A Note on the Potential Function of an Arbitrary Graph $H$

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Abstract. Given a graph $H$, a graphic sequence $\pi$ is potentially $H$-graphic if there is a realization of $\pi$ containing $H$ as a subgraph. In 1991, Erdős et al. introduced the following problem: determine the minimum even integer $\alpha(H,n)$ such that each $\pi$-term graphic sequence with sum at least $\alpha(H,n)$ is potentially $H$-graphic. This problem can be viewed as a “potential” degree sequence relaxation of the Turán problems. Let $H$ be an arbitrary graph of order $k$. Ferrara et al. [Combinatorica, 36(2016)687–702] established an upper bound on $\alpha(H,n)$: if $\omega = \omega(n)$ is an increasing function that tends to infinity with $n$, then there exists an $N = N(\omega,H)$ such that $\alpha(H,n) \leq \overline{\sigma}(H)n + \omega(n)$ for any $n \geq N$, where $\overline{\sigma}(H)$ is a parameter only depending on the graph $H$. Recently, Yin [European J. Combin., 85(2020)103061] obtained a new upper bound on $\alpha(H,n)$: there exists an $M = M(k,\alpha(H))$ such that $\alpha(H,n) \leq \overline{\sigma}(H)n + k^2 - 3k + 4$ for any $n \geq M$. In this paper, we investigate the precise behavior of $\alpha(H,n)$ for arbitrary $H$ with $\overline{\sigma}(H)+\alpha(H) < \overline{\sigma}(H)$ or $V_{\alpha(H)+1}(H) \geq 2$, where $V_{\alpha(H)+1}(H) = \min\{\Delta(F) | F \text{ is an induced subgraph of } H \text{ and } |V(F)| = \alpha(H)+1\}$ and $\overline{\sigma}(H)+\alpha(H) = 2(k-\alpha(H)-1)+V_{\alpha(H)+1}(H)-1$. Moreover, we also show that $\alpha(H,n) = (k-\alpha(H)-1)(2n-k+\alpha(H)+2)$ for those $H$ so that $V_{\alpha(H)+1}(H) = 1$, $\overline{\sigma}(H)+\alpha(H) < \overline{\sigma}(H)$ for $\alpha(H)+2 < p \leq k$ and there is an $F \subset H$ with $|V(F)| = \alpha(H)+1$ and $\pi(F) = (1^2,0^{\alpha(H)-1})$.

1. Introduction

A sequence $\pi = (d_1, \ldots, d_n)$ of non-negative integers is said to be a graphic sequence if it is realizable by a simple graph $G$ on $n$ vertices. In this case, $G$ is referred to as a realization of $\pi$. The set of all sequences $\pi = (d_1, \ldots, d_n)$ of non-negative, non-increasing integers with $d_1 \leq n - 1$ is denoted by $NS_n$. The set of all graphic sequences in $NS_n$ is denoted by $GS_n$. For a sequence $\pi = (d_1, \ldots, d_n)$, we denote $\sigma(\pi) = d_1 + \cdots + d_n$ and $p(\pi) = \max\{\lceil d_i \rceil \geq 1\}$. Given a (simple) graph $H$, a graphic sequence $\pi$ is said to be potentially (respectively, forcibly) $H$-graphic if there exists a realization of $\pi$ containing $H$ as a subgraph (respectively, each realization of $\pi$ contains $H$ as a subgraph).

One of the classical extremal problems is to determine the minimum integer $m$ such that every graph $G$ on $n$ vertices with edge number $e(G) \geq m$ contains $H$ as a subgraph. This $m$ is denoted by $e(H,n)$, and is called the Turán number of $H$. In terms of graphic sequences, the number $2e(H,n)$ is the minimum even...
integer such that each sequence $\pi \in GS_n$ with $\sigma(\pi) \geq 2ex(H, n)$ is forcibly $H$-graphic. In 1991, Erdős et al. [2] introduced the following problem: determine the minimum even integer $\sigma(H, n)$ such that each sequence $\pi \in GS_n$ with $\sigma(\pi) \geq \sigma(H, n)$ is potentially $H$-graphic. We will refer to $\sigma(H, n)$ as the potential number or potential function of $H$. As $\sigma(\pi)$ is twice the number of edges in any realization of $\pi$, this problem can be viewed as a potential degree sequence relaxation of the Turán problems.

In [2], Erdős et al. conjectured that $\sigma(K_r, n) = (r - 2)(2n + r + 1) + 2$, where $K_r$ is the complete graph on $r$ vertices. The cases $r = 3, 4$ and 5 were proved separately (see respectively [2], and [7,11], and [12]), and Li et al. [13] proved the conjecture true for $r \geq 6$ and $n \geq \binom{r - 1}{2} + 3$. In addition to these results for complete graphs, the value of $\sigma(H, n)$ has been determined exactly for a number of specific graph classes (c.f. [3,4,6,7,10,15,16]). For an arbitrarily chosen $H$, Ferrara and Schmitt [6] gave a construction that yields the best known lower bound on $\sigma(H, n)$.

We assume that $H$ is an arbitrary graph of order $k$ and $\omega(H)$ is the independent number of $H$. Let $\Delta(F)$ denote the maximum degree of a graph $F$, and let $F < H$ denote that $F$ is an induced subgraph of $H$. For each $p \in [\omega(H) + 1, \ldots, k]$, let

$$V_p(H) = \min|\Delta(F)|F < H \text{ and } |V(F)| = p|.$$ 

Clearly, $1 \leq V_\omega(H) + 1 \leq \cdots \leq V_k(H) \leq k - 1$. Let

$$\pi_p(H, n) = ((n - 1)^{k-p}, (k - p + \Delta_p(H) - 1)^{n-k+p})$$

if $(n - k + p)(\Delta_p(H) - 1)$ is even, and

$$\pi_p(H, n) = ((n - 1)^{k-p}, (k - p + \Delta_p(H) - 1)^{n-k+p-1}, k - p + \Delta_p(H) - 2)$$

if $(n - k + p)(\Delta_p(H) - 1)$ is odd, where the symbol $x^y$ in a sequence stands for $y$ consecutive terms, each equal to $x$. Then,

$$\sigma(\pi_p(H, n)) = (2(k - p) + \Delta_p(H) - 1)n - (k - p)(k - p + \Delta_p(H))$$

if $(n - k + p)(\Delta_p(H) - 1)$ is even, and

$$\sigma(\pi_p(H, n)) = (2(k - p) + \Delta_p(H) - 1)n - (k - p)(k - p + \Delta_p(H)) - 1$$

if $(n - k + p)(\Delta_p(H) - 1)$ is odd. Ferrara et al. [5] showed that $\pi_p(H, n)$ is graphic and is not potentially $H$-graphic for all $p \in [\omega(H) + 1, \ldots, k]$, thus establishing a lower bound on $\sigma(H, n)$.

**Proposition 1.1** $\sigma(H, n) \geq \sigma(\pi_p(H, n)) + 2$ for $p \in [\omega(H) + 1, \ldots, k]$.

Let

$$\overline{\sigma}_p(H) = 2(k - p) + \Delta_p(H) - 1,$$

and let

$$\overline{\sigma}(H) = \max\{\overline{\sigma}_p(H) | p = \omega(H) + 1, \ldots, k\}.$$ 

Ferrara et al. [5] established an upper bound on $\sigma(H, n)$ and determined $\sigma(H, n)$ asymptotically.

**Theorem 1.1** [5] Let $H$ be a graph, and let $\omega = \omega(n)$ be an increasing function that tends to infinity with $n$. There exists an $N = N(\omega, H)$ such that for any $n \geq N$,

$$\sigma(H, n) \leq \overline{\sigma}(H)n + \omega(n).$$

**Theorem 1.2** [5] Let $H$ be a graph of order $k$ and let $n$ be a positive integer. Then

$$\sigma(H, n) = \overline{\sigma}(H)n + \omega(n).$$

Recently, Yin [14] established a new upper bound on $\sigma(H, n)$ as follows.

**Theorem 1.3** [14] Let $H$ be a graph of order $k$. There exists an $M = M(k, \omega(H))$ such that for any $n \geq M$,

$$\sigma(H, n) \leq \overline{\sigma}(H)n + k^2 - 3k + 4.$$
The focus of this paper is the precise behavior of the potential number for arbitrary \( H \). As such, for \( p \in \{\alpha(H) + 1, \ldots, k\} \) and a graph \( F \) with \( |V(F)| = p \), we denote \( \pi(F) = (d_1, \ldots, d_p) \) to be the degree sequence of \( F \) with \( d_1 \geq \cdots \geq d_p \). We say that \( (d_1, \ldots, d_p) \geq (d'_1, \ldots, d'_p) \) if \( d_i \geq d'_i \) for \( 1 \leq i \leq p \). We now choose \( \rho_p(H) = (d_1, \ldots, d_p) \in NS_p \) with \( d_1 \leq p - 2 \) and \( d_p \geq \nu_p(H) - 1 \) so that \( \rho_p(H) \neq \pi(F) \) for each \( F < H \) with \( |V(F)| = p \) and \( \sigma(\rho_p(H)) \) is maximal, and let

\[
\pi_p(H, n) = ((n - 1)^{2-p}, k - p + d_1, \ldots, k - p + d_p, (k - p + \nu_p(H) - 1)^{n-k})
\]

if \( \sum_{i=1}^{p} d_i + (\nu_p(H) - 1)(n-k) \) is even, and

\[
\pi_p(H, n) = ((n - 1)^{2-p}, k - p + d_1, \ldots, k - p + d_p, (k - p + \nu_p(H) - 1)^{n-k-1}, k - p + \nu_p(H) - 2)
\]

if \( \sum_{i=1}^{p} d_i + (\nu_p(H) - 1)(n-k) \) is odd.

Clearly, \( \sigma_p(H) \) is also the leading coefficient of \( \sigma(\pi_p(H, n)) \), and \( \nu_p(H) - 1 \leq p - 2 \) and \( (\nu_p(H) - 1)^p \neq \pi_n(H) \) for each \( F < H \) with \( |V(F)| = p \). Thus \( \sigma(\pi_p(H, n)) + 2 \geq \sigma(\sigma_p(H, n)) + 2 \) for all \( p \in \{\alpha(H) + 1, \ldots, k\} \).

For \( \nu_p(H) \geq 2 \), applying Erdős-Gallai characterization, we can see that \( \pi_p(H, n) \) is graphic for \( n \) sufficiently large. Every realization \( G \) of \( \pi_p(H, n) \) is a complete graph on \( k - p \) vertices joined to an \( (n - k + p) \)-vertex graph \( G_p \) with degree sequence \( (d_1, \ldots, d_p, (\nu_p(H) - 1)^{n-k}) \) or \( (d_1, \ldots, d_p, (\nu_p(H) - 1)^{n-k-1}, \nu_p(H) - 2) \). Any \( k \)-vertex subgraph of \( G \) contains at least \( p \) vertices in \( G_p \). If \( G \) contains \( H \) as a subgraph, then \( G_p \) contains an \( F < H \) with \( |V(F)| = p \) as a subgraph. This implies \( \rho_p(H) \geq \pi_n(H) \), a contradiction. Thus \( H \) is not a subgraph of \( G \). In other words, \( \pi_p(H, n) \) is not potentially \( H \)-graphic. This also establishes a lower bound on \( \sigma(H, n) \) as follows.

**Proposition 1.2** \( \sigma(H, n) \geq \sigma(\pi_p(H, n)) + 2 \) for \( \alpha(H) + 1 \leq p \leq k \) and \( \nu_p(H) \geq 2 \).

For \( \alpha(H) + 1 \leq i \leq k \), we can see that \( \nu_i(H) = 1 \) implies \( \nu_{\alpha(H)+1}(H) = 1 \) and \( \sigma(H) \leq \sigma(\alpha(H)+1)(H) \). Therefore, if \( \sigma_{\alpha(H)+1}(H) < \sigma(H) \) or \( \nu_{\alpha(H)+1}(H) \geq 2 \), then

\[
\max(\sigma(\pi_p(H, n)), 2(\alpha(H) + 1 \leq p \leq k \text{ and } \nu_p(H) \geq 2)) \geq \sigma(\pi_p(H, n)) + 2
\]

for \( \alpha(H) + 1 \leq i \leq k \) and \( n \) sufficiently large. In this paper, we determine the precise value of \( \sigma(H, n) \) if \( \sigma_{\alpha(H)+1}(H) < \sigma(H) \) or \( \nu_{\alpha(H)+1}(H) \geq 2 \).

**Theorem 1.4** Let \( H \) be a graph of order \( k \), with \( \pi_p(H, n) \) as given in (1) or (2) for each \( p \in \{\alpha(H) + 1, \ldots, k\} \), and let \( n \) be a sufficiently large integer. If \( \nu_{\alpha(H)+1}(H) < \sigma(H) \) or \( \nu_{\alpha(H)+1}(H) \geq 2 \), then

\[
\sigma(H, n) = \max(\sigma(\pi_p(H, n)), 2(\alpha(H) + 1 \leq p \leq k \text{ and } \nu_p(H) \geq 2)).
\]

Moreover, we also prove the following Theorem 1.5.

**Theorem 1.5** Let \( H \) be a graph of order \( k \) with \( \nu_{\alpha(H)+1}(H) = 1 \), \( \nu_{\alpha(H)+1}(H) = \sigma(H) \) and \( \nu_p(H) < \sigma(H) \) for \( \alpha(H) + 2 \leq p \leq k \), and let \( n \) be a sufficiently large integer. If there is an \( F < H \) with \( |V(F)| = \alpha(H) + 1 \) so that \( \pi(F) = (1^2, \ldots, 1^{\alpha(H)-1}) \), then

\[
\sigma(H, n) = (k - \alpha(H) - 1)(2n - k + \alpha(H)) + 2.
\]

We can see that Theorem 1.5 covers a number of specific graph families, including complete graphs, disjoint unions of cliques, matchings, odd cycles, (generalized) friendship graphs, intersecting cliques, etc. We will adopt the method of the reference [14] to prove Theorem 1.4–1.5.
2. Proof of Theorem 1.4

The following known results will be useful. For $\pi = (d_1, \ldots, d_n) \in NS_n$, let $d'_1 \geq \cdots \geq d'_{n-1}$ be the rearrangement in non-increasing order of $d_2 - 1, \ldots, d_{d+1} - 1, d_{d+2}, \ldots, d_n$. We say that $\pi' = (d'_1, \ldots, d'_{n-1})$ is the residual sequence of $\pi$.

**Theorem 2.1** [8,9] Let $\pi = (d_1, \ldots, d_n) \in NS_n$. Then $\pi$ is graphic if and only if $\pi'$ is graphic.

**Theorem 2.2** [1] Let $\pi = (d_1, \ldots, d_n) \in NS_n$ with even $\sigma(\pi)$. Then $\pi$ is graphic if and only if $\sum_{i=1}^{h} d_i \leq h(h-1) + \sum_{j=h+1}^{n} \min\{h, d_j\}$ for each $h$ with $1 \leq h \leq n-1$.

**Theorem 2.3** [16] Let $\pi = (d_1, \ldots, d_n) \in NS_n$, $x = d_1$ and $\sigma(\pi)$ be even. If there is an integer $n_1$ with $1 \leq n_1 \leq n$ such that $d_{n_1} \geq y \geq 1$ and $n_1 \geq \frac{\lfloor \frac{(x+y+1)^2}{4} \rfloor}{2}$, then $\pi$ is graphic.

**Theorem 2.4** [17] Let $n \geq r$ and $\pi = (d_1, \ldots, d_n) \in GS_n$ with $d_r \geq r-1$. If $d_i \geq 2r-2-i$ for $i = 1, \ldots, r-2$, then $\pi$ is potentially $K_r$-graphic.

In this section, we always assume that $H$ is a graph of order $k$ with $\bar{\sigma}_{\alpha+1}(H) < \bar{\sigma}(H)$ or $V_{\alpha+1}(H) \geq 2$, and $n$ is a sufficiently large integer relative to $k$ and $\alpha(H)$. We need some lemmas. For convenience, we denote $\Sigma = \max\{\sigma(\pi_n^*(H, n)) + 2|\alpha(H)| + 1 \leq p \leq k \text{ and } V_p(H) \geq 2\}$ and $\alpha = \alpha(H)$.

**Lemma 2.1** Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq \Sigma$. Then
- (a) If $d_k \leq k - \alpha - 1$, by Theorem 2.2, then
  $$\sigma(\pi) = \sum_{i=1}^{k-1} d_i + \sum_{i=k}^{n} d_i \leq (k-1)(k-2) + \sum_{i=k}^{n} \min\{k-1, d_i\} + \sum_{i=k}^{n} d_i = (k-1)(k-2) + 2\sum_{i=k}^{n} d_i \leq (k-1)(k-2) + 2(n-k+1)(k-\alpha-1) = 2(k-\alpha-1)n + (k-1)(2n-k).$$

  However, if $\bar{\sigma}_{\alpha+1}(H) < \bar{\sigma}(H)$ or $V_{\alpha+1}(H) \geq 2$, then $\bar{\sigma}(H) > 2(k-\alpha-1)$. This implies that $\sigma(\pi) \geq \Sigma > 2(k-\alpha-1)n + (k-1)(2n-k)$, a contradiction.
- (b) If $d_k \leq h + V_{k-h}(H) - 2$, then
  $$\sigma(\pi) \leq (n-1)h + (k-2)(k-h-1) + (h + V_{k-h}(H) - 2)(n-k+1) \leq \sigma(\pi_{k-h}(H, n)) + 2 \leq \sigma(\pi_{k-h}(H, n)) + 2,$$
  a contradiction. Hence $d_k \geq h + V_{k-h}(H) - 1$. If $d_k \geq h + V_{k-h}(H)$, then there is an $F < H$ with $|V(F)| = k-h$ and $\Delta(F) = V_{k-h}(H)$ so that $(d_{h+1} - h, \ldots, d_k - h) \geq \pi(F)$. Assume $d_k = h + V_{k-h}(H) - 1$. By $\sigma(\pi) \geq \sigma(\pi_{k-h}(H, n)) + 2$, we have
  $$\sum_{i=1}^{k-h} (d_{h+i} - h) = \sigma(\pi) - \sum_{i=1}^{h} d_i - h(k-h) - \sum_{i=h+1}^{n} d_i \geq \sigma(\pi_{k-h}(H, n)) + 2 - (n-1)h - h(k-h) - (h + V_{k-h}(H) - 1)(n-k) \geq \sigma(\rho_{k-h}(H)) + 1.$$
Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq \Sigma$. By Lemma 2.1(a), $d_k \geq k - \alpha$. Denote $\pi_0 = (d_1^{(0)}, \ldots, d_n^{(0)})$, where $d_i^{(0)} = d_i$ for $1 \leq i \leq n$. We construct $\pi_1, \ldots, \pi_k$ depending on two cases.

**Case 1.** $d_{k-\alpha} \geq k - 1$.

For $i = 1, \ldots, k$ in turn, we construct $\pi_i = (d_1^{(i)}, \ldots, d_k^{(i)}, \ldots, d_n^{(i)})$, by deleting $d_i^{(i-1)}$ from $\pi_{i-1} = (d_1^{(i-1)}, \ldots, d_k^{(i-1)}, \ldots, d_n^{(i-1)})$, reducing the first $d_i^{(i-1)}$ nonzero remaining terms of $\pi_{i-1}$ by one, and then reordering the last $n - k$ terms to be non-increasing.

**Case 2.** There is an $h$ with $0 \leq h \leq k - \alpha - 1$ so that $d_h \geq k - 1$ and $d_{h+1} \leq k - 2$.

By Lemma 2.1(b), there is an $F < H$ with $|V(F)| = k - h$ so that $(d_{h+1}, h, \ldots, d_h - h) \geq \pi(F) = (d_1, \ldots, d_n)$. Let $d_{h+j} - h = d_j + f_j$ for $j = 1, \ldots, k - h$. In this case, we first construct $\pi_i$, $1 \leq i \leq h$ as above, and then we construct $\pi_i$, $h + 1 \leq i \leq k$ from $\pi_{i-1}$ by deleting $d_i^{(i-1)}$, reducing the first $f_i$, nonzero terms, starting with $d_i^{(i-1)}$ by one, and then reordering the last $n - k$ terms to be non-increasing.

Thus by Lemmas 2.2 and 2.3 of [14], the following Lemmas 2.2 and 2.3 are obvious and immediately.

**Lemma 2.2** [14] Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq \Sigma$, and let $d_{k-\alpha} \geq k - 1$. Then

(i) If $\pi_{k-\alpha} = (d_1^{(k-\alpha)}, \ldots, d_k^{(k-\alpha)}, \ldots, d_n^{(k-\alpha)})$, satisfies $d_k^{(k-\alpha)} \geq d_{k+1}^{(k-\alpha)}$, then $\pi$ is potentially $H$-graphic;

(ii) If $d_k^{(k-\alpha)} < d_{k+1}^{(k-\alpha)}$ and $\pi_k$ is graphic, then $\pi$ is potentially $H$-graphic.

**Lemma 2.3** [14] Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq \Sigma$. Assume that there is an $h$ with $0 \leq h \leq k - \alpha - 1$ so that $d_h \geq k - 1$ and $d_{h+1} \leq k - 2$. If $\sigma_h$ is graphic, then $\pi$ is potentially $H$-graphic.

**Lemma 2.4** Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq \Sigma$. If $d_{k-\alpha} \geq k - 1$, then $\pi$ is potentially $H$-graphic.

**Proof.** By Theorem 2.4, we may assume $d_k \leq 2k - 4$. By Lemma 2.2, we further assume $d_k^{(k-\alpha)} < d_{k+1}^{(k-\alpha)}$, and only need to check that $\pi_k$ is graphic. Let $d_i^{(l)} < d_{k+1}^{(l)}$ so that $\ell$ is minimal. Then $1 \leq \ell \leq k - \alpha - 1$ and $d_k^{(\ell)} \geq d_{k+1}^{(\ell)}$. Moreover, $d_j^{(\ell)} \geq d_{k+1}^{(\ell)}$ for $1 \leq j \leq \ell - 1$. This implies that $\pi_{\ell+1}$ is the residual sequence of $\pi$, for $0 \leq j \leq \ell - 2$, where $\pi_0 = \pi$. By Theorem 2.1, $\pi_i$ is graphic for $1 \leq j \leq \ell - 1$. Moreover, by $\overline{\sigma}_{\ell+1}(H) < \overline{\sigma}(H)$ or $\overline{V}_{\ell+1}(H) \geq 2, \pi_{\ell+1} = (d_1^{(\ell)}, \ldots, d_n^{(\ell)})$, satisfies that $d_i^{(\ell)} \geq \cdots \geq d_n^{(\ell)}$, $d_j^{(\ell)} = d_j - (\ell - 1)$ for $j = \ell, \ldots, k$.

$$
\sigma(\pi_{\ell+1}) = \sigma(\pi) - 2d_1 - 2(d_2 - 1) - \cdots - 2(d_{\ell-1} - \ell + 2) \\
\geq \sigma(\pi) - 2(n - 1) - 2(n - 2) - \cdots - 2(n - \ell + 1) \\
= \sigma(\pi) - 2(\ell - 1)n + \ell(\ell - 1) \\
\geq 2(k - \ell - \alpha) \\
\geq 2n,
$$

and

$$
n - \ell - 1 \geq d_i^{(\ell)} \geq \cdots \geq d_k^{(\ell)} = \cdots \geq d_{k+1}^{(\ell)} \geq d_{k+2}^{(\ell)} \geq \cdots \geq d_n^{(\ell)}.
$$

The rest proof is the same as the proof of Lemma 2.4 of [14], we omit it here. □

**Lemma 2.5** Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq \Sigma$. If there is an $h$ with $0 \leq h \leq k - \alpha - 1$ so that $d_h \geq k - 1$ and $d_{h+1} \leq k - 2$, then $\pi$ is potentially $H$-graphic.

**Proof.** By Lemma 2.3, it is enough to check that $\pi_k$ is graphic. If $\overline{V}_{\ell+1}(H) \geq 2$, then $\overline{V}_{k-\alpha}(H) \geq 2$, and hence

$$
\sigma(\pi_k) = \sigma(\pi) - 2d_1 - 2(d_2 - 1) - \cdots - 2(d_h - h + 1) \\
\geq \sigma(\pi) - 2(n - 1) - 2(n - 2) - \cdots - 2(n - h) \\
= \sigma(\pi) - 2hn + h(h + 1) \\
\geq \overline{\sigma}(H)n - 2hn + \frac{n}{2} \\
\geq \overline{\sigma}_{k-\alpha}(H)n - 2hn + \frac{n}{2} \\
= (\overline{V}_{k-\alpha}(H) - 1)n - \frac{n}{2} \\
\geq \frac{n}{2}.
$$
If \( \bar{\sigma}_{n+1}(H) < \bar{\sigma}(H) \), then similarly

\[
\sigma(\pi_k) \geq \begin{array}{l}
\sigma(\pi) - 2(n - 1) - 2(n - 2) - \cdots - 2(n - h) \\
= \sigma(\pi) - 2hn + h(h + 1) \\
> \bar{\sigma}(H)n - 2hn - \frac{h}{2} \\
\geq \bar{\sigma}_{n+1}(H)n - 2hn - \frac{n}{2} \\
\geq 2(k - \alpha - 1)h + \frac{n}{2} \\
\geq \frac{n}{2}.
\end{array}
\]

Thus by \( (p(\pi_k) - h)(k - 2) \geq \sigma(\pi_k) \), we can see that \( p(\pi_k) - h \) is sufficiently large. This implies that \( p(\pi_k) - k \) is also sufficiently large as \( f_j \leq k - 2 \) for \( 1 \leq j \leq k - h \). Therefore, \( \pi_k \) is graphic by \( d_{k+1}^{(h)} \leq k - 2 \) and Theorem 2.3.

**Proof of Theorem 1.4.** Let \( \pi = (d_1, \ldots, d_n) \in GS_n \) with \( \sigma(\pi) \geq \Sigma \). It is enough to show that \( \pi \) is potentially \( H \)-graphic. It is trivial for \( k = 1, 2 \). If \( \alpha = k \), then \( H = K_k \), and so \( \pi \) is clearly potentially \( H \)-graphic. Assume \( k \geq 3 \) and \( \alpha \leq k - 1 \). By Lemma 2.4, then \( \pi \) is potentially \( H \)-graphic. If there is an \( h \) with \( 0 \leq h \leq k - \alpha - 1 \) so that \( d_h \geq k - 1 \) and \( d_{h+1} \leq k - 2 \), by Lemma 2.5, then \( \pi \) is potentially \( H \)-graphic. \( \Box \)

3. **Proof of Theorem 1.5.**

In this section, we always assume that \( H \) is a graph of order \( k \) with \( V_{\alpha(n)}(H) = 1 \), \( \bar{\sigma}_{\alpha(n)}(H) = \bar{\sigma}(H) \) and \( \bar{\sigma}(H) < \bar{\sigma}(H) \) for \( \alpha(H) + 2 < p \leq k, n \) is a sufficiently large integer relative to \( k \) and \( \alpha(H) \) and there is an \( F \) with \( |V(F)| = \alpha(H) + 1 \) so that \( \pi(F) = (1^2, 0^{n-1}) \). Clearly, \( \bar{\sigma}_{\alpha(n)}(H) = 2(k - \alpha - 1) \). We also need some lemmas. For convenience, we denote \( \alpha = \alpha(H) \).

**Lemma 3.1** Let \( \pi = (d_1, \ldots, d_n) \in GS_n \) with \( \sigma(\pi) \geq (k - \alpha - 1)(2n - k + \alpha) + 2 \). Then

(a) \( d_k \geq k - \alpha - 1 \);

(b) \( d_{k-\alpha+1} \geq k - \alpha \);

(c) If there is an \( h \) with \( 0 \leq h \leq k - \alpha - 2 \) so that \( d_h \geq k - 1 \) and \( d_{h+1} \leq k - 2 \), then there is an \( F < H \) with \( |V(F)| = k - h \) and \( \Delta(F) = V_{k-h}(H) \) so that \( (d_{k-h+1}, \ldots, d_k - h) \geq \pi(F) \).

**Proof.** (a) If \( d_k \leq k - \alpha - 2 \), by Theorem 2.2, then

\[
\sigma(\pi) = \sum_{i=1}^{k-1} d_i + \sum_{i=k}^{n} d_i \\
\leq (k - 1)(k - 2) + \sum_{i=k}^{n} \min(k - 1, d_i) + \sum_{i=k}^{n} d_i \\
= (k - 1)(k - 2) + 2 \sum_{i=k}^{n} d_i \\
\leq (k - 1)(k - 2) + 2(n - k + 1)(k - \alpha - 2) \\
= 2(k - \alpha - 2)n + (k - 1)(2\alpha - k + 2) \\
< \sigma(\pi),
\]

a contradiction.

(b) If \( d_{k-\alpha+1} \leq k - \alpha - 1 \), by Theorem 2.2, then

\[
\sigma(\pi) = \sum_{i=1}^{k-\alpha} d_i + \sum_{i=k-\alpha+1}^{n} d_i \\
\leq (k - \alpha)(k - \alpha - 1) + \sum_{i=k-\alpha+1}^{n} \min(k - \alpha, d_i) + \sum_{i=k-\alpha+1}^{n} d_i \\
= (k - \alpha)(k - \alpha - 1) + 2 \sum_{i=k-\alpha+1}^{n} d_i \\
\leq (k - \alpha)(k - \alpha - 1) + 2(n - k + \alpha)(k - \alpha - 1) \\
= (k - \alpha - 1)(2n - k + \alpha) \\
< \sigma(\pi),
\]
a contradiction.

(c) If \( d_k \leq h + V_{k-h}(H) - 1 \), by \( \overline{\sigma}_{k+1}(H) = \overline{\sigma}(H) > \overline{\sigma}_{k-h}(H) \), then

\[
\sigma(\pi) \leq (n - 1)h + (k - 2)(k - h - 1) + (h + V_{k-h}(H) - 1)(n - k + 1)
\]

\[
= (2h + V_{k-h}(H) - 1)n - h + (k - 2)(k - h - 1) - (h + V_{k-h}(H) - 1)(k - 1)
\]

\[
= \overline{\sigma}_{k-h}(H)n - h + (k - 2)(k - h - 1) - (h + V_{k-h}(H) - 1)(k - 1)
\]

\[
< \sigma(\overline{\pi}_{k+1}(H, n))
\]

\[
= (k - \alpha - 1)(2n - k + \alpha),
\]

a contradiction. Hence \( d_k \geq h + V_{k-h}(H) \). This implies that there is an \( F < H \) with \( |V(F)| = h - k \) and \( \Delta(F) = V_{k-h}(H) \) so that \( (d_{h+1} - h, \ldots, d_h - h) \geq \pi(F) \). □

Let \( \pi = (d_1, \ldots, d_n) \in G_{\pi n} \) with \( \sigma(\pi) \geq (k - \alpha - 1)(2n - k + \alpha) + 2 \). By Lemma 3.1(a), \( d_k \geq k - \alpha - 1 \). Denote \( \pi_0 = (d_1^{(0)}, \ldots, d_n^{(0)}) \), where \( d_i^{(0)} = d_i \) for \( 1 \leq i \leq n \). We construct \( \pi_1, \ldots, \pi_\ell \) depending on two cases.

Case 1. \( d_{k+1} - k \geq k - 1 \).

For \( i = 1, \ldots, k \) in turn, we construct \( \pi_i = (d_1^{(i)}, \ldots, d_k^{(i)}, d_{k+1}^{(i)}, \ldots, d_n^{(i)}) \), by deleting \( d_{j}^{(i-1)} \) from \( \pi_{i-1} = (d_1^{(i-1)}, \ldots, d_k^{(i-1)}, d_{k+1}^{(i-1)}, \ldots, d_n^{(i-1)}) \), reducing the first \( d_{j}^{(i-1)} \) nonzero remaining terms of \( \pi_{i-1} \) by one, and then reordering the last \( n - k \) terms to be non-increasing.

Case 2. There is an \( h \) with \( 0 \leq h < k - \alpha - 2 \) so that \( d_h \geq k - 1 \) and \( d_{h+1} \leq k - 2 \).

By Lemma 3.1(c), there is an \( F < H \) with \( |V(F)| = k - h \) and \( \Delta(F) = V_{k-h}(H) \) so that \( (d_{h+1} - h, \ldots, d_h - h) \geq \pi(F) = (d_1, \ldots, d_h) \). Let \( d_{h+1} - h = f_1 + \ell \) for \( j = 1, \ldots, k - h \). In this case, we first construct \( \pi_i, 1 \leq i < \ell \) as above, and then we construct \( \pi_\ell, h + 1 \leq i < k \) from \( \pi_{i-1} \) by deleting \( d_{j}^{(i-1)} \), reducing the first \( f_\ell \) nonzero terms, starting with \( d_{k+1}^{(i-1)} \) by one, and then reordering the last \( n - k \) terms to be non-increasing.

Thus by Lemmas 2.2 and 2.3 of [14], and Lemma 3.1(b), the following Lemmas 3.2 and 3.3 are also obvious and immediately.

**Lemma 3.2** [14] Let \( \pi = (d_1, \ldots, d_n) \in G_{\pi n} \) with \( \sigma(\pi) \geq (k - \alpha - 1)(2n - k + \alpha) + 2 \), and let \( d_{k+1} - k \geq k - 1 \).

Then

(i) If \( \pi_{k+1} = (d_1^{(k-1)}, \ldots, d_k^{(k-1)}, d_{k+1}^{(k-1)}, \ldots, d_n^{(k-1)}) \) satisfies \( d_k^{(k-1)} \geq d_{k+1}^{(k-1)} \), then \( \pi \) is potentially \( H \)-graphic;

(ii) If \( d_k^{(k-1)} < d_{k+1}^{(k-1)} \) and \( \pi_k \) is graphic, then \( \pi \) is potentially \( H \)-graphic.

**Lemma 3.3** [14] Let \( \pi = (d_1, \ldots, d_n) \in G_{\pi n} \) with \( \sigma(\pi) \geq (k - \alpha - 1)(2n - k + \alpha) + 2 \). Assume that there is an \( h \) with \( 0 \leq h < k - \alpha - 2 \) so that \( d_h \geq k - 1 \) and \( d_{h+1} \leq k - 2 \). If \( \pi_k \) is graphic, then \( \pi \) is potentially \( H \)-graphic.

**Lemma 3.4** Let \( \pi = (d_1, \ldots, d_n) \in G_{\pi n} \) with \( \sigma(\pi) \geq (k - \alpha - 1)(2n - k + \alpha) + 2 \). If \( d_{k+1} - k \geq k - 1 \), then \( \pi \) is potentially \( H \)-graphic.

**Proof.** By Theorem 2.4, we may assume \( d_k \leq 2k - 4 \). By Lemma 3.2, we further assume \( d_k^{(k-1)} < d_{k+1}^{(k-1)} \), and only need to check that \( \pi_k \) is graphic. Let \( d_k^{(\ell)} < d_{k+1}^{(\ell)} \) so that \( \ell \) is minimal. Then \( 1 \leq \ell \leq k - \alpha - 1 \) and \( d_k^{(\ell)} \geq d_{k+1}^{(\ell)} \). Moreover, \( d_k^{(j)} \geq d_{k+1}^{(j)} \) for \( 1 \leq j \leq \ell - 1 \). This implies that \( \pi_{j+1} \) is the residual sequence of \( \pi_j \) for \( 0 \leq j \leq \ell - 2 \), where \( \pi_0 = \pi \). By Theorem 2.1, \( \pi_j \) is graphic for \( 1 \leq j \leq \ell - 1 \). Moreover, \( \pi_{\ell+1} = (d_1^{(\ell-1)}, \ldots, d_n^{(\ell-1)}) \) satisfies \( d_{\ell}^{(\ell-1)} \geq \cdots \geq d_{\ell}^{(\ell-1)} \), \( d_j^{(\ell-1)} = d_j - (\ell - 1) \) for \( j = \ell, \ldots, k \),

\[
\sigma(\pi_{\ell+1}) \geq \sigma(\pi) - 2(n - 1) - 2(n - 2) - \cdots - 2(n - \ell + 1)
\]

\[
= \sigma(\pi) - 2\ell - (\ell - 1)n + \ell(\ell - 1)
\]

\[
\geq (k - \alpha - 1)(2n - k + \alpha) + 2 - 2(\ell - 1)n + \ell(\ell - 1)
\]

\[
> n
\]

and

\[
\ell \geq 1 \geq d_{\ell}^{(\ell-1)} \geq \cdots \geq d_k^{(\ell-1)} = \cdots = d_{\ell}^{(\ell-1)} \geq d_{\ell}^{(\ell-1)}(d_{\ell}^{(\ell-1)} + d_{\ell+1}^{(\ell-1)} + \cdots + d_n^{(\ell-1)}) \geq \cdots \geq d_n^{(\ell-1)}.
\]

The rest proof is the same as the proof of Lemma 2.4 of [14], we omit it here. □

**Lemma 3.5** Let \( \pi = (d_1, \ldots, d_n) \in G_{\pi n} \) with \( \sigma(\pi) \geq (k - \alpha - 1)(2n - k + \alpha) + 2 \). If there is an \( h \) with \( 0 \leq h < k - \alpha - 2 \) so that \( d_h \geq k - 1 \) and \( d_{h+1} \leq k - 2 \), then \( \pi \) is potentially \( H \)-graphic.
Proof. By Lemma 3.3, it is enough to check that \( \pi_k \) is graphic. Clearly,

\[
\sigma(\pi_k) \geq \sigma(\pi) - 2(n - 1) - 2(n - 2) - \cdots - 2(n - h) \\
= \sigma(\pi) - 2hn + h(h + 1) \\
\geq (k - \alpha - 1)(2n - k + \alpha) + 2 - 2hn + h(h + 1) \\
\geq n.
\]

Thus by \((p(\pi_k) - h)(k - 2) \geq \sigma(\pi_k))\), we can see that \( p(\pi_k) - h \) is sufficiently large. This implies that \( p(\pi_k) - k \) is also sufficiently large as \( f_j \leq k - 2 \) for \( 1 \leq j \leq k - h \). Therefore, \( \pi_k \) is graphic by \( d^{(k)}_{k+1} \leq k - 2 \) and Theorem 2.3. □

Proof of Theorem 1.5. Let \( \pi = (d_1, \ldots, d_n) \in GS_\alpha \) with \( \sigma(\pi) \geq (k - \alpha - 1)(2n - k + \alpha) + 2 \). It is enough to show that \( \pi \) is potentially \( H \)-graphic. It is trivial for \( k = 1, 2 \). If \( \alpha = k \), then \( H = K_\alpha \), and so \( \pi \) is clearly potentially \( H \)-graphic. Assume \( k \geq 3 \) and \( \alpha \leq k - 1 \). If \( d_{k-\alpha} \geq k - 1 \), by Lemma 3.4, then \( \pi \) is potentially \( H \)-graphic. If there is an \( h \) with \( 0 \leq h \leq k - \alpha - 2 \) so that \( d_h \geq k - 1 \) and \( d_{h+1} \leq k - 2 \), by Lemma 3.5, then \( \pi \) is potentially \( H \)-graphic. □

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