorem 2.7. Let $H=\begin{pmatrix} A & X\\ X^* & B\end{pmatrix}\in\mathbb{M}_2(\mathbb{M}_2)$ be positive semidefinite. Then

$$2s_j(\Psi(X)) \le s_j(\Psi(A)+\Psi(B)),\quad j=1,2,$$

where $\Psi : X \mapsto 2\mathrm{tr}(X)I_2 - X$.

Proof. Since the linear map $\overline{\Psi} : X \mapsto \mathrm{tr}(X)I_2 - X$ is completely copositive [6, Proposition 2.1], then

$$\begin{pmatrix}\overline{\Psi}(A) & \overline{\Psi}(X^*)\\ \overline{\Psi}(X) & \overline{\Psi}(B)\end{pmatrix} = \begin{pmatrix}\mathrm{tr}(A)I_2 - A & \mathrm{tr}(X^*)I_2 - X^*\\ \mathrm{tr}(X)I_2 - X & \mathrm{tr}(B)I_2 - B\end{pmatrix}\ge 0.$$

Thus, by [5, (1.1)], we have

$$2s_j(\Phi(\mathrm{tr}(X^*)I_2 - X^*)) \le s_j(\Phi(\mathrm{tr}(A)I_2 - A)+\Phi(\mathrm{tr}(B)I_2 - B)),\quad j=1,2,$$

where $\Phi : X \mapsto \mathrm{tr}(X)I_2 + X$.
Notice that $\mathrm{tr}(\mathrm{tr}(X)I_2 - X) = \mathrm{tr}(X)$ for any $X\in\mathbb{M}_2$. Then for each j, a simple calculation gives

$$\begin{aligned}&2s_j(\Phi(\mathrm{tr}(X^*)I_2 - X^*))\\ &= 2s_j(\mathrm{tr}(\mathrm{tr}(X^*)I_2 - X^*)I_2 + (\mathrm{tr}(X^*)I_2 - X^*))\\ &= 2s_j(2\mathrm{tr}(X^*)I_2 - X^*)\\ &= 2s_j(\Psi(X^*))\\ &= 2s_j(\Psi(X))\\ &\le s_j(\mathrm{tr}(\mathrm{tr}(A)I_2 - A)I_2 + (\mathrm{tr}(A)I_2 - A) + \mathrm{tr}(\mathrm{tr}(B)I_2 - B)I_2 + (\mathrm{tr}(B)I_2 - B))\\ &= s_j(2\mathrm{tr}(A)I_2 - A + 2\mathrm{tr}(B)I_2 - B)\\ &= s_j(\Psi(A+B)).\end{aligned}$$

Hence, the result follows. □

Remark 2.8. *Although we have not solved Lin's conjecture in Theorem 2.7, our result is a step closer to the solution of the conjecture.*

References

[1] R. Bhatia, Positive Definite Matrices, Princeton University Press, Princeton, 2007.
[2] M. D. Choi, Completely positive linear maps on complex matrices, Linear Algebra Appl. 10 (1975) 285-290.
[3] M. Lin, A completely PPT map, Linear Algebra Appl. 459 (2014) 404-410.
[4] M. Lin, Inequalities related to 2×2 block PPT matrices, Oper Matrices 94 (2015) 917-924.
[5] M. Lin, A singular value inequality related to a linear map, Electron. J. Linear Algebra 31 (2016) 120-124.
[6] M. Lin, A determinantal inequality involving partial traces, Canad. Math. Bull. 59 (2016) 585-591.
[7] M. Lin, New properties for certain positive semidefinite matrices, Linear Algebra Appl. 520 (2017) 32-43.