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Hemi-slant $\xi^{\perp}-$Riemannian Submersions in Contact Geometry

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Abstract. M. A. Akyol and R. Sarı [On semi-slant $\xi^{\perp}$-Riemannian submersions, Mediterr. J. Math. 14(6) (2017) 234.] defined semi-slant $\xi^{\perp}-$Riemannian submersions from Sasakian manifolds onto Riemannian manifolds. As a generalization of the above notion and natural generalization of anti-invariant $\xi^{\perp}-$Riemannian submersions, semi-invariant $\xi^{\perp}-$Riemannian submersions and slant submersions, we study hemi-slant $\xi^{\perp}-$Riemannian submersions from Sasakian manifolds onto Riemannian manifolds. We obtain the geometry of foliations, give some examples and find necessary and sufficient condition for the base manifold to be a locally product manifold. Moreover, we obtain some curvature relations from Sasakian space forms between the total space, the base space and the fibres.

1. Introduction

Riemannian submersions betwen Riemannian manifolds were studied by O'Neill and Gray [14, 23]. After this kind of submersions were studied between manifolds endowed with differentiable structures. Many authors studied different geometric properties of the Riemannian submersions, anti-invariant submersion [18, 30, 33], semi-invariant submersion [4, 31], paraquaternionic 3-submersion [37], statistical submersion [38], slant submersion [11, 12, 15, 27, 32], semi-slant submersion [16, 25, 26], conformal slant submersion [2, 17] , conformal semi-slant submersion [1], bi-slant submersion [34] and Quasi bi-slant submersion [28].

On the other hand, Riemannian submersions have some aplications in physics and in mathematics. More precisely, Riemannian submersions have applications in supergravity and superstring theories [21, 22], Kaluza-Klein theory [9, 20] and Yang-Mills theory [8, 39].

As a generalization of anti-invariant, semi-invariant and slant submersion, Taştan et al. defined the notion of hemi-slant Riemannian submersion in [36] (see also [3], [19], [24]).

Recently, Akyol et al. defined and studied of semi-invariant $\xi^{\perp}$-Riemannian submersion and semi-slant $\xi^{\perp}$-Riemannian submersion from almost contact manifolds onto Riemannian manifold [4, 5, 35]. They studied the geometry of this new submersions on almost contact manifolds. Our motivation is to fill a gap in the geometry of hemi-slant $\xi^{\perp}-$Riemannian submersions in contact geometry.

The paper consists of five sections. Sect. 2, we mention fundamental basic notions related to Riemannian submersions and Sasakian manifolds. Sect. 3, we define hemi-slant $\xi^{\perp}-$Riemannian submersions from

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Sasakian manifolds onto Riemannian manifolds and demostrate lots of examples of such submersions. Sect. 4, we investigate the geometry of leaves of the horizontal distribution and the vertical distribution of a hemi-slant $\xi^{\perp}-$Riemannian submersion. In the final section, we obtain curvature properties of distribution for a hemi-slant $\xi^{\perp}-$Riemannian submersion from Sasakian space forms.

2. Preliminaries

Let $(M,<,>_M)$ be an almost contact metric manifold with structure tensors $(\varphi,\xi,\eta,<,>_M)$ where φ is a tensor field of type (1,1), ξ is a characteristic vector field, η is a 1-form and $<,>_M$ is the Riemannian metric on M. Then these tensors satisfy [7]

$$\varphi\xi = 0, \ \ \eta\circ\varphi = 0, \ \ \eta(\xi) = 1 \tag{1}$$

$$\varphi^2 = -I + \eta\otimes\xi \ \ \text{and} \ \ <\varphi X, \varphi Y>_M = <X, Y>_M - \eta(X)\eta(Y), \tag{2}$$

where I denotes the identity endomorphism of TM and X, Y are any vector fields on M, where $\eta(X) = <X, \xi>_M$. Moreover, if M is Sasakian [29], then we have

$$(\nabla^M_X\varphi)Y = <X, Y>_M \xi + \eta(Y)X \ \ \text{and} \ \ \nabla^M_X\xi = -\varphi X, \tag{3}$$

where ∇^M is the connection of Levi-Civita covariant differentiation.

Let $(M^m_1, <,>_1)$ and $(M^n{}_2, <,>_2)$ Riemannian manifolds, where $\dim(M_1) = m$, $\dim(M_2) = n$ and $m > n$. A Riemannian submersion $\phi : M_1 \to M_2$ is a map of M_1 onto M_2 satisfying the following axioms:

(i) ϕ has maximal rank.
(ii)The differential ϕ_* preserves the lenghts of horizontal vectors.

For each $q \in M_2$, $\phi^{-1}(q)$ is an $(m-n)$ dimensional submanifold of M_1. The submanifolds $\phi^{-1}(q)$, $q \in M_2$, are called fibers. A vector field on M_1 is called vertical if it is always tangent to fibres. A vector field on M_1 is called horizontal if it is always orthogonal to fibres. A vector field X on M_1 is called basic if X is horizontal and $\phi-$related to a vector field X' on M_2, i.e., $\phi_*X_p = X'_{\phi_*(p)}$ for all $p \in M_1$. Note that we denote the projection morphisms on the distributions $ker\phi_*$ and $(ker\phi_*)^{\perp}$ by $\mathcal{V}$ and $\mathcal{H}$, respectively. We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A Riemannian submersion $\phi : M_1 \to M_2$ determines two $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ on M_1, by the formulas:

$$\mathcal{T}(E, F) = \mathcal{T}_E F = \mathcal{H}\nabla^{M_1}_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla^{M_1}_{\mathcal{V}E}\mathcal{H}F \tag{4}$$

and

$$\mathcal{A}(E, F) = \mathcal{A}_E F = \mathcal{V}\nabla^{M_1}_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla^{M_1}_{\mathcal{H}E}\mathcal{V}F \tag{5}$$

for any $E, F \in \Gamma(TM_1)$, where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections (see [13]). From (4) and (5), one can obtain

$$\nabla^{M_1}_V W = \mathcal{T}_V W + \hat{\nabla}_V W; \tag{6}$$

$$\nabla^{M_1}_V X = \mathcal{T}_V X + \mathcal{H}(\nabla^{M_1}_V X); \tag{7}$$

$$\nabla^{M_1}_X V = \mathcal{V}(\nabla^{M_1}_X V) + \mathcal{A}_X V; \tag{8}$$

$$\nabla^{M_1}_X Y = \mathcal{A}_X Y + \mathcal{H}(\nabla^{M_1}_X Y), \tag{9}$$

for any $X, Y \in \Gamma((ker\phi_*)^{\perp})$ and $V, W \in \Gamma(ker\phi_*)$. Moreover, if X is basic then

$$\mathcal{H}(\nabla^{M_1}_V X) = \mathcal{A}_X V. \tag{10}$$

We note that for $U, V \in \Gamma(ker\phi_*)$, $\mathcal{T}_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X, Y \in \Gamma((ker\phi_*)^\perp)$, $\mathcal{A}_X Y = \frac{1}{2}\mathcal{V}[X, Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that $\mathcal{A}$ is alternating on the horizontal distribution: $\mathcal{A}_X Y = -\mathcal{A}_Y X$, for $X, Y \in \Gamma((ker\phi_*)^\perp)$ and $\mathcal{T}$ is symmetric on the vertical distribution: $\mathcal{T}_U V = \mathcal{T}_V U$ for $U, V \in \Gamma(ker\phi_*)$.

Lemma 2.1. *(see [13], [23]). If $\phi : M_1 \to M_2$ is a Riemannian submersion and X, Y basic vector fields on M_1, $\phi-$related to X' and Y' on M_2, then we have the following properties*

1. $\mathcal{H}[X, Y]$ *is a basic vector field and* $\phi_*\mathcal{H}[X, Y] = [X', Y'] \circ \phi$;
2. $\mathcal{H}(\nabla^{M_1}_X Y)$ *is a basic vector field $\phi-$related to $(\nabla^{M_2}_{X'} Y')$, where ∇^{M_1} and ∇^{M_2} are the Levi-Civita connection on M_1 and M_2;*
3. $[E, U] \in \Gamma(ker\phi_*)$, *for any $U \in \Gamma(ker\phi_*)$ and for any basic vector field E.*

Let $(M_1, <,>_{M_1})$ and $(M_2, <,>_{M_2})$ be Riemannian manifolds and $\phi : M_1 \to M_2$ is a smooth map. Then the second fundamental form of ϕ is given by

$$(\nabla\phi_*)(X, Y) = \nabla^{M_2}_X \phi_* Y - \phi_*(\nabla^{M_1}_X Y) \tag{11}$$

for $X, Y \in \Gamma(TM_1)$, where we denote the Levi-Civita connections of the metrics $<,>_{M_1}$ and $<,>_{M_2}$ conveniently by ∇. Recall that ϕ is called a *totally geodesic* map if $(\nabla\phi_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM_1)$ [6]. It is known that the second fundamental form is symmetric.

We note that, the tensor fields $\mathcal{A}, \mathcal{T}$ their covariant derivatives play a fundemantal role in expressing the Riemannian curvaure R^{M_1} of M_1. In 1966 ([23]), O'Neill are given

$$R^{M_1}(U, V, W, S) = \widehat{R}(U, V, W, S) + < \mathcal{T}_U W, \mathcal{T}_V S >_{M_1} - < \mathcal{T}_V W, \mathcal{T}_U S >_{M_1} \tag{12}$$

where $\widehat{R}$ is Riemannian curvature tensor of any fibre $(\pi^{-1}(x), <,>_{M_1})$. Moreover if $\{U, V\}$ is orthonormal basis of the vertical 2-plane, then from (12) we have

$$K^{M_1}(U, V) = \widehat{K}(U, V) + \|\mathcal{T}_U V\|^2 - < \mathcal{T}_U U, \mathcal{T}_V V >_{M_1}$$

where K^{M_1} and $\widehat{K}$ is sectional curvature of M_1 and $\phi^{-1}(x)$.

3. Hemi-slant $\xi^\perp-$Riemannian submersions

Definition 3.1. *Let $(M, \varphi, \xi, \eta, <,>_M)$ be a Sasakian manifold and $(N, <,>_N)$ be a Riemannian manifold. Suppose that there exists a Riemannian submersion $\phi : M \longrightarrow N$ such that ξ is normal to $ker\phi_*$. Then ϕ is called a hemi-slant $\xi^\perp-$Riemannian submersion if the vertical distribution $ker\phi_*$ of ϕ admits two orthogonal complementary distributions $\mathcal{D}_\perp$ and $\mathcal{D}_\theta$ such that $\mathcal{D}_\perp$ is anti-invariant and $\mathcal{D}_\theta$ is slant, i.e, we have*

$$\ker\phi_* = \mathcal{D}_\perp \oplus \mathcal{D}_\theta.$$

In this case, the angle θ is called the hemi slant angle of the hemi-slant $\xi^\perp-$Riemannian submersion.

If $\theta \neq 0, \frac{\pi}{2}$ then we say that the submersion is proper hemi-slant $\xi^\perp-$Riemannian submersion. Now, we are going to give some proper examples in order to guarantee the existence of hemi-slant $\xi^\perp-$Riemannian submersions in Sasakian manifolds and demonstrate that the method presented in this paper is effective. Note that, $(\mathbb{R}^{2n+1}, \varphi, \eta, \xi, <,>_{\mathbb{R}^{2n+1}})$ will denote the manifold $\mathbb{R}^{2n+1}$ with its usual contact structure given by [10]

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i), \ \ \xi = 2\partial z,$$

$$<,>= \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\varphi(\sum_{i=1}^{n}(X_i\partial x^i + Y_i\partial y^i) + Z\partial z) = \sum_{i=1}^{n}(Y_i\partial x^i - X_i\partial y^i) + \sum_{i=1}^{n} Y_i y^i \partial z$$

where $(x_1, .., x_n, y_1, ..., y_n, z)$ denotes the Cartesian coordinates on $\mathbb{R}^{2n+1}$.

Example 3.2. *Every anti-invariant $\xi^\perp-$Riemannian submersion from a Sasakian manifold onto a Riemannian manifold is a hemi-slant $\xi^\perp-$Riemannian submersion with $\mathcal{D}_\theta = \{0\}$.*

Example 3.3. *Every slant $\xi^\perp-$Riemannian submersion from a Sasakian manifold onto a Riemannian manifold is a hemi-slant $\xi^\perp-$Riemannian submersion with $\mathcal{D}_\perp = \{0\}$.*

Example 3.4. *Let ϕ be a submersion defined by*

$$\begin{array}{cccc} \phi: & (\mathbb{R}^9, <,>_{\mathbb{R}^9}) & \to & (\mathbb{R}^5, <,>_{\mathbb{R}^5}) \\ & (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z) & & (\frac{x_1+y_2}{\sqrt{2}}, \frac{x_2+y_1}{\sqrt{2}}, \sin\gamma x_3 - \cos\gamma x_4, y_4, z) \end{array}$$

with $\gamma \in (0, \frac{\pi}{2})$. Then it follows that

$$\ker\phi_* = Sp\{V_1 = -\partial x_1 + \partial y_2, V_2 = -\partial x_2 + \partial y_1, V_3 = -\cos\gamma\partial x_3 - \sin\gamma\partial x_4, \\ V_4 = \partial y_3\}$$

and

$$(\ker\phi_*)^\perp = Sp\{W_1 = \partial x_1 + \partial y_2, W_2 = \partial x_2 + \partial y_1, W_3 = \sin\gamma\partial x_3 - \cos\gamma\partial x_4, \\ W_4 = \partial y_4, W_5 = \partial z\}$$

hence we have $\varphi V_1 = W_2, \varphi V_2 = W_1$. Thus it follows that $\mathcal{D}_\perp = sp\{V_1, V_2\}$ and $\mathcal{D}_\theta = sp\{V_3, V_4\}$ is a slant distribution with hemi-slant angle $\theta = \gamma$. Thus ϕ is a hemi-slant $\xi^\perp-$ submersion. Also by direct computations, we obtain

$$< W_i, W_i >_{\mathbb{R}^9} = < \varphi W_i, \varphi W_i >_{\mathbb{R}^5}, \quad i = 1, ..., 5$$

which show that ϕ is a hemi-slant $\xi^\perp-$Riemannian submersion.

Example 3.5. *Let F be a submersion defined by*

$$\begin{array}{cccc} F: & (\mathbb{R}^9, <,>_{\mathbb{R}^9}) & \longrightarrow & (\mathbb{R}^5, <,>_{\mathbb{R}^5}) \\ & (x_1, ..., y_1, ..., z) & & (\frac{x_1+y_2}{\sqrt{2}}, \frac{x_2+y_1}{\sqrt{2}}, \frac{x_3+x_4}{\sqrt{2}}, \frac{y_3+y_4}{\sqrt{2}}, z). \end{array}$$

The submersion F is hemi-slant $\xi^\perp-$Riemannian submersion such that $\mathcal{D}_\perp = sp\{\partial x_1 - \partial y_2, \partial x_2 - \partial y_1\}$ and $\mathcal{D}_\theta = sp\{\partial x_3 + \partial x_4, \partial y_3 + \partial y_4\}$ with hemi-slant angle $\theta = 0$.

Example 3.6. *Let π be a submersion defined by*

$$\begin{array}{cccc} \pi: & (\mathbb{R}^7, <,>_{\mathbb{R}^7}) & \longrightarrow & (\mathbb{R}^4, <,>_{\mathbb{R}^4}) \\ & (x_1, ..., y_1, ..., z) & & (\frac{x_1+x_2}{\sqrt{2}}, \sin\gamma x_3 - \cos\gamma y_4, \cos\beta x_4 - \sin\beta y_3, z). \end{array}$$

The submersion π is a hemi-slant $\xi^\perp-$Riemannian submersion such that $\mathcal{D}_\perp = sp\{\partial x_1 - \partial x_2\}$ and $\mathcal{D}_\theta = sp\{\cos\gamma\partial x_3 - \sin\gamma\partial y_4, \sin\beta\partial x_4 - \cos\beta\partial y_3\}$ with hemi-slant angle $\theta = \alpha + \beta$.

Let ϕ be a hemi-slant $\xi^{\perp}-$Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, <,>_M)$ onto a Riemannian manifold $(N, <,>_N)$. Then, for $U \in \Gamma(ker\phi_*)$, we put

$$U = \mathcal{P}U + \mathcal{Q}U$$

where $\mathcal{P}U \in \Gamma(\mathcal{D}_{\perp})$ and $\mathcal{Q}U \in \Gamma(\mathcal{D}_{\theta})$. For $Z \in \Gamma(TM)$, we have

$$Z = \mathcal{V}Z + \mathcal{H}Z$$

where $\mathcal{V}Z \in \Gamma(ker\phi_*)$ and $\mathcal{H}Z \in \Gamma(ker\phi_*)^{\perp}$.

We denote the complementary distribution to $\varphi\mathcal{D}_{\perp}$ in $(ker\phi_*)^{\perp}$ by μ. Then we have

$$(ker\phi_*)^{\perp} = \varphi\mathcal{D}_{\perp} \oplus \mu,$$

where $\varphi(\mu) \subset \mu$. Hence μ contains ξ. For $V \in \Gamma(ker\phi_*)$, we write

$$\varphi V = \rho V + \omega V \tag{13}$$

where ρV and ωV are vertical (resp. horizontal) components of φV, respectively. Also for $X \in \Gamma((ker\phi_*)^{\perp})$, we have

$$\varphi X = \mathcal{B}X + \mathcal{C}X, \tag{14}$$

where $\mathcal{B}X$ and $\mathcal{C}X$ are vertical (resp. horizontal) components of φX, respectively. Then the horizontal distribution $(ker\phi_*)^{\perp}$ is decomposed as

$$(ker\phi_*)^{\perp} = \varphi\mathcal{D}_{\perp} \oplus \mu,$$

here μ is the orthogonal complementary distribution of $\mathcal{D}_{\perp}$ and it is both invariant distribution of $(ker\phi_*)^{\perp}$ with respect to φ and contains ξ. Then by using (6), (7), (13) and (14), we get

$$(\overset{M}{\nabla}_V \rho)W = \mathcal{B}\mathcal{T}_V W - \mathcal{T}_V \omega W \tag{15}$$

$$(\overset{M}{\nabla}_V \omega)W = \mathcal{C}\mathcal{T}_V W - \mathcal{T}_V \rho W \tag{16}$$

for $V, W \in \Gamma(ker\phi_*)$, where

$$(\overset{M}{\nabla}_V \rho)W = \hat{\nabla}_V \rho W - \rho\hat{\nabla}_V W$$

and

$$(\overset{M}{\nabla}_V \omega)W = \mathcal{H}\overset{M}{\nabla}_V \omega W - \omega\hat{\nabla}_V W.$$

The proof of the following is exactly same with slant immersions (see [10]), therefore we omit its proof.

Theorem 3.7. *Let $\phi : (M, \varphi, \eta, \xi, <,>_M) \longrightarrow (N, <,>_N)$ be a hemi-slant $\xi^{\perp}-$Riemannian submersion, where $(M, \varphi, \eta, \xi, <,>_M)$ is a Sasakian manifold and $(N, <,>_N)$ is a Riemannian manifold. Then we have*

$$\rho^2 W = \cos^2\theta W, \ W \in \Gamma(\mathcal{D}_{\theta}), \tag{17}$$

where θ denotes the hemi-slant angle of $ker\phi_*$.

By using above theorem, it is easy to see the following.

Lemma 3.8. *Let $\phi : M \to N$ be a hemi-slant $\xi^{\perp}-$Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, <,>_M)$ onto a Riemannian manifold $(N, <,>_N)$. Then we have*

$$< \rho U, \rho V >_M = \cos^2\theta < U, V >_M \tag{18}$$

$$< \omega U, \omega V >_M = \sin^2\theta < U, V >_M \tag{19}$$

for all $U, V \in \Gamma(ker\phi_*)$.

4. Integrability, Totally Geodesicness and Decomposition Theorems

Theorem 4.1. *Let $\phi : (M, \varphi, \eta, \xi, <,>_M) \longrightarrow (N, <,>_N)$ be a hemi-slant $\xi^{\perp}-$Riemannian submersion, where $(M, \varphi, \eta, \xi, <,>_M)$ is a Sasakian manifold and $(N, <,>_N)$ is a Riemannian manifold. Then the distribution $\mathcal{D}_{\perp}$ is integrable if and only if we have*

$$< \mathcal{T}_U\varphi V - \mathcal{T}_V\varphi U, \rho Z >_M=< (\nabla\phi_*)(U, \varphi V) - (\nabla\phi_*)(V, \varphi U), \phi_*(\omega Z) >_N$$

for any $U, V \in \Gamma(\mathcal{D}_{\perp})$ and $Z \in \Gamma(\mathcal{D}_\theta)$.

Proof. For $U, V \in \Gamma(TM)$, by using (2) and (3), we have

$$< \nabla^M_U V, Z >_M=< \nabla^M_U \varphi V, \varphi Z >_M . \tag{20}$$

For $U, V \in \Gamma(\mathcal{D}_{\perp})$, $Z \in \Gamma(\mathcal{D}_\theta)$, using (2) and (20), we have

$$< [U, V], Z >_M=< \nabla^M_U \varphi V, \varphi Z >_M - < \nabla^M_V \varphi U, \varphi Z >_M .$$

On the other hand, by using (6) we get

$$< [U, V], Z >_M=< \mathcal{T}_U\varphi V - \mathcal{T}_V\varphi U, \rho Z >_M + < \mathcal{H}(\nabla^M_U \varphi V) - \mathcal{H}(\nabla^M_V \varphi U), wZ >_M .$$

Now, using the property of ϕ, we obtain

$$\begin{aligned} < [U, V], Z >_M =& < \mathcal{T}_U\varphi V - \mathcal{T}_V\varphi U, \rho Z >_M \\ &+ < \phi_*(\nabla^M_U \varphi V) - \phi_*(\nabla^M_V \varphi U), \phi_*(\omega Z) >_N \end{aligned}$$

which gives the proof. □

Theorem 4.2. *Let ϕ be a hemi-slant $\xi^{\perp}-$Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, <,>_M)$ onto Riemannian manifold $(N, <,>_N)$ with a hemi-slant angle θ. Then the distribution $\mathcal{D}_\theta$ is integrable if and only if we have*

$$< (\nabla\phi_*)(Z, \omega W) - (\nabla\phi_*)(W, \omega Z), \phi_*(\varphi U) >_N=< \mathcal{T}_Z\omega\rho W - \mathcal{T}_W w\rho Z, U >_M$$

for any $Z, W \in \Gamma(\mathcal{D}_\theta)$ and $U \in \Gamma(\mathcal{D}_{\perp})$.

Proof. For $Z, W \in \Gamma(\mathcal{D}_\theta)$ and $U \in \Gamma(\mathcal{D}_{\perp})$, using (2) and (20) we have

$$< [Z, W], U >_M \quad = \quad < \nabla^M_Z \varphi W, \varphi U >_M - < \nabla^M_W \varphi Z, \varphi U >_M .$$

Therefore by using (13), we get

$$\begin{aligned} < [Z, W], U >_M \quad = \quad & - < \nabla^M_Z \rho^2 W, U >_M - < \nabla^M_Z \omega\rho W, U >_M \\ & + < \nabla^M_Z \omega W, \varphi U >_M + < \nabla^M_W \rho^2 Z, U >_M \\ & + < \nabla^M_W \omega\rho Z, U >_M - < \nabla^M_W \omega Z, \varphi U >_M . \end{aligned}$$

Now, by using (17) we obtain

$$\begin{aligned} < [Z, W], U >_M \quad = \quad & \cos^2\theta < [Z, W], U >_M - < \nabla^M_Z \omega\rho W, U >_M \\ & + < \nabla^M_Z \omega W, \varphi U >_M + < \nabla^M_W \omega\rho Z, U >_M \\ & - < \nabla^M_W \omega Z, \varphi U >_M . \end{aligned}$$

Then we have,

$$\begin{aligned} \sin^2\theta < [Z, W], U >_M =& < \nabla^M_W \omega\rho Z - \nabla^M_Z \omega\rho W, U >_M \\ &+ < \nabla^M_Z \omega W - \nabla^M_W \omega Z, \varphi U >_M . \end{aligned}$$

On the other hand, by using (7) we get

$$\begin{aligned}\sin^2\theta < [Z,W],U >_M &= < \mathcal{T}_W\omega\rho Z - \mathcal{T}_Z\omega\rho W, U >_M \\ &+ < H(\nabla^M_Z\omega W) - \mathcal{H}(\nabla^M_W\omega Z), \varphi U >_M \\ &= < \mathcal{T}_W\omega\rho Z - \mathcal{T}_Z\omega\rho W, U >_M \\ &+ < \phi_*(\nabla^M_Z\omega W) - \phi_*(\nabla^M_W\omega Z), \phi_*(\varphi U) >_N\end{aligned}$$

which gives desired result. This completes the proof. □

Theorem 4.3. *Let $\phi : (M,\varphi,\eta,\xi,<,>_M) \longrightarrow (N,<,>_N)$ be a hemi-slant $\xi^\perp-$Riemannian submersion, where $(M,\varphi,\eta,\xi,<,>_M)$ is a Sasakian manifold and $(N,<,>_N)$ is a Riemannian manifold. Then the distribution $D_\perp$ is parallel if and only if*

$$< \phi_*(\nabla_U V), \phi_*(\omega\rho Z) >_M = < \varphi\nabla_U V, \omega Z >_M$$

and

$$< \hat{\nabla}_U\rho V + \mathcal{T}_U\omega V, BX >_M = - < \mathcal{T}_U\rho V + \mathcal{H}(\nabla_U\omega V), CX >_M$$

for any $U,V \in \Gamma(D_\perp)$, $Z \in \Gamma(D_\theta)$, $X \in \Gamma((\ker\phi_*)^\perp)$.

Proof. For $U,V \in \Gamma(D_\perp)$, $Z \in \Gamma(D_\theta)$ using (2) we get

$$\begin{aligned}< \nabla_U V, Z >_M &= < \varphi\nabla_U V, \varphi Z >_M + \eta(\nabla_U V)\eta(Z) \\ &= < \varphi\nabla_U V, \varphi Z >_M .\end{aligned}$$

By using (13) we have,

$$< \nabla_U V, Z >_M = - < \nabla_U V, \rho^2 Z + \omega\rho Z + \varphi\omega Z >_M .$$

Then, using (7) and (17), we obtain

$$\sin^2\theta < \nabla_U V, Z >_M = - < \mathcal{H}(\nabla_U V), \omega\rho Z >_M + < \varphi\nabla_U V, \omega Z >_M .$$

If we take into account the property of ϕ, we get

$$\sin^2\theta < \nabla_U V, Z >_M = - < \phi_*(\nabla_U V), \phi_*(\omega\rho Z) >_N + < \varphi\nabla_U V, \omega Z >_M .$$

On the other hand, for any $U,V \in \Gamma(D_\perp)$ and $X \in \Gamma((\ker\phi_*)^\perp)$, from (2) and (3) we find

$$< \nabla_U V, X >_M = < \nabla_U\varphi V, \varphi X >_M .$$

Then by using (6), (7), (13) and (14), we get

$$\begin{aligned}< \nabla_U V, X >_M &= < \mathcal{T}_U\rho V, CX >_M + < \hat{\nabla}\rho V, BX >_M \\ &+ < \mathcal{T}_U\omega V, BX >_M + < \mathcal{H}(\nabla_U\omega V), CX >_M\end{aligned}$$

which completes the proof. □

Theorem 4.4. *Let ϕ be a hemi-slant $\xi^\perp-$Riemannian submersion from a Sasakian manifold $(M,\varphi,\eta,\xi,<,>_M)$ onto Riemannian manifold $(N,<,>_N)$ with a hemi-slant angle θ. Then the distribution D_θ is parallel if and only if*

$$< \phi_*(\omega W), (\nabla\phi_*)(Z,\varphi U) >_N = < \varphi U, \mathcal{T}_Z\rho W >_M$$

and

$$\begin{aligned}&< (\nabla\phi_*)(Z,\omega W), \phi_*(X) >_N - < (\nabla\phi_*)(Z,\omega\rho W), \phi_*(CX) >_N \\ &= < \mathcal{T}_Z\omega W, BX >_M + < Z, \rho W >_M \eta(X)\end{aligned}$$

for all $Z,W \in \Gamma(D_\theta)$, $U \in \Gamma(D_\perp)$, $X \in \Gamma((\ker\phi_*)^\perp)$.

Proof. For $Z, W \in \Gamma(D_\theta)$ and $U \in \Gamma(D_\perp)$, from (2) and (3), we have

$$< \nabla_Z W, U >_M =< \nabla_Z \varphi W, \varphi U >_M .$$

Moreover, we get

$$< \nabla_Z W, U >_M = Z < \varphi W, \varphi U >_M - < \varphi W, \nabla_Z \varphi U >_M .$$

Then using (7) and (13), we have

$$< \nabla_Z W, U >_M = - < \rho W, \mathcal{T}_Z \varphi U >_M + < \phi_*(\omega W), \phi_*(\nabla_Z \varphi U) >_N .$$

On the other hand, for any $Z, W \in \Gamma(D_\theta)$, $X \in \Gamma((\ker \phi_*)^\perp)$, by using (2) and (3), we get

$$\begin{aligned} < \nabla_Z W, X >_M &= < \varphi \nabla_Z W, \varphi X >_M + \eta(\nabla_Z W)\eta(X) \\ &= < -(\nabla_Z \varphi)W + \nabla_Z \varphi W, \varphi X >_M + < \nabla_Z W, \xi >_M \eta(X) \\ &= < \nabla_Z \varphi W, \varphi X >_M + < W, \varphi Z >_M \eta(X). \end{aligned}$$

By using (7), (17) and (18), we obtain

$$\begin{aligned} < \nabla_Z W, X >_M = \cos^2\theta < \nabla_Z W, X >_M - < \mathcal{H}(\nabla_Z \omega \rho W), X >_M \\ + < Z, \rho W >_M \eta(X) + < \mathcal{T}_Z \omega W, BX >_M \\ + < \mathcal{H}(\nabla_Z \omega W), CX >_M . \end{aligned}$$

Then we have

$$\begin{aligned} \sin^2\theta < \nabla_Z W, X >_M = - < \phi_*(\nabla_Z \omega \rho W), \phi_*(X) >_N + < Z, \rho W >_M \eta(X) \\ + < T_Z \omega W, BX >_M + < \phi_*(\nabla_Z \omega W), \phi_*(CX) >_N \end{aligned}$$

which proves assertion. □

Theorem 4.5. *Let $\phi : (M, \varphi, \eta, \xi, <,>_M) \longrightarrow (N, <,>_N)$ be a hemi-slant $\xi^\perp-$Riemannian submersion, where $(M, \varphi, \eta, \xi, <,>_M)$ is a Sasakian manifold and $(N, <,>_N)$ is a Riemannian manifold. Then $D_\perp$ defines a totally geodesic foliation on M if and only if*

$$< (\nabla \phi_*)(U, \varphi V), \phi_*(\omega Z) >_N =< \mathcal{T}_U \omega \rho Z, V >_M$$

and

$$< \mathcal{T}_U \varphi V, BX >_M =< (\nabla \phi_*)(U, \varphi V), \phi_*(CX) >_N$$

for any $U, V \in \Gamma(D_\perp)$, $Z \in \Gamma(D_\theta)$ and $X \in \Gamma((\ker \phi_*)^\perp)$.

Proof. For any $U, V \in \Gamma(D_\perp)$ and $Z \in \Gamma(D_\theta)$, by using (2), (6), (7), (13) and (17), we have

$$< \nabla_U V, Z >_M = \cos^2\theta < \nabla_U V, Z >_M - < \mathcal{T}_U V, \omega \rho Z >_M + < \mathcal{H}(\nabla_U \varphi V), wZ >_M .$$

Taking into account the property of ϕ, we obtain

$$\sin^2\theta < \nabla_U V, Z >_M = - < \mathcal{T}_U V, w \rho Z >_M - < \phi_*(\nabla_U \varphi V), \phi_*(\omega Z) >_N .$$

On the other hand, for any $X \in \Gamma((\ker \phi_*)^\perp)$, by using (2), (6), (7) and (14), we get

$$< \nabla_U V, X >_M =< \mathcal{T}_U \varphi V, BX >_M + < \mathcal{H}(\nabla_U \varphi V), CX >_M .$$

By using the property of ϕ, we get

$$< \nabla_U V, X >_M =< \mathcal{T}_U \varphi V, BX >_M + < \phi_*(\nabla_U \varphi V), \phi_*(CX) >_N$$

which completes proof. □

Theorem 4.6. *Let ϕ be a hemi-slant $\xi^{\perp}-$Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, <,>_M)$ onto Riemannian manifold $(N, <,>_N)$ with a hemi-slant angle θ. Then D_θ defines a totally geodesic foliation on M if and only if*

$$<(\nabla\phi_*)(Z, \omega W), \phi_*(\varphi U)>_N = <\mathcal{T}_Z U, \omega\rho W>_M$$

and

$$<(\nabla\phi_*)(Z, \omega W), \phi_*(X)>_N + <(\nabla\phi_*)(Z, \omega\rho W), \phi_*(CX)>_N = <\mathcal{T}_Z \omega W, BX>_M$$

for any $Z, W \in \Gamma(D_\theta)$, $U \in \Gamma(D_\perp)$ and $X \in \Gamma((\ker\phi_*)^\perp)$.

Proof. By using (2), (3) and (13), we obtain

$$<\nabla_Z W, U>_M = <\nabla_Z \rho^2 W, U>_M - <\nabla_Z \omega\rho W, U>_M + <\nabla_Z \omega W, \varphi U>_M$$

for any $Z, W \in \Gamma(D_\theta)$ and $U \in \Gamma(D_\perp)$. Now, using (7) and (17) we get

$$<\nabla_Z W, U>_M = \cos^2\theta <\nabla_Z W, U>_M - <\mathcal{T}_Z \omega\rho W, U>_M + <\mathcal{H}(\nabla_Z \omega W), \varphi U>_M .$$

Then we have

$$\sin^2\theta <\nabla_Z W, U>_M = - <\mathcal{T}_Z \omega\rho W, U>_M + <\phi_*(\nabla_Z \omega W), \phi_*(\varphi U)>_N .$$

On the other hand, by using (2), (3), (13) and (14), we have

$$<\nabla_Z W, X>_M = - <\nabla_Z \varphi\rho W, X>_M + <\nabla_Z \omega W, BX>_M + <\nabla_Z \omega W, CX>_M .$$

for any $X \in \Gamma((\ker\phi_*)^\perp)$. Hence, again using (7) and (17) we obtain

$$\begin{aligned}<\nabla_Z W, X>_M &= \cos^2\theta <\nabla_Z W, X>_M - <\mathcal{H}(\nabla_Z w\rho W), X>_M \\ &\quad + <\mathcal{T}_Z \omega W, BX>_M + <\mathcal{H}(\nabla_Z \omega W), CX>_M .\end{aligned}$$

Taking into account the property of ϕ, we obtain

$$\begin{aligned}\sin^2\theta <\nabla_Z W, X>_M &= - <\phi_*(\nabla_Z \omega\rho W), \phi_*(X)>_N + <\mathcal{T}_Z \omega W, BX>_M \\ &\quad + <\phi_*(\nabla_Z \omega W), \phi_*(CX)>_N\end{aligned}$$

which proves assertion. □

5. Hemi-slant $\xi^{\perp}-$Riemannian submersions from Sasakian Space Forms

A plane section in the tangent space T_pM at $p \in M$ is called a φ-section if it is spanned by a vector X orthogonal to ξ and φX. The sectional curvature of φ-section is called φ-sectional curvature. A Sasakian manifold with constant φ-sectional curvature c is a Sasakian space form. The Riemannian curvature tensor of a Sasakian space form is given by

$$\begin{aligned}R^M(X, Y, Z, W) &= \frac{c+3}{4}\{<Y, Z>_M <X, W>_M - <X, Z>_M <Y, W>_M\} \\ &\quad + \frac{c-1}{4}\{<Y, W>_M \eta(X)\eta(Z) - <X, W>_M \eta(Y)\eta(Z) \\ &\quad + <X, Z>_M \eta(Y)\eta(W) - <Y, Z>_M \eta(X)\eta(W) \\ &\quad + <\varphi Y, Z>_M <\varphi X, W>_M - <\varphi X, Z>_M <\varphi Y, W>_M \\ &\quad - 2 <\varphi X, Y>_M <\varphi Z, W>_M\} \end{aligned} \tag{21}$$

for any $X, Y, Z, W \in \Gamma(TM)$ [7].

Theorem 5.1. *Let $\phi : (M, \varphi, \eta, \xi, <,>_M) \longrightarrow (N, <,>_N)$ be a hemi-slant $\xi^\perp$−Riemannian submersion, where $(M, \varphi, \eta, \xi, <,>_M)$ is a Sasakian manifold and $(N, <,>_N)$ is a Riemannian manifold. Then we have*

$$\widehat{R}(U, V, W, S) = \frac{c+3}{4}\{< V, S >_M< U, W >_M - < U, S >_M< V, W >_M\} \\ + < \mathcal{T}_V W, \mathcal{T}_U S >_M - < \mathcal{T}_U W, \mathcal{T}_V S >_M \tag{22}$$

and

$$\widehat{K}(U, V) = \frac{c+3}{4}\{< U, V >_M^2 -1\} + < \mathcal{T}_V U, \mathcal{T}_U V >_M - < \mathcal{T}_U U, \mathcal{T}_V V >_M \tag{23}$$

for all $U, V, S, W \in \Gamma(\mathcal{D}^\perp)$.

Proof. For any $U \in \Gamma(\mathcal{D}_\perp)$, we have $\varphi U \in \Gamma((\ker \phi_*)^\perp)$ and $\eta(U) = 0$. Now, from (21) we obtain

$$R^M(U, V, W, S) = \frac{c+3}{4}\{< V, S >_M< U, W >_M - < U, S >_M< V, W >_M\} \tag{24}$$

for any $V, S, W \in \Gamma(\mathcal{D}_\perp)$. Then, by using (12) we get

$$\widehat{R}(U, V, W, S) = \frac{c+3}{4}\{< V, S >_M< U, W >_M - < U, S >_M< V, W >_M\} \\ + < \mathcal{T}_V W, \mathcal{T}_U S >_M - < \mathcal{T}_U W, \mathcal{T}_V S >_M$$

which gives (22). If we take $W = V$ and $S = U$ in (22), then we get (23). This completes the proof. □

From above theorem, we have the following result.

Corollary 5.2. *Let ϕ be a hemi-slant $\xi^\perp$−Riemannian submersion from a Sasakian manifold $(M^m, \varphi, \eta, \xi, <,>_M)$ onto Riemannian manifold $(N, <,>_N)$ with a hemi-slant angle θ and $m \geq 3$. If $\mathcal{D}_\perp$ is totally geodesic, then M is flat if and only if $c = -3$.*

Theorem 5.3. *Let $\phi : (M, \varphi, \eta, \xi, <,>_M) \longrightarrow (N, <,>_N)$ be a hemi-slant $\xi^\perp$−Riemannian submersion, where $(M, \varphi, \eta, \xi, <,>_M)$ is a Sasakian manifold and $(N, <,>_N)$ is a Riemannian manifold. If $\mathcal{D}_\perp$ is totally geodesic, then*

$$\widehat{\tau}_\perp = \frac{c+3}{2} q(1-2q)$$

where $\widehat{\tau}_\perp$ is the scaler curvature of fibres.

Proof. We know that the trace of scalar curvature is Ricci curvature. So, we have

$$\widehat{S}_\perp(U, V) = \sum_{i=1}^{2q} \widehat{R}(E_i, U, V, E_i)$$

where $\{E_1, ..., E_{2q}\}$ is ortonormal basis on $\Gamma(\mathcal{D}_\perp)$ and $U, V \in \Gamma(\mathcal{D}_\perp)$. Now if $\mathcal{D}_\perp$ is totally geodesic, then using (22), we obtain

$$\widehat{S}_\perp(U, V) = \sum_{i=1}^{2q} \{\frac{c+3}{4}\{< U, E_i >_M< E_i, V >_M - < E_i, E_i >_M< U, V >_M\}\}.$$

From above equation, we get

$$\widehat{S}_\perp(U, V) = \frac{c+3}{4}(1-2q) < U, V >_M . \tag{25}$$

Now, if we take $U = V = E_k, k = 1, ..., 2q$ and taking the trace of (25), then we obtain the proof. □

From the above theorem, we have the following:

Corollary 5.4. *Let ϕ be a hemi-slant $\xi^{\perp}-$Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, <,>_M)$ onto Riemannian manifold $(N, <,>_N)$ with a hemi-slant angle θ. If $\mathcal{D}_{\perp}$ is totally geodesic distribution, then $\mathcal{D}_{\perp}$ is Einstein.*

Proof. The proof follows from (25). □

Theorem 5.5. *Let $\phi : (M, \varphi, \eta, \xi, <,>_M) \longrightarrow (N, <,>_N)$ be a hemi-slant $\xi^{\perp}-$Riemannian submersion, where $(M, \varphi, \eta, \xi, <,>_M)$ is a Sasakian manifold and $(N, <,>_N)$ is a Riemannian manifold. Then we have*

$$\begin{aligned}\widehat{R}(K,L,P,W) = &\frac{c+3}{4}\{<L,P>_M<K,W>_M - <K,P>_M<L,W>_M\}\\ &+\frac{c-1}{4}\{<\varphi L,P>_M<\varphi K,W>_M\\ &-<\varphi K,P>_M<\varphi L,W>_M - 2<\varphi K,L>_M<\varphi P,W>_M\}\\ &+<\mathcal{T}_L P,\mathcal{T}_K W>_M - <\mathcal{T}_K P,\mathcal{T}_L W>_M\end{aligned} \tag{26}$$

and

$$\begin{aligned}\widehat{K}(K,L) = &\frac{c+3}{4}\{<L,K>_M<K,L>_M - <K,K>_M<L,L>_M\}\\ &-3\frac{c-1}{4}<\varphi K,L>_M + <T_L K,T_K L>_M - <\mathcal{T}_K K,\mathcal{T}_L L>_M\end{aligned} \tag{27}$$

for all $K, L, P, N \in \Gamma(\mathcal{D}_\theta)$.

Proof. For any $K, L, P, W \in \Gamma(\mathcal{D}_\theta)$, by using (21), we obtain (26) and (27) which gives the proof. □

Theorem 5.6. *Let ϕ be a hemi-slant $\xi^{\perp}$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, <,>_M)$ onto Riemannian manifold $(N, <,>_N)$ with a hemi-slant angle θ. If $\mathcal{D}_\theta$ is totally geodesic, then we have*

$$\widehat{\tau}_\theta = p\frac{(c+3)(2p-1)+3(c-1)\cos^2\theta}{2}.$$

Proof. By using (26), we have

$$\widehat{S}_\theta(K,L) = \frac{c+3}{4}(2p-1)<K,L>_M + 3\frac{c-1}{4}\cos^2\theta<K,L>_M \tag{28}$$

where $K, L \in \Gamma(\mathcal{D}_\theta)$. Taking $K = L = E_k,\ k = 1, 2, ..., 2p$ in (28), then we obtain the proof. □

Finally, from the above theorem we have the following result.

Corollary 5.7. *Let $\phi : (M, \varphi, \eta, \xi, <,>_M) \longrightarrow (N, <,>_N)$ be a hemi-slant $\xi^{\perp}-$Riemannian submersion, where $(M, \varphi, \eta, \xi, <,>_M)$ is a Sasakian manifold and $(N, <,>_N)$ is a Riemannian manifold. If $\mathcal{D}_\theta$ is totally geodesic distribution, then $\mathcal{D}_\theta$ is Einstein.*

Proof. The proof follows from (28). □

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