Gray’s Decomposition on Doubly Warped Product Manifolds and Applications

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Abstract. A. Gray presented an interesting \(O(n)\) invariant decomposition of the covariant derivative of the Ricci tensor. Manifolds whose Ricci tensor satisfies the defining property of each orthogonal class are called Einstein-like manifolds. In the present paper, we answered the following question: Under what condition(s), does a factor manifold \(M_i, i = 1, 2\) of a doubly warped product manifold \(M = f_2 M_1 \times f_1 M_2\) lie in the same Einstein-like class of \(M\)? By imposing sufficient and necessary conditions on the warping functions, an inheritance property of each class is proved. As an application, Einstein-like doubly warped product space-times of type \(A, B\) or \(P\) are considered.

1. An introduction

Alfred Gray in \[22\] presented \(O(n)\) invariant orthogonal irreducible decomposition of the space \(W\) of all \((0, 3)\) tensors satisfying only the identities of the gradient of the Ricci tensor \(\nabla_i R_{ij}\). The space \(W\) is decomposed into three orthogonal irreducible subspaces, that is, \(W = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{I}\). This decomposition produces seven classes of Einstein-like manifolds, that is, manifolds whose Ricci tensor satisfies the defining identity of each subspace. They are the trivial class \(P\), the classes \(\mathcal{A}, \mathcal{B}, \mathcal{I}\) and three composite classes \(\mathcal{I} \oplus \mathcal{A}, \mathcal{I} \oplus \mathcal{B}\) and \(\mathcal{A} \oplus \mathcal{B}\).

In class \(P\), the Ricci tensor is parallel i.e. \(\nabla_i R_{ij} = 0\) whereas class \(\mathcal{A}\) contains manifolds whose Ricci tensor is Killing. The Ricci tensor of manifolds in class \(\mathcal{B}\) is a Codazzi tensor i.e. \(\nabla_i R_{ij} = \nabla_k R_{ik}\). The traceless part of the Ricci tensor vanishes in class \(\mathcal{I}\) i.e. class \(\mathcal{I}\) contains Sinyukov manifolds\[26\]. The tensor

\[ L_{ij} = R_{ij} - \frac{2R}{n + 2} g_{ij} \]

is Killing in class \(\mathcal{I} \oplus \mathcal{A}\) whereas the tensor

\[ H_{ij} = R_{ij} - \frac{R}{2(n - 1)} g_{ij} \]

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is a Codazzi tensor in class $I \oplus B$. The class $A \oplus B$ is identified by having constant scalar curvature. The same decomposition is discussed extensively in [3 Chapter 16] (see also [24, 25] and Section 3 for more details and equivalent conditions). Thereafter, Einstein-like manifolds have been studied by many authors such as G. Calvaruso in [7-10] Mantica et al in [24, 25] and many others [2, 5, 6, 30, 36]. An interesting study in [26] shows Einstein-like generalized Robertson-Walker space-times are perfect fluid space-times except those in class $I$ which are not restricted. Sufficient conditions on generalized Robertson-Walker space-times in this class to be a perfect fluid are derived in [13].

Doubly warped products is a generalization of singly warped products introduced in [4]. The geometric properties of doubly warped product manifolds have been investigated by many authors such as pseudo-convexity in [1], harmonic Weyl conformal curvature tensor in [18], conformal flatness in [20, 21], geodesic completeness in [15], doubly warped product submanifolds in [17, 28, 29, 31] and conformal vector fields in [1]. Doubly warped space-times are widely used as exact solutions of Einstein’s field equations. Recently, the existence of compact Einstein doubly warped product manifolds is considered in [15].

Inspired by the above studies of Einstein-like metrics and doubly warped product manifolds, we studied doubly warped product manifolds equipped with Einstein-like metrics. The inheritance properties of the Einstein-like class type $P, A, B, I \oplus A, I \oplus B$ or $A \oplus B$ are investigated. To assure that factor manifolds of a doubly warped product manifold inherits the Einstein-like class type, sufficient and necessary conditions are derived on the warping functions. Finally, we apply the results to doubly warped space-times.

2. Preliminaries

A doubly warped product manifold is the (pseudo-)Riemannian product manifold $M = M_1 \times M_2$ of two (pseudo-)Riemannian manifolds $(M_i, g_i, D_i), i = 1, 2$, furnished with the metric tensor

$$g = (f_2 \circ \pi_2)^2 \pi_1^\ast (g_1) \oplus (f_1 \circ \pi_1)^2 \pi_2^\ast (g_2),$$

where the functions $f_i : M_i \to (0, \infty), i = 1, 2$ are the warping functions of $M$. $M$ is denoted by $f_\ast M_1 \times f_\ast M_2$. The maps $\pi_i : M_1 \times M_2 \to M_i$ are the natural projections $M$ onto $M_i$ whereas $\ast$ denotes the pull-back operator on tensors. In particular, if for example $f_2 = 1$, then $M = M_1 \times f_\ast M_2$ is called a singly warped product manifold (see [15, 33] for doubly warped products and [4, 14, 16, 27, 33, 34] for singly warped products).

Notation 2.1. Throughout this work, we use the following notations

1. All tensor fields on $M_i$ are identified with their lifts to $M$. For example, we use $f_i$ for a function on $M_i$ and for its lift $(f_i \circ \pi_i)_{\ast}^\pi M$.
2. The manifolds $M_i$ has dimensions $n_i$ where $n = n_1 + n_2$.
3. Ric is the Ricci curvature tensor on $M$ and $\text{Ric}^i$ is the Ricci tensor on $M_i$.
4. The gradient of $f_i$ on $M_i$ is denoted by $\nabla f_i$ and the Laplacian by $\Delta f_i$, whereas $f_i^\ast = f_i \Delta f_i + (n_i - 1) g_i \left( \nabla f_i, \nabla f_i \right),$ $i \neq j$.
5. The indices $i$ and $j$ to denote the geometric objects of the factor manifolds $M_i$ and $M_j$.
6. The $(0, 2)$ tensors $\mathcal{F}^i$ is defined as

$$\mathcal{F}^i (X_i, Y_i) = \frac{n_i}{f_i} H^i (X_i, Y_i),$$

for $X_i, Y_i \in \mathfrak{X} (M_i)$ and $i, j = 1, 2, i \neq j$.

The Levi-Civita connection $D$ on $M = f_\ast M_1 \times f_\ast M_2$ is given by

$$D_{X_i} X_j = X_i (\ln f_j) X_j + X_j (\ln f_i) X_i,$$

$$D_{X_i} Y_i = D^i X_i - \frac{f_i^2}{f_j} g_i (X_i, Y_i) \nabla (\ln f_i),$$
where \( i \neq j \) and \( X_i, Y_i \in \mathfrak{X} (M_i) \). Then the Ricci curvature tensor \( \text{Ric} \) on \( M \) is given by

\[
\text{Ric} (X_i, Y_i) = \text{Ric}^i (X_i, Y_i) - \frac{n_i}{f_i} H^i (X_i, Y_i) - f_i \frac{f_i}{f_i} g_i (X_i, Y_i),
\]

\[
\text{Ric} \left( X_i, Y_i \right) = (n - 2) X_i \left( \ln f_i \right) Y_i \left( \ln f_i \right),
\]

where \( i \neq j \) and \( X_i, Y_i \in \mathfrak{X} (M_i) \). The reader is referred to [11] [12] [19] for some studies of curvature conditions on warped product manifolds.

3. Einstein-like doubly warped product manifolds

The Einstein-like doubly warped product manifolds \( M =_{f_2} M_1 \times_{f_1} M_2 \) are investigated in this section. Every subsection is devoted to the study of a class of Einstein-like doubly warped product manifolds. Sufficient and necessary conditions are derived on the warping functions \( f_i \) for factor manifolds \( M_i \) to acquire the same Einstein-like class type.

3.1. Class \( \mathcal{A} \)

A doubly warped product manifold \((M, g)\) whose Ricci tensor is Killing, that is,

\[
(D_X \text{Ric}) (Y, Z) + (D_Y \text{Ric}) (Z, X) + (D_Z \text{Ric}) (X, Y) = 0,
\]

for any vector fields \( X, Y, Z \in \mathfrak{X} (M) \) is called Einstein-like doubly warped product manifold of class \( \mathcal{A} \). This condition equivalent to

\[
(D_X \text{Ric}) (X, X) = 0,
\]

for any vector field \( X \in \mathfrak{X} (M) \) and the Ricci tensor is also called cyclic parallel. The legacy of factor manifolds of \( M \) in class \( \mathcal{A} \) is as follows.

**Theorem 3.1.** In a doubly warped product manifold \( M =_{f_2} M_1 \times_{f_1} M_2 \) where \( M \) is of class type \( \mathcal{A} \), a factor manifold \((M_i, g_i)\) is an Einstein-like manifold of class \( \mathcal{A} \) if and only if

\[
\left( D^X_{X^i} F^i \right) (X_i, X_i) = \frac{2}{f_i^2} X_i (f_i) g_i (X_i, X_i) \left[ f_i^o + (n - 2) \left( \nabla^i f_i \right) (f_i) \right],
\]

where \( i, j = 1, 2, i \neq j \) and \( X_i \in \mathfrak{X} (M_i) \).

**Proof.** In a doubly warped product manifold \( M =_{f_2} M_1 \times_{f_1} M_2 \) of class \( \mathcal{A} \), it is

\[
0 = (D_X \text{Ric}) (X, X) = X (\text{Ric} (X, X)) - 2 \text{Ric} (D_X X, X).
\]

Thus, for the special case where \( X = X_i \) lands on one factor, one may get

\[
0 = (D_{X_i} \text{Ric}) (X_i, X_i) = X_i \left( \text{Ric}^i (X_i, X_i) - F^i (X_i, X_i) - \frac{f_i^o}{f_i^2} g_i (X_i, X_i) \right)
\]

\[
-2 \text{Ric} \left( D^i_{X_i} X_i, X_i \right) + 2 F^i \left( D^i_{X_i} X_i, X_i \right) + 2 \frac{f_i^o}{f_i^2} g_i \left( D^i_{X_i} X_i, X_i \right)
\]

\[
+ 2 (n - 2) \frac{1}{f_i^o} \left( \nabla^i f_i \right) (f_i) X_i (f_i) g_i (X_i, X_i).
\]
Thus, after lengthy computations, it is

\[ 0 = \left( D_{X_i} \text{Ric} \right)(X_i, X_i) - \left( D_{X_i} \mathcal{F}^i \right)(X_i, X_i) \]

\[ + \frac{2}{f_i} X_i (f_i) g_i (X_i, X_i) \left[ f_i^2 + (n - 2) \left( \nabla^i f_i \right) (f_i) \right]. \]

These equations complete the proof. \( \square \)

It is now easy to recover a similar result on singly warped product manifolds.

**Corollary 3.2.** In a singly warped product manifold \( M = M_1 \times f \_ \ M_2 \) where \( M \) is of class type \( \mathcal{A} \), \( (M_1, g_1) \) is an Einstein-like manifold of class \( \mathcal{A} \) if and only if \( \mathcal{F}^i \) is Killing. In addition, \( (M_2, g_2) \) is of class type \( \mathcal{A} \).

### 3.2 Class \( \mathcal{B} \)

Let \( M \) be as Einstein-like doubly warped product manifold of class \( \mathcal{B} \). Then, the Ricci tensor is a Codazzi tensor, that is,

\[ \left( D_{Y_i} \text{Ric} \right)(Y_i, Z_i) = \left( D_{Y_i} \text{Ric} \right)(X_i, Z_i). \]

The above condition is equivalent to:

1. \( M \) has a harmonic Riemann tensor, that is, \( \nabla \epsilon R^\epsilon_{\alpha \beta \gamma} = 0 \), or
2. \( M \) admits a harmonic Weyl conformal tensor and the scalar curvature is constant, that is, \( \nabla \epsilon C^\epsilon_{\alpha \beta \gamma} = 0 \) and \( \nabla \epsilon R = 0 \).

The base manifold and the fiber manifold gain the Einstein-like class type \( \mathcal{B} \) according to.

**Theorem 3.3.** In a doubly warped product manifold \( M = f_i M_1 \times f_i M_2 \) where \( M \) is of class type \( \mathcal{B} \), the factor manifold \( (M_i, g_i) \) is an Einstein-like manifold of class \( \mathcal{B} \) if and only if

\[ \left( D_{X_i} \mathcal{F}^i \right)(Y_i, Z_i) = \left( D_{Y_i} \mathcal{F}^i \right)(X_i, Z_i) \]

\[ + \frac{1}{f_i} X_i (f_i) g_i (Y_i, Z_i) \left( 2f_i^2 - (n - 2) \left( \nabla^i f_i \right) (f_i) \right) \]

\[ - \frac{1}{f_i} Y_i (f_i) g_i (X_i, Z_i) \left( 2f_i^2 - (n - 2) \left( \nabla^i f_i \right) (f_i) \right), \]

where \( i, j = 1, 2, i \neq j \) and \( X_i, Y_i, Z_i \in \mathfrak{X}(M_i) \).

**Proof.** Let us define the deviation tensor \( B(X, Y, Z) \) as follows

\[ B(X, Y) Z = \left( D_X \text{Ric} \right)(Y, Z) - \left( D_Y \text{Ric} \right)(X, Z). \]

There are three different cases. Let us consider the first case, that is,

\[ B(X_i, Y_i, Z_i) = \left( D_{X_i} \text{Ric} \right)(Y_i, Z_i) - \left( D_{Y_i} \text{Ric} \right)(X_i, Z_i). \] (1)
It is enough to find \((D_X \text{Ric})(Y_i, Z_i)\) as

\[
(D_X \text{Ric})(Y_i, Z_i) = X_i \left( \text{Ric}^i (Y_i, Z_i) \right) - X_i \left( F^i (Y_i, Z_i) \right) - f_i^2 X_i \left( \frac{1}{f_i} g_i (Y_i, Z_i) \right)
\]

\[
- \text{Ric}^i \left( D_i Y_i, Z_i \right) + F^i \left( D_i Y_i, Z_i \right) + \frac{f_i^2}{f_i} g_i \left( D_i Y_i, Z_i \right)
\]

\[
- \text{Ric}^i \left( Y_i, D_i Z_i \right) + F^i \left( Y_i, D_i Z_i \right) + \frac{f_i^2}{f_i} g_i \left( Y_i, D_i Z_i \right)
\]

\[
+ (n - 2) \frac{1}{f_i} g_i (X_i, Y_i) \nabla f \left( f_i \right) Z_i \left( f_i \right)
\]

\[
+ (n - 2) \frac{1}{f_i} g_i (X_i, Z_i) \nabla f \left( f_i \right) Y_i \left( f_i \right).
\]

Simplifying this expression, it is

\[
(D_X \text{Ric})(Y_i, Z_i) = \left( D_i \text{Ric}^i \right)(Y_i, Z_i) - \left( D_i \text{F}^i \right)(Y_i, Z_i) + 2 \frac{f_i^2}{f_i} X_i \left( f_i \right) g_i (Y_i, Z_i)
\]

\[
+ (n - 2) \frac{1}{f_i} g_i (X_i, Y_i) \nabla f \left( f_i \right) Z_i \left( f_i \right)
\]

\[
+ (n - 2) \frac{1}{f_i} g_i (X_i, Z_i) \nabla f \left( f_i \right) Y_i \left( f_i \right).
\]

(2)

By exchanging \(X_i\) and \(Y_i\) in the last equation and substitution in Equation (1), one gets the deviation tensor. For Einstein-like manifolds of class \(B\), the deviation tensor vanishes from which the result hold. □

It is easy to retrieve a similar result on a singly warped product manifold.

**Corollary 3.4.** In a singly warped product manifold \(M = M_1 \times f_i M_2\) where \(M\) is of class type \(B\), \((M_1, g_1)\) is an Einstein-like manifold of class \(B\) if and only if

\[
\left( D_i \text{F} \right)(Y_1, Z_1) = \left( D_i \text{F} \right)(X_1, Z_1),
\]

where \(X_i, Y_1, Z_1 \in \mathfrak{X}(M_1)\). In addition, \((M_2, g_2)\) is Einstein-like of class type \(B\).

### 3.3. Class \(P\)

Let \(M\) be an Einstein-like doubly warped product manifold of class \(P\). Thus, \(M\) has a parallel Ricci tensor, that is,

\[
(D_X \text{Ric})(Y, Z) = 0.
\]

Manifolds in this class are usually called Ricci symmetric.

**Theorem 3.5.** In a doubly warped product manifold \(M = f_i M_1 \times f_j M_2\) where \(M\) is of class type \(P\), \((M_i, g_i)\) is an Einstein-like manifold of class \(P\) if and only if

\[
\left( D_i \text{F} \right)(Y_i, Z_i) = \frac{n - 2}{f_i} \left[ g_i (X_i, Y_i) Z_i \left( f_i \right) + g_i (X_i, Z_i) Y_i \left( f_i \right) \right] \nabla f \left( f_i \right) f_i
\]

\[
+ 2 \frac{f_i^2}{f_i} X_i \left( f_i \right) g_i (Y_i, Z_i),
\]

where \(i, j = 1, 2, i \neq j\) and \(X_i, Y_i, Z_i \in \mathfrak{X}(M_i)\).
Proof. Let \( M = f_2 M_1 \times f_1 M_2 \) be a Ricci symmetric doubly warped product manifold, that is,
\[
0 = (D_X \text{Ric})(Y, Z)
\]
Equation (2) infers
\[
(D_X \text{Ric})(Y_i, Z_i) = \left( D_X^i \text{Ric}^i \right)(Y_i, Z_i) - \frac{f_i}{f_j} X_i(f_j) g_i(Y_j, Z_j)
\]
\[
+ (n - 2) \frac{1}{f_j^2} g_j(X_i, Y_i) \nabla^j f_j Z_i(f_j)
\]
\[
+ (n - 2) \frac{1}{f_j^2} g_j(X_i, Z_i) \nabla^j f_j Y_i(f_j).
\]
Thus, having a parallel Ricci tensor implies
\[
\left( D_X^i \text{Ric}^i \right)(Y_i, Z_i) = \left( D_X^i \text{Ric}^i \right)(Y_i, Z_i) - \frac{f_i}{f_j} X_i(f_j) g_i(Y_j, Z_j)
\]
\[
- \frac{n - 2}{f_j^2} \left[ g_j(X_i, Y_i) Z_i(f_j) + g_j(X_i, Z_i) Y_i(f_j) \right] \nabla^j f_j (f_j).
\]
This equation completes the proof. \( \Box \)

The corresponding result on singly warped product manifolds is as follows.

**Corollary 3.6.** In a singly warped product manifold \( M = M_1 \times f_1 M_2 \) where \( M \) is of class type \( \mathcal{P} \). Then \( (M_1, g_1) \) is an Einstein-like manifold of class \( \mathcal{P} \) if and only if
\[
\left( D_X^i \text{F}^i \right)(Y_i, Z_i) = 0,
\]
where \( X_i, Y_1, Z_1 \in \mathcal{X}(M_1) \). Also, \( (M_2, g_2) \) is Einstein-like of class type \( \mathcal{P} \).

### 3.4. Class \( I \oplus B \)

A doubly warped product manifold \( M \) is of class type \( I \oplus B \) if its Ricci tensor satisfies
\[
\nabla_\nu \left[ R_{\alpha \beta} - \frac{R}{2(n - 1)} g_{\alpha \beta} \right] = \nabla_\alpha \left[ R_{\nu \beta} - \frac{R}{2(n - 1)} g_{\nu \beta} \right],
\]
that is, the tensor \( \mathcal{H}_{\alpha \beta} = R_{\alpha \beta} - \frac{R}{2(n - 1)} g_{\alpha \beta} \) is a Codazzi tensor. This condition is equivalent to
\[
\nabla_\nu C_{\alpha \beta}^{\epsilon} = 0,
\]
where \( C \) is the Weyl conformal curvature tensor and \( n \geq 3 \), i.e., \( M \) has a harmonic Weyl tensor. Let \( g_{\alpha \beta} = \phi^2 g_{\alpha \beta} \) be a conformal change of on a manifold \( M \). It is well known that the Weyl tensor \( C_{\alpha \beta}^{\epsilon} \) remains invariant, that is, \( \bar{C}_{\alpha \beta}^{\epsilon} = C_{\alpha \beta}^{\epsilon} \) however \( C_{\alpha \beta}^{\epsilon} = \phi^2 \bar{C}_{\alpha \beta}^{\epsilon} \). The divergence of the Weyl tensor is given by
\[
\nabla_\nu C_{\alpha \beta}^{\epsilon} = \nabla_\nu \bar{C}_{\alpha \beta}^{\epsilon} - \frac{n - 3}{\phi} \left( \nabla_\nu \phi \right) \bar{C}_{\alpha \beta}^{\epsilon}
\]
(3)

The doubly warped product metric may be rewritten as follows
\[
g = f_1^2 f_2^2 \left( f_1^2 g_1 + f_2^2 g_2 \right)
\]
\[
= f_1^2 f_2^2 (g_1 + g_2)
\]
\[
= f_1^2 f_2^2 g
\]
where \( g_i = f_i^2 \bar{g}_i \) and \( \bar{g} = \bar{g}_1 + \bar{g}_2 \). The doubly warped product manifold \((M, \bar{g})\) has harmonic Weyl tensor if and only
\[
\nabla_\varepsilon C_{\alpha\beta\gamma\delta} = \frac{n - 3}{q} (\nabla_\varepsilon \varphi) C_{\alpha\beta\gamma\delta}
\]
(4)
where \( \varphi = f_1 f_2 \). Assume that \( \nabla_\varepsilon (f_1 f_2) C_{\alpha\beta\gamma\delta} = 0 \), then
\[
\nabla_\varepsilon C_{\alpha\beta\gamma\delta} = 0.
\]
(5)

having a harmonic Weyl tensor is equivalent to the condition
\[
0 = \bar{\nabla}_\varepsilon \bar{R} = -\frac{1}{2(n-1)} \left[ (\nabla_\gamma \bar{R}) g_{\alpha\beta} - (\nabla_\gamma \bar{R}) g_{\alpha\beta} \right],
\]
where \( \bar{T} \) is the Cotton tensor. The metric \( \bar{g} \) splits as \( \bar{g} = \bar{g}_1 + \bar{g}_2 \) and consequently the divergence of the Cotton tensor \( \bar{T} \) splits on the factor manifolds \((M_i, \bar{g}_i)\) as
\[
0 = \bar{T}^\iota_{\alpha\beta\gamma} + \frac{n_2}{2(n-1)(n-1-1)} \left[ (\nabla_\gamma \bar{R}^\iota) (\bar{g}_i)_{\alpha\beta} - (\nabla_\gamma \bar{R}^\iota) (\bar{g}_i)_{\alpha\beta} \right].
\]
(6)

In this case, \( (\nabla_\gamma \bar{R}^\iota) (\bar{g}_i)_{\alpha\beta} - (\nabla_\gamma \bar{R}^\iota) (\bar{g}_i)_{\alpha\beta} = 0 \), that is, \( \bar{R}^\iota \) is constant if and only if the cotton tensor \( \bar{T}^\iota \) on the doubly warped factor manifolds \((M^\iota, \bar{g}_i)\) vanishes i.e.
\[
\nabla_\varepsilon C^\iota_{\alpha\beta\gamma\delta} = 0.
\]

The Weyl tensors \( C^\iota \) on doubly warped product factor manifolds \((M_i, g_i)\) satisfy
\[
0 = \nabla_\varepsilon C^\iota_{\alpha\beta\gamma\delta} = \nabla_\gamma C^\iota_{\alpha\beta\gamma\delta} + \frac{n - 3}{f_i} (\nabla_\gamma f_i) C^\iota_{\alpha\beta\gamma\delta}
\]
(7)

It is time now to write the following result.

**Theorem 3.7.** In a doubly warped product manifold \( M = f_1 M_1 \times f_2 M_2 \) where \( M \) is of class type \( I \oplus B \). Assume that \( \nabla_\varepsilon (f_1 f_2) C^\iota_{\alpha\beta\gamma\delta} = 0 \) and the conformal change \((M_i, f_i^2 g_i)\) has a constant scalar curvature. Then \((M_i, g_i)\) is an Einstein-like manifold of class \( I \oplus B \) if and only if \( (\nabla_\gamma f_i) C^\iota_{\alpha\beta\gamma\delta} = 0 \) for each \( i = 1, 2 \).

A. Gebarowski proved an inheritance property of this class in [18, Theorem 2].

3.5. Class \( I \oplus A \)

Doubly warped product manifolds where the tensor
\[
\mathcal{L} = \text{Ric} - \frac{2R}{n+2} \bar{g}
\]
is Killing lies the class \( I \oplus A \). The above condition is equivalent to
\[
0 = (D_X \mathcal{L})(X, X).
\]
The following theorem draw the inheritance property of this class.
Theorem 3.8. In a doubly warped product manifold $M = f_1 M_1 \times f_2 M_2$ where $M$ is of class type $I \oplus \mathfrak{A}$, the factor manifold $(M, g_i)$ is of class type $I \oplus \mathfrak{A}$ if and only if

$$\left(D_{X}^{i} F^{i}\right)(X_i, X_i) = \frac{2}{f_i} X_i (f_i) g_i (X_i, X_i) \left[f_i + (n - 2) \left(\nabla^i f_i\right) (f_i)\right]$$

$$- \frac{2}{n + 2} \left(D_X R - \frac{n + 2}{n_i + 2} D_X R^{i}\right) g_i (X_i, X_i).$$

Proof. Assume that $M = f_1 M_1 \times f_2 M_2$ be a doubly warped product manifold of class type $I \oplus \mathfrak{A}$. Then

$$0 = (D_X) \left(Ric - \frac{2 R}{n + 2} g\right) (X, X)$$

$$= (D_X \text{Ric}) (X, X) - \frac{2}{n + 2} g(X, X) D_X R.$$

Using equation (2), it is

$$0 = \left(D_{X}^{i} \text{Ric}^{i}\right)(X_i, X_i) - \left(D_{X}^{i} F^{i}\right)(X_i, X_i)$$

$$+ \frac{2}{f_i} X_i (f_i) g_i (X_i, X_i) \left[f_i + (n - 2) \left(\nabla^i f_i\right) (f_i)\right]$$

$$- \frac{2}{n + 2} \left(D_X R - \frac{n + 2}{n_i + 2} D_X R^{i}\right) g_i (X_i, X_i)$$

and consequently, one has

$$0 = \left(D_{X}^{i} \text{Ric}^{i}\right)(X_i, X_i) - \frac{2}{n_i + 2} g_i (X_i, X_i) D_X^{i}$$

$$- \left(D_{X}^{i} F^{i}\right)(X_i, X_i)$$

$$+ \frac{2}{f_i} X_i (f_i) g_i (X_i, X_i) \left[f_i + (n - 2) \left(\nabla^i f_i\right) (f_i)\right]$$

$$- \frac{2}{n + 2} \left(D_X R - \frac{n + 2}{n_i + 2} D_X R^{i}\right) g_i (X_i, X_i)$$

which completes the proof. □

3.6. Class $\mathfrak{A} \oplus \mathcal{B}$

This class is identified by having a constant scalar curvature. Let $M = f_1 M_1 \times f_2 M_2$ be a doubly warped product manifold of class type $\mathfrak{A} \oplus \mathcal{B}$, that is, the scalar curvature $R$ of $M$ is constant, say $c$. The use of Equation 7 in [18] implies that $M_j$ is of class $\mathfrak{A} \oplus \mathcal{B}$ if there are two constants $c_i$ and $c_j$ such that

$$\frac{c_i}{f_i^2} + \frac{c_j}{f_j^2} - \frac{n_i (n_i - 1)}{f_i^2} \Delta_i f_i - \frac{n_j (n_j - 1)}{f_j^2} \Delta_j f_j - \frac{2 n_i}{f_i} F_i - \frac{2 n_j}{f_j} F_j = c,$$

where $F_i = g_i^{\alpha \beta} \nabla^\alpha_i \nabla^\beta_i f_i$. 
4. Einstein-like doubly warped Relativistic space-times

Let $(M, g)$ be a Riemannian manifold, $f : M \to (0, \infty)$ and $\sigma : I \to (0, \infty)$ are smooth functions. The manifold $\bar{M} = I \times_o M$ furnished with the metric tensor $\bar{g} = -f^2 dt^2 \oplus \sigma^2 g$ is called a doubly warped space-time. For $U, V \in \mathfrak{X}(M)$, the covariant derivative $\bar{D}$ on $\bar{M}$ is given by

$$\bar{D}_\partial_t \partial_t = \frac{f}{\sigma^2} \nabla f,$$
$$\bar{D}_\partial_t U = \bar{D}_U \partial_t = \frac{\sigma^2}{\sigma} U + \frac{1}{f} U(f) \partial_t,$$
$$\bar{D}_U V = \bar{D}_V U = \frac{\sigma^2}{f^2} g(U, V) \partial_t,$$

whereas the Ricci tensor $\bar{\text{Ric}}$ on $\bar{M}$ is given by

$$\bar{\text{Ric}}(\partial_t, \partial_t) = \frac{n}{\sigma} + \frac{f^2}{\sigma^2},$$
$$\text{Ric}(U, V) = \text{Ric}(U, V) - \frac{1}{f} H(U, V) - \frac{\sigma^2}{f^2} g(U, V),$$
$$\text{Ric}(\partial_t, U) = \left(n - 1\right) \frac{\sigma^2}{\sigma} U(\ln f).$$

For the definition and relativistic significance of doubly warped space-times, the reader is referred to [15, 32] and references therein.

**Theorem 4.1.** In a doubly warped space-time $\bar{M} = I \times_o M$ of class type $\mathcal{A}$, $M$ is an Einstein-like manifold of class type $\mathcal{A}$ if and only if

$$(\bar{D}_V F)(V, V) = \left(n - 1\right) \frac{\sigma^2}{\sigma} + \sigma^2 \right) \frac{2}{f^2} V(f) g(V, V).$$

**Theorem 4.2.** In a doubly warped space-time $\bar{M} = I \times_o M$ of class type $\mathcal{B}$, $M$ is an Einstein-like manifold of class type $\mathcal{B}$ if and only if

$$(\bar{D}_V F)(U, V) = (\bar{D}_U F)(W, V) + \left(2\sigma^2 - (n - 1) \frac{\sigma^2}{f^2}\right) \frac{1}{f^3} W(f) g(U, V)$$
$$- \left(2\sigma^2 + (n - 1) \frac{\sigma^2}{f^2}\right) \frac{1}{f^3} U(f) g(W, V).$$

**Theorem 4.3.** In a doubly warped space-time $\bar{M} = I \times_o M$ of class type $\mathcal{P}$, $M$ is an Einstein-like manifold of class type $\mathcal{P}$ if and only if

$$(\bar{D}_V F)(U, V) = 2 \frac{\sigma^2}{f^3} W(f) g(U, V) + \frac{\sigma^2}{f^3} (n - 1) (g(W, V) U(f) + g(W, U) V(f)).$$

References


