Covering Properties of $C_p(X)$ and $C_k(X)$

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Abstract. Let $X$ be a Tychonoff space. We survey some classic and recent results that characterize the topology or cardinality of $X$ when $C_p(X)$ or $C_k(X)$ is covered by certain families of sets (sequences, resolutions, closure-preserving coverings, compact coverings ordered by a second countable space) which swallow or not some classes of sets (compact sets, functionally bounded sets, pointwise bounded sets) in $C(X)$.

1. Preliminaries

Unless otherwise stated, $X$ will stand for an infinite Tychonoff space. We denote by $C_p(X)$ the linear space $C(X)$ of real-valued continuous functions on $X$ equipped with the pointwise topology $\tau_p$. The topological dual of $C_p(X)$ is denoted by $L(X)$, or by $L_p(X)$ when provided with the weak* topology. We denote by $C_k(X)$ the space $C(X)$ equipped with the compact-open topology $\tau_k$. A family $\{A_\alpha : \alpha \in \mathbb{N}\}$ of subsets of a set $X$ is a resolution for $X$ if it covers $X$ and verifies that $A_\alpha \subseteq A_\beta$ for $\alpha \leq \beta$. A family of bounded sets in a locally convex space $E$ that swallows the bounded sets is called a fundamental family of bounded sets. Definitions not included in this paper can be found in [6, 18, 49].

2. Countable coverings for $C_p(X)$

The following folklore result can be found in [49, Proposition 9.18]. Velichko’s theorem can be found in [1, I.2.1 Theorem] or in [49, Theorem 9.12].

Theorem 2.1. The space $C_p(X)$ admits a fundamental sequence of pointwise bounded sets if and only if $X$ is finite.

Theorem 2.2 (Velichko). The space $C_p(X)$ is covered by a sequence of compact sets if and only if $X$ is finite.

Next theorem extends Velichko’s result to relatively countably compact sets.

Theorem 2.3 (Tkachuk-Shakhmatov [75]). $C_p(X)$ is covered by a sequence of relatively countably compact sets if and only if $X$ is finite.

2010 Mathematics Subject Classification. 54C30, 46A03

Keywords. Lindelöf $\Sigma$-space, $K$-analytic space, analytic space, cosmic space

Received: 20 August 2020; Accepted: 04 September 2020

Communicated by Vladimir Rakočević

Research supported by Grant PGC2018-094431-B-I00 of Ministry of Science, Innovation & Universities of Spain.

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Theorem 2.5 below extends Tkachuk-Shakhmatov theorem to pointwise bounded relatively sequentially complete sets. Recall that a sequence \( \{f_n\}_{n=1}^{\infty} \) of real-valued functions defined on \( X \) is pointwise eventually constant [34] if for each \( x \in X \) there is a constant \( f(x) \) such that \( f_n(x) = f(x) \) for all but finitely many \( n \in \mathbb{N} \).

**Theorem 2.4 (Ferrando-Kąkol-Saxon [34, Theorem 3.1]).** \( C_p(X) \) is covered by a sequence of relatively sequentially complete sets if and only if \( X \) is a \( P \)-space.

**Proof.** Assume that \( C_p(X) = \bigcup_{n=1}^{\infty} Q_n \) with \( Q_n \) relatively sequentially complete for every \( n \in \mathbb{N} \) and let \( \{f_n\}_{n=1}^{\infty} \) be a uniformly bounded pointwise eventually constant sequence in \( C_p(X) \) with limit \( f \) in \( R^X \). Let us denote by \( C^\circ(X) \) the Banach space of all continuous and bounded functions on \( X \) equipped with the supremum norm \( \|\cdot\|_{\infty} \). Fix \( k > 0 \) such that \( \sup_{n \in \mathbb{N}} \|f_n\|_{\infty} \leq k \).

Since \( \big[ C^\circ(X) \cap Q_n : n \in \mathbb{N} \big] \) is a countable covering of \( C^\circ(X) \), according to the Baire category theorem there is \( p \in \mathbb{N} \) such that the closure \( B_p \) of \( C^\circ(X) \cap Q_p \) in \( C^\circ(X) \) has an interior point in the norm topology. So, if \( D \) denotes the closed unit ball of \( C^\circ(X) \), there are \( \epsilon > 0 \) and \( h \in Q_p \) with \( h + \epsilon D \subseteq B_p \). Since \( f_n \in kD \) for each \( n \in \mathbb{N} \), we have \( h + ek^{-1}f_n : n \in \mathbb{N} \subseteq B_p \). As \( C^\circ(X) \cap Q_n \) is norm dense in \( B_p \), for each \( n \in \mathbb{N} \), there is \( g_n \in C^\circ(X) \cap Q_n \) with \( g_n(x) - (h + ek^{-1}f_n)(x) \) \( < n^{-1} \) for all \( x \in X \). Since \( \{h + ek^{-1}f_n\}_{n=1}^{\infty} \) is a pointwise eventually constant sequence that converges to \( h + ek^{-1}f \) pointwise on \( X \). Using the fact that \( Q_n \) is relatively sequentially complete, it turns out that \( h + ek^{-1}f \in C(X) \). Hence \( f \in C(X) \).

But, as follows from [34, Theorem 1.1], a Tychonoff space \( X \) is a \( P \)-space if and only if each uniformly bounded pointwise eventually constant sequence in \( C_p(X) \) converges in \( C_p(X) \). So, \( X \) is a \( P \)-space. For the converse note that if \( X \) is a \( P \)-space, then \( C_p(X) \) is sequentially complete [8].

**Theorem 2.5 (Ferrando-Kąkol-Saxon [34, Corollary 3.2]).** \( C_p(X) \) is covered by a sequence of pointwise bounded relatively sequentially complete sets if and only if \( X \) is finite.

**Proof.** If \( C_p(X) = \bigcup_{n=1}^{\infty} Q_n \) with \( Q_n \) pointwise bounded and relatively sequentially complete, Theorem 2.4 ensures that \( X \) is a \( P \)-space. If \( \{x_n\}_{n=1}^{\infty} \) is an infinite sequence in \( X \), for each \( n \in \mathbb{N} \) there is \( \alpha_n > 0 \) with \( \sup_{n \in \mathbb{N}} \|g(x_n)\| < \alpha_n \). But [49, Lemma 9.5] provides \( f \in C(X) \) with \( f(x_n) = \alpha_n \), i.e., such that \( f \not\in Q_n \) for every \( n \in \mathbb{N} \), a contradiction. Thus \( X \) must be finite.

**Theorem 2.6 (Tkachuk, [69, 3.11 Theorem]).** If \( C_p(X) \) is covered by a sequence of functionally bounded sets, then \( X \) is pseudocompact and each countable subset of \( X \) is closed, discrete and \( C^\circ \)-embedded in \( X \).

**Proof.** (Sketch) Let us call \( \sigma \)-bounded a space which is covered by countably many functionally bounded sets and assume that \( C_p(X) \) is \( \sigma \)-bounded. If \( X \) is not pseudocompact, it contains a closed homeomorphic copy \( Y \) of \( \mathbb{N} \), hence \( C^\circ \)-embedded [39, Problem 31]. Since the restriction map \( T : C_p(X) \to C_p(Y) \) defined by \( Tf = f|_Y \) is continuous and onto, this implies that \( C_p(Y) \) is \( \sigma \)-bounded. Hence \( C_p(N) = R^N \) is covered by a sequence of compact sets and Velichko’s theorem ensures that \( N \) must be finite, a contradiction. On the other hand, since \( C_p(X, I) = \{f \in C(X) : -1 \leq f \leq 1\} \) is a retract of \( C_p(X) \), it turns out that \( C_p(X, I) \) is \( \sigma \)-bounded. If \( Z \) is a non-closed countable subset of \( X \) and \( y \in \bar{Z} \setminus Z \), it is not hard to show that \( M = \{f \in C_p(X, I) : f(y) = 0\} \) is also covered by countably many functionally bounded sets \( \{F_n : n \in \mathbb{N}\} \). But one can determine a function \( f \in M \) such that \( f \not\in F_n \) for every \( n \in \mathbb{N} \) (see [69, 3.7 Lemma] for details). So, such \( Z \) does not exist. Finally, it is well-known that a subspace \( S \) of \( X \) is \( C^\circ \)-embedded if and only if \( S^{\text{cl}} = \beta S \). If each countable set in \( X \) is closed, it can be seen that each countable set \( A \) is discrete and \( C^\circ \)-embedded if and only if \( A^{\text{cl}} = \beta A \) [69, 3.8 Proposition]. With the help of this result one can show that if \( C_p(X, I) \) is \( \sigma \)-bounded, every countable subset of \( X \) is discrete and \( C^\circ \)-bounded [69, 3.9 Theorem].
3. Uncountable coverings for \( \mathcal{C}_p(X) \)

Recall that \( X \) is a Lindelöf \( \Sigma \)-space if it is a continuous image of a space that can be perfectly mapped onto a second countable space [1, 57]. Also, \( X \) is a Lindelöf \( \Sigma \)-space if and only if it is countably \( K \)-determined [63], i.e., if there is an upper semi-continuous (usc) map \( T \) from a subspace \( \Sigma \) of \( \mathbb{N}^\mathbb{N} \) into the family \( \mathcal{K}(X) \) of compact subsets of \( X \) such that \( \bigcup \{ T(\alpha) : \alpha \in \Sigma \} = X \). This is equivalent to saying that (i) \( \{ T(\alpha) : \alpha \in \Sigma \} \) covers \( X \) and (ii) if \( a_n \to a \in \Sigma \) and \( x_n \in T(a_n) \) for every \( n \in \mathbb{N} \) the sequence \( \{ x_n \}_{n=1}^\infty \) has a cluster point in \( T(a) \). A space \( X \) is \( K \)-analytic (resp. quasi-Suslin) if there is a map \( T \) from \( \mathbb{N}^\mathbb{N} \) into \( \mathcal{K}(X) \) (resp. into the family of countably compact sets in \( X \)) such that (i) \( \{ T(\alpha) : \alpha \in \mathbb{N}^\mathbb{N} \} \) covers \( X \) and (ii) if \( a_n \to a \) in \( \mathbb{N}^\mathbb{N} \) and \( x_n \in T(a_n) \) for each \( n \in \mathbb{N} \) then the sequence \( \{ x_n \} \) has a cluster point contained in \( T(a) \) (see [76, I.4.2 and I.4.3]). Each \( \sigma \)-compact (\( \sigma \)-countably compact) space is \( K \)-analytic (resp. quasi-Suslin). A space \( X \) is analytic if it is a continuous image of \( \mathbb{N}^\mathbb{N} \). Each analytic space is \( K \)-analytic, each \( K \)-analytic space is quasi-Suslin and Lindelöf \( \Sigma \), and each Lindelöf \( \Sigma \)-space is Lindelöf. A family \( \mathcal{N} \) of subsets of \( X \) is a network for \( X \) if for any \( x \in X \) and any open set \( U \) in \( X \) with \( x \in U \) there is some \( P \in \mathcal{N} \) such that \( x \in P \subseteq U \). The network weight \( nw(X) \) of \( X \) is the least cardinality of a network of \( X \), and a space \( X \) is called cosmic if \( nw(X) = \aleph_0 \). Alternatively, \( X \) is a cosmic space if and only if it is a continuous image of a separable metric space [56]. So, each analytic space is cosmic. Conversely, every \( K \)-analytic cosmic space is analytic [49, Proposition 6.4]. Moreover, \( \mathcal{C}_p(X) \) is a cosmic space if and only if \( X \) is cosmic [56, Proposition 10.5]. A family \( \mathcal{N} \) of subsets of a space \( \mathcal{X} \) is a network modulo a family \( \mathcal{A} \) of subsets of \( \mathcal{X} \) if for each open set \( V \) of \( \mathcal{X} \) and for every \( A \in \mathcal{A} \) with \( A \subseteq V \) there exists \( N \in \mathcal{N} \) such that \( A \subseteq N \subseteq V \). A space \( \mathcal{X} \) is Lindelöf \( \Sigma \)-space if and only if it admits a countable network modulo a covering by compact sets [49, Proposition 3.5]. Hence, every cosmic space is a Lindelöf \( \Sigma \)-space. A space \( \mathcal{X} \) is angelic if relatively countably compact sets in \( \mathcal{X} \) are relatively compact and for every relatively compact subset \( A \) of \( \mathcal{X} \) each point of \( \mathcal{A} \) is the limit of a sequence of \( \mathcal{A} \)-points in \( \mathcal{X} \). A space \( \mathcal{X} \) is projectively \( \sigma \)-compact if each separable metrizable space \( \mathcal{Y} \) that is a continuous image of \( \mathcal{X} \) is \( \sigma \)-compact. Clearly, every \( \sigma \)-bounded space (in the sense of Theorem 2.6) is projectively \( \sigma \)-compact [3, Proposition 1.1], and every projectively \( \sigma \)-compact cosmic space is \( \sigma \)-compact (see [49, Proposition 9.4] or [60]). A space \( \mathcal{C}_p(X) \) is said to be Lindelöf \( \Sigma \)-framed (or \( K \)-analytic-framed) in \( \mathbb{R}^\mathbb{R} \) if there is a Lindelöf \( \Sigma \)-space \( \mathcal{X} \) (resp. a \( K \)-analytic space) \( \mathcal{S} \) in \( \mathbb{R}^\mathbb{R} \) such that \( \mathcal{C}(\mathcal{X}) \subseteq \mathcal{S} \). A family \( \mathcal{N} \) of subsets of a topological space \( \mathcal{X} \) is called a \( \mathcal{C}_\mathcal{S} \)-network at a point \( x \in \mathcal{X} \) if for each sequence \( \{ x_n \}_{n=1}^\infty \) in \( \mathcal{X} \) converging to \( x \) and for each neighborhood \( \mathcal{O}_x \) of \( x \) there is a set \( N \in \mathcal{N} \) such that \( x \in N \subseteq \mathcal{O}_x \) and the set \( \{ n \in \mathbb{N} : x_n \in N \} \) is infinite [38]. \( \mathcal{N} \) is a \( \mathcal{C}_\mathcal{S} \)-network in \( \mathcal{X} \) if \( \mathcal{N} \) is a \( \mathcal{C}_\mathcal{S} \)-network at each point \( x \in \mathcal{X} \).

**Lemma 3.1.** If \( \mathcal{C}_p(X) \) is Lindelöf \( \Sigma \)-framed in \( \mathbb{R}^\mathbb{R} \), then \( \nu X \) is a Lindelöf \( \Sigma \)-space and \( \mathcal{C}_p(X) \) is angelic.

**Proof.** First statement after the conditional comes from [59, Theorem 3.5] or [22, Theorem 3]. For the second use the first and [62, Theorem 3], since \( \mathcal{C}_p(X) \) is angelic whenever \( \mathcal{C}_p(\nu X) \) is angelic. \( \square \)

**Lemma 3.2 (Ferrando-Kąkol, [29, Lemma 1]).** Let \( X \) be nonempty and \( Z \) be a subspace of \( \mathbb{R}^\mathbb{R} \). If \( Z \) has a countable network modulo a cover \( \mathcal{B} \) of \( Z \) by pointwise bounded subsets, then \( Y = \bigcup \{ \overline{B} : B \in \mathcal{B} \} \), closures in \( \mathbb{R}^\mathbb{R} \), is a Lindelöf \( \Sigma \)-space such that \( Z \subseteq Y \subseteq \mathbb{R}^\mathbb{R} \).

**Proof.** Let \( \mathcal{N}_1 = \{ T_n : n \in \mathbb{N} \} \) be a countable network modulo a cover \( \mathcal{B} \) of \( Z \) consisting of pointwise bounded sets. Set \( \mathcal{N}_1 = \{ T_n : n \in \mathbb{N} \} \), \( B_1 = \{ \overline{B} : B \in \mathcal{B} \} \), closures in \( \mathbb{R}^\mathbb{R} \), and \( Y = \bigcup B_1 \). Let us show that \( \mathcal{N}_1 \) is a network in \( Y \) modulo the compact cover \( B_1 \) of \( Y \). In fact, if \( U \) is a neighborhood in \( \mathbb{R}^\mathbb{R} \) of \( B_1 \), use \( B \) compactness to get a closed neighborhood \( V \) of \( B \) in \( \mathbb{R}^\mathbb{R} \) contained in \( U \). Since \( \mathcal{N} \) is a network modulo \( \mathcal{B} \) in \( Z \) there is \( n \in \mathbb{N} \) with \( B \subseteq T_n \subseteq V \cap Z \), which implies that \( \overline{B} \subseteq T_n \subseteq U \). According to Nagami’s criterion [1, IV.9.1 Proposition], \( Y \) is a Lindelöf \( \Sigma \)-space such that \( Z \subseteq Y \subseteq \mathbb{R}^\mathbb{R} \). \( \square \)

**Theorem 3.3 (Ferrando-Kąkol, [29, Proposition 1]).** The following asserts are equivalent

1. \( \mathcal{C}_p(X) \) admits a resolution of pointwise bounded sets.
2. \( \mathcal{C}_p(X) \) is \( K \)-analytic-framed in \( \mathbb{R}^\mathbb{R} \).
Proof. Let $\{A_{\alpha} : \alpha \in \mathbb{N}^n\}$ be a resolution for $C_p(X)$ of bounded sets, denote by $B_{\alpha}$ the closure of $A_{\alpha}$ in $\mathbb{R}^X$ and put $Z = \bigcup \{B_{\alpha} : \alpha \in \mathbb{N}^n\}$. Clearly each $B_{\alpha}$ is a compact subset of $\mathbb{R}^X$ and $Z$ is a quasi-Suslin space [11, Proposition 1] such that $C_p(X) \subseteq Z \subseteq \mathbb{R}^X$. As each quasi-Suslin space $Z$ has a countable network modulo a resolution $B$ of $Z$ consisting of countably compact sets (see [20, Proof Theorem 8]) and every countable compact subset of $\mathbb{R}^X$ is pointwise bounded, Lemma 3.2 assures that $Y = \bigcup \{B \in B : B \in B\}$ is a Lindelöf $\Sigma$-space, hence Lindelöf, such that $Z \subseteq Y \subseteq \mathbb{R}^X$. As each set $\overline{B}$ with $B \in B$ is compact, and $[\overline{B} : B \in B]$ is a resolution for $Y$, again $Y$ is a quasi-Suslin space. Since every Lindelöf quasi-Suslin space is K-analytic and $C_p(X) \subseteq Y \subseteq \mathbb{R}^X$, it turns out that $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$. For the converse, note that each K-analytic space has a resolution consisting of compact sets [67].

Theorem 3.4 (Arkhangelskii-Calbrix, [4, Theorem 2.3]). If $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$, then $X$ is projectively $\sigma$-compact.

Proof. Assume $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$. Let $Y$ be a separable metric space that is a continuous image of $X$, say $f : X \to Y$. Consider the pullback $f^* : \mathbb{R}^Y \to \mathbb{R}^X$ defined by $f^*(g) = g \circ f$, which is a linear homeomorphism onto $f^*(\mathbb{R}^X)$ with closed range [1, 0.4.6 Proposition]. If $S$ is a K-analytic space such that $C(X) \subseteq S \subseteq \mathbb{R}^X$, then $f^*(C(Y)) \subseteq S \cap f^*(\mathbb{R}^X)$, which is a K-analytic subspace of $\mathbb{R}^X$, since $S \cap f^*(\mathbb{R}^X)$ is closed in $S$. Hence $T := (f^*)^{-1}(S) \cap \mathbb{R}^X$ is a K-analytic subspace of $\mathbb{R}^Y$ such that $C(Y) \subseteq T \subseteq \mathbb{R}^Y$, i.e., $C_p(Y)$ is K-analytic-framed in $\mathbb{R}^Y$. So, if $\mathbb{R}^X$ are the nonnegative reals, since there exists a (strictly increasing) homeomorphism from $\mathbb{R}$ onto $\mathbb{R}^+$, there exists a K-analytic subspace $M$ of $\mathbb{R}^X$ such that $C^+(Y) := C(Y) \cap \mathbb{R}^+_X$ is contained in $M$. Let $\varphi : \mathbb{N}^X \to \mathcal{K}(M)$, where $\mathcal{K}(M)$ designates the family of compact sets of $M$, an usc map such that $\bigcup \varphi(a) : a \in \mathbb{N}^X = M$. Define $\lambda : \mathbb{N}^X \to \mathbb{R}^+_X$ by $\lambda(a) = \inf \{\beta \in \mathbb{N}^X : \beta \leq a\}$. As $[\beta \in \mathbb{N}^X : \beta \leq a]$ is a compact set in $\mathbb{N}^X$, $\varphi(\{\beta \in \mathbb{N}^X : \beta \leq a\})$ is a compact set in $M$ and the infimum is with respect to the pointwise ordering of $\mathbb{R}^X$, hence $\lambda(a)(y) = \inf \{\varphi(\beta)(y) : \beta \leq a\} > 0$ for each $y \in Y$. Clearly $\lambda(a)(y) = \lambda(\beta)(y)$ whenever $\beta \leq a$, and if $f \in C^+(Y) \subseteq M$ there is $y \in \mathbb{N}^X$ such that $f \in \varphi(\gamma)$, so that $\lambda(\gamma) \leq f$. Let $\{\overline{\gamma}, d\}$ be a metric compactification of $Y$. For each $a \in \mathbb{N}^X$ set $K_a = \cap \{\overline{\gamma} \setminus B(y, \lambda(a)(y)) : y \in Y\}$, where $B(y, \lambda(a)(y)) = \{z \in \overline{\gamma} : d(y, z) < \lambda(a)(y)\}$ is the open ball in $\overline{\gamma}$ of center $y$ and radius $\lambda(a)(y) \geq 0$. Clearly $K_a$ is a compact set in $\overline{\gamma} \setminus Y$, and we claim that $[K_a : a \in \mathbb{N}^X]$ is a compact resolution for $\overline{\gamma} \setminus Y$ that swallows the compact sets in $\overline{\gamma} \setminus Y$. The relation $K_a \subseteq K_\beta$ comes from $\lambda(\beta) \leq \lambda(a)$ whenever $\beta \leq a$. In addition, if $Q$ is a compact set in $\overline{\gamma} \setminus Y$, the function $h : \overline{\gamma} \setminus Y \to \mathbb{R}_X$, defined by $h(y) = d(y, Q)$ belongs to $C^+(Y)$ when restricted to $Y$. So, there is $y \in \mathbb{N}^X$ such that $\lambda(\gamma) \leq h(y)$. Thus $d(y, z) \geq \lambda(\gamma)(y)$ for every $y \in Y$ and $z \in Q$. In other words, $Q \cap \bigcup \{B(y, \lambda(a)(y)) : y \in Y\} = \emptyset$, which means that $Q \subseteq K_a$. In this circumstances, Christensen’s theorem [15, Theorem 3.3] shows that $\overline{\gamma} \setminus Y$ is a Polish space, so an absolute $G_\delta$ [51, Chapter 6, Problem K]. Consequently, $Y$ is an $F_\sigma$ of the compact space $\overline{\gamma}$, i.e., $Y$ is a $\sigma$-compact space.

Corollary 3.5. If $C_p(X)$ admits a resolution consisting of pointwise bounded sets, then $X$ is projectively $\sigma$-compact.

Proof. This is a straightforward consequence of Theorems 3.3 and 3.4.

Theorem 3.6 (Ferrando-Kąkol, [29, Corollary 1]). Let $X$ be a cosmic space. $C_p(X)$ has a resolution of pointwise bounded sets if and only if $X$ is $\sigma$-compact.

Proof. The ‘only if’ statement is consequence of Corollary 3.5 and the fact, mentioned earlier, that each projectively $\sigma$-compact cosmic space is $\sigma$-compact. For the ‘if’ part note that if $X = \bigcup_{n=1}^\infty K_n$ with each $K_n$ compact, the family $\{A_{\alpha} : \alpha \in \mathbb{N}^n\}$ with

$$A_{\alpha} = \{f \in C(X) : \sup_{x \in K_n} |f(x)| \leq \alpha(n), n \in \mathbb{N}\}$$

is a resolution for $C(X)$ consisting of pointwise bounded sets.

Theorem 3.7 (Calbrix [9, Theorem 2.3.1]). If $C_p(X)$ is analytic, then $X$ is $\sigma$-compact.
I.4.3. (21). This forces to C IV .6.15 Proposition. it turns out that X is not true. Since [15, Theorem 3.7] (σ a metrizable bounded sets entails that X is a cosmic space. So, Theorem 3.6 yields the implication 3 ⇒ 2. Finally, if X is a metrizable σ-compact space then X is separable. Thus C p (X) is analytic by a classic result of Christensen [15, Theorem 3.7] (cf. Theorem 4.4 below). Hence 2 ⇒ 1.

Corollary 3.8. If X is metrizable, the following are equivalent.
1. C p (X) is analytic.
2. X is σ-compact.
3. C p (X) has a resolution of pointwise bounded sets.

Proof. 1 ⇒ 2 follows from Theorem 3.7 and, as mentioned above, 2 ⇒ 3 always holds true. On the other hand, if C p (X) has a resolution of pointwise bounded sets, then C p (X) is K-analytic-framed in Rω by Theorem 3.3 and angelic by Lemma 3.1. But if X is metrizable, C p (X) is angelic if and only if X is separable [49, Corollary 6.10]. Consequently, for metrizable X, the fact that C p (X) has a resolution of pointwise bounded sets entails that X is a cosmic space. So, Theorem 3.6 yields the implication 3 ⇒ 2. Finally, if X is a metrizable σ-compact space then X is separable. Thus C p (X) is analytic by a classic result of Christensen [15, Theorem 3.7] (cf. Theorem 4.4 below). Hence 2 ⇒ 1.

Corollary 3.9. If C p (C p (X)) has a resolution consisting of pointwise bounded sets, then X is pseudocompact.

Proof. If X is not pseudocompact, then C p (X) contains a complemented (linearly homeomorphic) copy of Rω. If P is a continuous linear projection from C p (X) onto the linear subspace Rω the (linear) restriction map T : C p (C p (X)) → C p (Rω) given by Tψ = ϕ|Rω is continuous and onto, for if ψ ∈ C(Rω) then ψ ◦ P ∈ C(C p (X)) and T(ψ ◦ P) = ψ due to Pg = g for every g ∈ Rω. Hence T carries a resolution from C p (C p (X)) onto C p (Rω) made up of pointwise bounded sets. Since Rω is metrizable, Corollary 3.8 shows that Rω is a σ-space, which is not true.

Theorem 3.10 (Tkachuk [71, 2.8 Theorem]). C p (X) has a resolution consisting of compact sets if and only if it is K-analytic.

Proof. If C p (X) has a resolution consisting of compact sets, then C p (X) is a quasi-Suslin space [11, Proposition 1]. But, according to Lemma 3.1, the space C p (X) is angelic, and every quasi-Suslin angelic space is K-analytic [11]. The converse can be found in [67] or in [49, Theorem 3.2].

The following result was stated and proved by Tkachuk, [71, 3.9 Theorem]. However, it can also be derived as a consequence of Valdivia’s closed graph theorem for K-analytic spaces [76, Chapter I] (as mentioned in [71]), which is the approach we choose.

Theorem 3.11. Assume C p (X) is a Baire space. C p (X) has a resolution of compact sets if and only if X is countable and discrete.

Proof. According to Theorem 3.10, if C p (X) has a resolution of compact sets then C p (X) is K-analytic. Hence C p (X) is a locally convex space which is both Baire and K-analytic, so a separable Fréchet space by [76, I.4.3.21]). This forces to C p (X) = Rω with X countable. Hence X is countable and discrete.

Theorem 3.12 (Arkhangelskii). If C p (X) is both Baire and a Lindelöf Σ-space, then X is countable.

Proof. Let us prove this result with the additional assumption that X is realcompact. A proof of the general case can be found in [71, 3.8 Theorem]. If C p (X) is a Baire space, it is barrelled, i.e., each closed absorbing absolutely convex set is a neighborhood of the null function. Hence, by the Buchwalter-Schmets theorem, the functionally bounded sets in X are finite [8] (see also [1, I.3.4 Theorem]). If C p (X) is a Lindelöf Σ-space, then νX is a Lindelöf Σ-space by Lemma 3.1 (see also [59, Theorem 3.5]). Since by assumption X = νX, it turns out that X is a Lindelöf Σ-space with finite compact sets. Consequently X must be countable [1, IV.6.15 Proposition].
Theorem 3.13. Let $C_p(X)$ be a Baire space. If $C_p(X)$ has a resolution of pointwise bounded sets, then $X$ is countable.

Proof. This follows from a general property of locally convex spaces which assures that each locally convex Baire space $E$ with a resolution of bounded sets is metrizable (see [50, Corollary 1]). Let us try a direct approach. Let $\{A_\alpha : \alpha \in N^N\}$ be a resolution for $C_p(X)$ consisting of absolutely convex pointwise bounded sets. Define $\beta (1) = n_1$, $\beta (i + 1) = \alpha (i)$ for each $i \in N$, and set $B_\beta := n_1 \text{abx} (A_\alpha)$ where abx $(A_\alpha)$ stands for the absolutely convex cover of $A_\alpha$ and the closure is in $R^X$. Thus $Z := \bigcup \{B_\beta : \beta \in N^N\}$ is a linear subspace of $R^X$, and each set $B_\beta$ is compact with $B_\alpha \subseteq B_\beta$ if $\alpha \leq \beta$. So, $Z$ is a locally convex Baire space with a resolution of compact sets. By [31, Theorem 1], $Z$ is a separable Fréchet space. Hence $C_p(X)$ is metrizable, so $X$ must be countable. 

Theorem 3.14. Let $X$ be a paracompact locally compact space. $C_p(X)$ has a resolution of pointwise bounded sets if and only if $X$ is $\sigma$-compact.

Proof. As follows from [7, 9, Theorem 5] the space $X$ is the topological sum $\bigoplus_{\alpha \in A} X_\alpha$ of a family $\{X_\alpha : \alpha \in A\}$ of locally compact $\sigma$-compact (pairwise disjoint) subspaces of $X$. Consequently, $C_p(X) = \prod_{\alpha \in A} C_p(X_\alpha)$ isomorphic. By Theorem 3.13, $C_p(X)$ contains a copy of $R^A$. If $C_p(X)$ has a resolution of pointwise bounded sets, the subspace $R^A$ of $C_p(X)$ also has a resolution of pointwise bounded sets. Since $R^A$ is a Baire space, Theorem 3.13 shows that $A$ must be countable. So, $X$ is $\sigma$-compact. The converse also holds as shown in the ‘if’ part of Theorem 3.6.

The preceding theorem was originally stated as a part of [10, Proposition 2.2] assuming $C_p(X)$ is $K$-analytic.

Theorem 3.15 (Tkachuk, [71, 3.7 Theorem]). $C_p(X)$ has a resolution of compact sets that swallows the compact sets if and only if $X$ is countable and discrete.

Proof. Assume $\{A_\alpha : \alpha \in N^N\}$ is a resolution for $C_p(X)$ of compact sets that swallows the compact sets of $C_p(X)$. We claim that compact subsets of $X$ are finite. Otherwise there exists an infinite compact set $K$ in $X$. Since, according to Theorem 3.10, $C_p(X)$ is $K$-analytic, it turns out that $C_p(C_p(X))$ is angelic [24, Theorem 78]. As $K$ is embedded in $C_p(C_p(X))$, it must be a Fréchet-Urysohn compact, so there is a non trivial sequence $\{x_n\}_{n=1}^\infty$ that converges to some $x \in K$. Let $S = \{x_n : n \in N\} \cup \{x\}$, so that $S$ is a countable compact set, hence metrizable. Thus, there is a linear extender map $\varphi : C_p(S) \to C_p(X)$, i.e., such that $\varphi (f|_S) = f$ for every $f \in C(X)$, which embeds $C_p(S)$ into a closed linear subspace of $C_p(X)$, [5, Proposition 4.1]. Therefore the metrizable space $C_p(S)$ also has a resolution of compact sets that swallows the compact sets in $C_p(S)$. According to Christensen’s theorem [24, Theorem 94] this means that $C_p(S)$ is a Polish space. Hence, [1, I.3.3 Corollary] ensures that the compact set $S$ is discrete, hence finite. This contradiction ensures that the compact sets in $X$ are finite.

Since $C_p(X)$ is $K$-analytic, Lemma 3.1 asserts that $\nu X$ is a Lindelöf $\Sigma$-space. But a Lindelöf $\Sigma$-space with finite compact sets is countable [1, IV.6.15 Proposition], so $X$ is countable. On the other hand, if $Q$ is a compact set in $C_p(X)$ there is $\gamma \in N^N$ such that $Q \subseteq A_\gamma$. Hence, $\{A_\alpha : \alpha \in N^N\}$ is a resolution of compact sets for the metrizable space $C_p(X)$ that swallows the compact sets of $C_p(X)$. So, again $C_p(X)$ is a Polish space by Christensen’s theorem, and one more time [1, I.3.3 Corollary] asserts that $X$ is discrete. For the converse, note that $C_p(X)$ coincides with $R^N$ whenever $X$ is countable and discrete. Then $\{A_\alpha : \alpha \in N^N\}$ with $A_\alpha = \{x \in R^N : |x|_\alpha \leq \alpha_\alpha\}$ is a resolution for $C_p(X) = R^N$ consisting of compact sets that swallows the compact sets in $R^N$.

Theorem 3.16 (Ferrando-Gabriyelyan-Kąkol [28, Theorem 3.3]). $C_p(X)$ has a resolution of pointwise bounded sets that swallows the pointwise bounded sets if and only if $X$ is countable. In other words, $C_p(X)$ has a fundamental resolution of pointwise bounded sets if and only if $X$ is countable.
Proof. (Sketch) If \( C_\alpha(X) \) admits a fundamental resolution of pointwise bounded sets one can fix [28, Theorem 3.3] a countable family of closed sets (some of them may be empty) \( \mathcal{K} = \{ K_\alpha : n \in \mathbb{N}, \alpha \in \mathbb{N}\} \) in \( X \) enjoying the properties:

1. \( K_\alpha \subseteq K_{\alpha+1} \) for every \( n \in \mathbb{N} \) and each \( \alpha \in \mathbb{N} \).
2. \( K_\alpha \supseteq K_\beta \) for every \( n \in \mathbb{N} \) whenever \( \alpha \leq \beta \).
3. \( \bigcup_{n \in \mathbb{N}} K_\alpha = X \) for each \( \alpha \in \mathbb{N} \).
4. For every increasing closed covering \( \{ V_n : n \in \mathbb{N} \} \) of \( X \) there exists \( \gamma \in \mathbb{N} \) such that \( K_\alpha \subseteq V_n \) for all \( n \in \mathbb{N} \).

Then it turns out that the family \( \mathcal{N} := \{ N_{m_n}(\alpha) : m, n \in \mathbb{N}, \alpha \in \mathbb{N}\} \), where

\[
N_{m_n}(\alpha) := \left\{ f \in C(X) : |f(x)| \leq \frac{1}{m} \ \forall x \in K_\alpha \right\}
\]

and \( N_{m_n}(\alpha) := \{ 0 \} \) if \( K_\alpha \) is empty, is a countable \( cs^* \)-network at the origin in \( C_\alpha(X) \) (see [28, Proposition 3.2] or [24, Claim 108] for details). So, according to [65, Theorem 2.3], \( X \) must be countable. \( \square \)

Recall that a locally convex space \( E \) is a quasi-(LB)-space if \( E \) has a resolution consisting of Banach disks, i.e., of absolutely convex bounded sets \( D \) whose linear span \( E_D \) is a Banach space when equipped with the Minkowski functional of \( D \) as a norm.

**Theorem 3.17 (Valdivia, [77]).** If \( E \) is a quasi-(LB)-space, there exists a resolution for \( E \) consisting of Banach disks that swallows the Banach disks of \( E \).

Proof. (Sketch) Let \( \{ D_\alpha : \alpha \in \mathbb{N}\} \) be a resolution for \( E \) consisting of Banach disks. For \( (n_1, \ldots, n_k) \in \mathbb{N}^k \) define the absolutely convex set

\[
C_{n_1 \ldots n_k} = \bigcup\{ D_\alpha : \alpha \in \mathbb{N}, \alpha (i) = n_i, 1 \leq i \leq k \}.
\]

If \( \alpha \in \mathbb{N}^N \) and \( U \) a neighborhood of the origin in \( E \) it can be easily seen that there exists \( k(\alpha, U) \in \mathbb{N} \) such that \( C_{\alpha(1), \ldots, \alpha(k)} \subseteq kU \). So, if we set \( F_{\alpha(1), \ldots, \alpha(k)} := \text{span} \{ C_{\alpha(1), \ldots, \alpha(k)} \} \) for every \( \alpha \in \mathbb{N} \) and \( F_\alpha := \bigcap \{ F_{\alpha(1), \ldots, \alpha(k)} : k \in \mathbb{N} \} \), the sequence

\[
\left\{ F_\alpha \cap k^{-1} C_{\alpha(1), \ldots, \alpha(k)} : k \in \mathbb{N} \right\}
\]

is a base of absolutely convex neighborhoods of the origin in the linear subspace \( F_\alpha \) of a locally convex topology \( \tau_\alpha \) stronger than the relative topology of \( E \). In fact, it turns out that \( \{ (F_\alpha, \tau_\alpha) : \alpha \in \mathbb{N}^N \} \) is a family Fréchet spaces [77, Proposition 21] which covers \( E \). Now, for \( \alpha \in \mathbb{N}^N \) set \( \overline{\alpha}(i) = \alpha (2i-1) \) for each \( i \in \mathbb{N} \) and define

\[
Q_\alpha = \bigcap_{k=1}^{\infty} \alpha (2k) \cdot \left( F_\alpha \cap C_{\alpha(1), \ldots, \alpha(k)} \right)
\]

The family \( \{ Q_\alpha : \alpha \in \mathbb{N}^N \} \) is clearly a resolution for \( E \), and consists of Banach disks. It remains to prove that this family swallows the Banach disks of \( E \). In order to establish this statement, choose a Banach disk \( D \) in \( E \) and consider the Banach space \( E_D \). Then consider the canonical inclusion \( J : E_D \rightarrow E \) and put \( U_{n_1 \ldots n_k} := J^{-1} \left( C_{n_1 \ldots n_k} \right) \). As \( E_D = \bigcup \{ U_n : n \in \mathbb{N} \} \) and \( U_{n_1 \ldots n_k} = \bigcup \{ U_{n_1 \ldots n_i n_{i+1}} : n_{i+1} \in \mathbb{N} \} \) for each \( k \in \mathbb{N} \), there is \( \beta \in \mathbb{N}^N \) such that \( U_{\beta(1), \ldots, \beta(k)} \subseteq E_D \) is a neighborhood of the origin in \( E_D \) for each \( k \in \mathbb{N} \). So, using the fact that \( \text{Int} \left( U_{\beta(1), \ldots, \beta(k)} \right) \subseteq U_{\beta(1), \ldots, \beta(k)} \) for each \( k \in \mathbb{N} \) (see [77] or [49, Proposition 3.21] for details), if \( x \in E_D \) and \( k \in \mathbb{N} \) there is \( \lambda > 0 \) such that \( \lambda x \in U_{\beta(1), \ldots, \beta(k)} \), which implies that \( J(x) \in F_{\beta(1), \ldots, \beta(k)} \). This shows that \( J(E) \subseteq F_\beta \). So, by the closed graph theorem \( J \) is a continuous linear map from \( E_D \) into \( F_\beta \). Hence, if we choose a sequence \( \{ m_k \}_{k=1}^{\infty} \) in \( \mathbb{N} \) such that

\[
D \subseteq m_k \cdot \left( F_\beta \cap C_{\beta(1), \ldots, \beta(k)} \right)
\]

for every \( k \in \mathbb{N} \), setting \( \gamma(2k) = m_k \) and \( \gamma(2k-1) = \beta(k) \) for each \( k \in \mathbb{N} \), it follows that \( D \subseteq Q_\gamma \). \( \square \)
Theorem 3.18 (Ferrando-Gabriyelyan-Ka˘kol [28, Proposition 3.6]). Let X be a P-space. \( C_p(X) \) has a resolution of pointwise bounded sets if and only if X is countable and discrete.

**Proof.** If X is a P-space then \( C_p(X) \) is locally complete [34, Theorem 1.1], i.e., each pointwise bounded set is contained in a Banach disk. So, according to Theorem 3.17 there exists a resolution for \( C_p(X) \) consisting of Banach disks that swallowing the pointwise bounded sets in \( C_p(X) \). Hence, X is countable by Theorem 3.16. But every countable P-space is discrete. \( \square \)

Alternatively, one may use the fact that \( C_p(X) \) is a Baire space (note that \( C_p(X) \) is pseudocomplete [72, Section 1.5, p. 46] whenever X is a P-space and use [72, Problem 464]). Then apply Theorem 3.13 to conclude that X must be countable, hence discrete.

Recall that a sequence \( \{x_n\}_{n=1}^\infty \) in a locally convex space E is called local null or Mackey convergent to zero [52, 28.3] if there is a closed disk \( B \) in E such that \( x_n \rightarrow 0 \) in the normed space \( EB \). Each local null sequence in E is a null sequence.

Theorem 3.19 (Ferrando, [25, Theorem 12]). \( C_p(X) \) admits a resolution of convex compact sets that swallows the local null sequences in \( C_p(X) \) if and only if X is countable and discrete.

**Proof.** We may assume that \( C_p(X) \) admits a resolution \( \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \) of absolutely convex compact sets swalling the local null sequences in \( C_p(X) \). If \( T : C_p(\nu X) \rightarrow C_p(X) \) denotes the restriction map \( Tg = g|_X \) we proceed as in [49, Proposition 9.14] to show that the family \( \mathcal{A} = \{T^{-1}(A_\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\} \) is a resolution for \( C_p(\nu X) \) consisting of (absolutely convex) compact sets, with the additional benefit that \( \mathcal{A} \) swallows the local null sequences in \( C_p(\nu X) \). So, we may assume without loss of generality that \( X \) is realcompact or, equivalently, that \( C_p(X) \) is bornological [8]. Hence, we denote as above by \( \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \) a resolution for \( C_p(X) \), with X realcompact, consisting of absolutely convex compact sets that swallows the local null sequences in \( C_p(X) \).

Let \( M \) denote the family of all local null sequences in \( C_p(X) \). Since \( \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \) swallows the members of \( M \), the Mackey* topology \( \mu(\sigma(L(X), C(X))) \) of \( L(X) \) is stronger than the topology \( \tau_{\omega_0} \) on \( L(X) \) of the uniform convergence on the local null sequences of \( C_p(X) \). As in addition \( \sigma(L(X), C(X)) \leq \tau_{\omega_0} \) we conclude that \( (L(X), \tau_{\omega_0})' = C(X) \). Moreover, since we are assuming that \( C_p(X) \) is bornological, its \( \tau_{\omega_0} \)-dual \( (L(X), \tau_{\omega_0}) \) is complete by [52, 28.5.1].

We claim that every compact set in \( X \) is finite. Indeed, if \( K \) is a compact set in \( X \), the homeomorphic copy \( \delta(K) \) of \( K \) in \( L_p(X) \) is compact, i.e., \( \delta(K) \) is a \( \sigma(L(X), C(X)) \)-compact set in \( L(X) \). So, the completeness of \( (L(X), \tau_{\omega_0}) \), together with Krein’s theorem and the fact that \( \tau_{\omega_0} \) is a locally convex topology of the dual pair \( (L(X), C(X)) \), ensures that the weak* closure \( Q = \text{ab}(\delta(K)) \) in \( L(X) \), where \( \text{ab}(\delta(K)) \) stands for the absolutely convex hull of \( \delta(K) \), is a compact set in \( L_p(X) \), hence a strongly bounded set. Since \( C_p(X) \) is quasi-barrelled [47, 11.7.3 Corollary], the strongly bounded sets in \( L(X) \) are finite-dimensional. Therefore the set \( \delta(K) \), as a linearly independent system of vectors in \( L(X) \), must be finite. Thus \( K \) is finite as well.

Since \( \nu X = X \) is a Lindelöf \( \Sigma \)-space by Lemma 3.1 and as we know each Lindelöf \( \Sigma \)-space with finite compact sets is countable [1, IV.6.15 Proposition], \( X \) is countable. So \( C_p(X) \) is a metrizable space. But in a metrizable locally convex space, the local null sequences and the null sequences are the same [52, 28.3.1]. Furthermore, if \( M \) is a compact set in the metrizable space \( C_p(X) \), then \( M \) lies in the closed absolutely convex cover of a null sequence \( \{f_\alpha\}_{\alpha=1}^\infty \) [52, 21.10.3]. So, if \( \{f_\alpha\}_{\alpha=1}^\infty \subseteq A_{\gamma} \), thanks to the fact that \( A_{\gamma} \) is a closed absolutely convex set, it turns out that \( M \subseteq A_{\gamma} \). Therefore \( \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \) is a compact resolution for \( C_p(X) \) that swallows the compact sets of \( C_p(X) \). So, \( C_p(X) \) is a Polish space by Christensen’s theorem [24, Theorem 94]. But then [1, I.3.3 Corollary] asserts that \( X \) is discrete. The converse is obvious. \( \square \)

Theorem 3.20 (Ferrando, [25, Theorem 16]). \( C_p(X) \) has a resolution of absolutely convex pointwise bounded sequentially complete sets that swallows the null sequences if and only if \( X \) is countable and discrete.

**Proof.** It can be readily seen that there is no loss of generality if we assume \( X \) to be realcompact. If \( C_p(X) \) has a resolution \( \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \) of the stated characteristics and \( \{f_\alpha\}_{\alpha=1}^\infty \) is a null sequence in \( C_p(X) \), there is
\( \gamma \in \mathbb{N}^\mathbb{N} \) such that \( f_{n} \in A_{\gamma} \) for every \( n \in \mathbb{N} \). Since \( \sum_{n=1}^{\infty} |x| f_{n} \in A_{\gamma} \) for every \( x \in \ell_{1} \) with \( \|x\|_{1} \leq 1 \) and \( A_{\gamma} \) is sequentially complete, it follows that \( \sum_{m=1}^{\infty} |x| f_{m} \in A_{\gamma} \) for every \( x \in \ell_{1} \) with \( \|x\|_{1} \leq 1 \). So, the Banach disk

\[
Q := \left\{ \sum_{i=1}^{n} |x| f_{i} : x \in \ell_{1}, \|x\|_{1} \leq 1 \right\}
\]

is contained in \( A_{\gamma} \). Now, it can be proved as in [52, 20.10.(6)] that \( Q = \{ f_{n} : n \in \mathbb{N} \} \), the absolute bipolar of the null sequence \( \{ f_{n} : n \in \mathbb{N} \} \). Since each local null sequence is a null sequence, the dual of \( (L(X), \tau_{\omega}) \) is \( C(X) \), so \( \sigma(L(X), C(X)) \leq \tau_{\omega} \leq \mu(L(X), C(X)) \). As \( C_{p}(X) \) is bornological, the space \( L(X) \) is \( \mu(L(X), C(X)) \)-complete. So, proceeding as in the proof of Theorem 3.19, with the help of Krein’s theorem we establish that each compact set in \( X \) is finite. Now, using the fact that the resolution \( \{ A_{\gamma} : \gamma \in \mathbb{N}^\mathbb{N} \} \) consists of pointwise bounded sets, Lemma 3.1 asserts that \( X \) is a Lindelöf \( \Sigma \)-space. Thus \( X \) must be countable, [1, IV.6.15 Proposition], so \( C_{p}(X) \) is metrizable.

If \( M \) is a compact set in the metrizable space \( C_{p}(X) \), as mentioned above \( M \) lies in the closed absolutely convex cover of a null sequence \( \{ f_{n} \}_{n=1}^{\infty} \). So, if \( \{ f_{n} \} \subseteq A_{\gamma} \) then \( M \subseteq A_{\gamma} \). Thus \( \{ A_{\gamma} : \gamma \in \mathbb{N}^\mathbb{N} \} \) is a resolution for \( C_{p}(X) \) that swallows the compact sets of \( C_{p}(X) \). Since each set \( A_{\gamma} \) is precompact in \( C_{p}(X) \) and sequentially complete, the metrizability of \( C_{p}(X) \) ensures that \( A_{\gamma} \) is compact in \( C_{p}(X) \). Hence \( C_{p}(X) \) is a Polish space by Christensen’s theorem. Thus \( X \) is discrete. The converse is clear, since each (absolutely convex) compact set in \( \mathbb{R}^{N} \) is pointwise bounded and sequentially complete. \( \Box \)

Another result of this type, which we state without proof is the following.

Theorem 3.21 (Ferrando, [25, Theorem 33]). Let \( X \) be first countable. \( C_{p}(X) \) has a resolution of pointwise bounded sets that swallows the Cauchy sequences if and only if \( X \) is countable.

4. Uncountable coverings for \( C_{k}(X) \)

Theorem 4.1 (Ferrando-Moll, [35, Corollary 5]). The space \( C_{k}(X) \) has a resolution consisting of compact sets if and only if it is \( K \)-analytic.

Proof. If \( C_{k}(X) \) has a resolution consisting of compact sets, so does \( C_{p}(X) \). So, Lemma 3.1 and Theorem 3.3 ensure that \( uX \) is a Lindelöf \( \Sigma \)-space and \( C_{k}(X) \) is angelic. Therefore \( C_{k}(X) \) is angelic as well [36, 3.3 Theorem]. Since \( C_{k}(X) \) is a quasi-Suslin space, necessarily \( C_{k}(X) \) must be \( K \)-analytic [11]. \( \Box \)

Theorem 4.2 (Gabriyelyan-Kačol [37, Corollary 2.10]). Let \( X \) be metrizable. \( C_{k}(X) \) has a resolution of compact sets that swallows the compact sets if and only if \( X \) is \( \sigma \)-compact.

Proof. If \( C_{k}(X) \) has a resolution of compact sets, \( C_{p}(X) \) has a resolution of pointwise bounded sets. So, Corollary 3.8 assures that \( X \) is \( \sigma \)-compact. Conversely, if \( \{ U_{m,n} : m \in \mathbb{N} \} \) is an increasing sequence of compact sets in \( X \) covering \( X \) then \( \Delta_{m} = \{(x, y) : x \in U_{m,n}\} \) is compact in the metric space \( (X \times X, d) \). Hence, the sequence \( \{ U_{m,n} : n \in \mathbb{N} \} \) where

\[
U_{m,n} = \left\{ (x, y) \in X \times X : d((x, y), \Delta_{m}) < n^{-1} \right\}
\]

is a basis of the system of neighborhoods of \( \Delta_{m} \). Let us encode in each \( \alpha \in \mathbb{N}^{\mathbb{N}} \) a whole sequence \( \{ x_{n} \}_{n=1}^{\infty} \) of elements of \( \mathbb{N}^{\mathbb{N}} \) by considering a bidimensional array whose \( i \)-th file is formed by coordinates \( (a_{1}(1), a_{2}(1), \ldots, a_{n}(n), \ldots) \) of \( a_{i} \) and defining \( a \) by setting \( a(1) = a_{1}(1) \), \( a(2) = a_{2}(2) \), \( a(3) = a_{3}(1) \), \( a(4) = a_{1}(3) \), \( a(5) = a_{2}(2) \), \( a(6) = a_{3}(1) \), \( a(7) = a_{1}(4) \), \ldots and so on. Conversely, given \( \alpha \in \mathbb{N}^{\mathbb{N}} \) we may extract a sequence \( \{ x_{m} \}_{m=1}^{\infty} \subseteq \mathbb{N}^{\mathbb{N}} \) from \( \alpha \) as indicated above. Then let \( A_{\alpha} \) be the absolutely convex set

\[
\left\{ f \in C(X) : \sup_{(x,y) \in U_{m,n}} |f(x) - f(y)| \leq \frac{1}{n}, \sup_{x \in A_{\alpha}} |f(x)| \leq \alpha_{m}(1) \quad \forall m, n \in \mathbb{N} \right\}
\]
Let $x \in X$ and $\epsilon > 0$ be given. Take $m \in \mathbb{N}$ such that $x \in K_m$ and $1/n < \epsilon$. Setting $U_{m,n}(x) := \{ y \in X : (x, y) \in U_{m,n} \}$, each $f \in A_\alpha$ satisfies
\[
\sup_{y \in U_{m,n}(x)} \left| f(x) - f(y) \right| \leq \sup_{(z, y) \in U_{m,n}} \left| f(z) - f(y) \right| \leq n^{-1} < \epsilon.
\]
As $U_{m,n}(\alpha)(x)$ is a neighborhood of $x$, this means that $A_\alpha$ is equicontinuous at $x$. So all sets $A_\alpha$ are equicontinuous. In addition, since $\sup_{f \in A_\alpha} \left| f(z) \right| \leq a_m(1)$ if $z \in K_m$, we see that $A_\alpha$ is pointwise bounded and closed. Hence $A_\alpha$ is a compact set in $C_k(X)$.

On the other hand, if $\mathcal{K}$ is a compact set in $C_k(X)$, the fact that $X$ is a $k_\mathbb{R}$-space guarantees that $\mathcal{K}$ is equicontinuous (Ascoli’s theorem). Since $\mathcal{K}$ is equicontinuous at each $x \in K_m$, for each $n \in \mathbb{N}$ there is $\epsilon(m, n, x) > 0$ such that
\[
\sup_{(x, y) \in U(x, m, n)} \left| f(x) - f(y) \right| \leq \frac{1}{2n}
\]
for all $f \in \mathcal{K}$, where $B(x, \epsilon)$ stands for the open ball of center at $x$ and radius $\epsilon > 0$.

Setting $U = \bigcup_{z \in K_m} B(z, \epsilon(m, n, z)) \times B(z, \epsilon(m, n, z))$, if $(x, y) \in U$ there is $z \in K_m$ such that $x, y \in B(z, \epsilon(m, n, z))$, so $\left| f(x) - f(y) \right| \leq \left| f(x) - f(z) \right| + \left| f(z) - f(y) \right| < n^{-1}$ for all $f \in \mathcal{K}$. As $\Delta_m \subseteq U$ there is $r(m, n) \in \mathbb{N}$ with $\Delta_m \subseteq U_{m,r(m,n)} \subseteq U$. Thus
\[
\sup_{(x, y) \in U_{m,r(m,n)}} \left| f(x) - f(y) \right| \leq \frac{1}{n}.
\]
On the other hand, the fact that $\mathcal{K}$ is a compact set for the compact-open topology ensures that for each $m \in \mathbb{N}$ there is $k_m \in \mathbb{N}$ such that $\sup_{f \in \mathcal{K}} \sup_{z \in K_m} \left| f(x) \right| \leq k_m$. Hence, setting $\alpha$ such that $\alpha_m(1) = r(m, n)$, we may assume that $\alpha_m(1) \geq k_m$. All this says that $\mathcal{K} \subseteq A_\alpha$. As $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$, the family $\{ A_\alpha : \alpha \in \mathbb{N}^\mathbb{N} \}$ is as stated.

**Corollary 4.3 (Ferrando [23, Proposition 3]).** Let $X$ be a metrizable space. $C_k(X)$ has a fundamental bounded resolution if and only if $X$ is $\sigma$-compact.

**Proof.** If $X$ is $\sigma$-compact, Theorem 4.2 ensures that $C_k(X)$ has a resolution consisting of compact sets that swallows the compact sets. So, $C_k(X)$ has a bounded resolution $\{ A_\alpha : \alpha \in \mathbb{N}^\mathbb{N} \}$ consisting of closed absolutely convex bounded sets. As $X$ is a $k_\mathbb{R}$-space, $C_k(X)$ is complete and consequently each $A_\alpha$ is a Banach disk. So, Theorem 3.17 provides a resolution $\{ A_\alpha : \alpha \in \mathbb{N}^\mathbb{N} \}$ for $C_k(X)$ consisting of Banach disks that swallows the Banach disks, hence the bounded sets in $C_k(X)$. Thus, $C_k(X)$ has a fundamental bounded resolution. The converse comes from Corollary 3.8.

**Theorem 4.4 (Christensen [15, Theorem 3.7]).** Let $X$ be a separable metric space. $C_k(X)$ is analytic if and only if $X$ is $\sigma$-compact.

**Proof.** If $C_k(X)$ is analytic then $C_\tau(X)$ is analytic as well, so Calbrix’s theorem ensures that $X$ is $\sigma$-compact. If $X$ is $\sigma$-compact then $C_k(X)$ has a resolution of compact sets by Theorem 4.2. Hence $C_k(X)$ is $K$-analytic by Theorem 4.1. As $X$ is a separable metric space, it is a cosmic space, and so is $C_\tau(X)$. So, $C_\tau(X)$ being $K$-analytic and cosmic is analytic. Hence $C_\tau(X)$ must be submetrizable by the second statement of [24, Theorem 85] (see [66, Proposition 6.3]). Consequently, $C_k(X)$ is $K$-analytic and submetrizable, hence analytic by the first statement of [24, Theorem 85].

If $\mathcal{N}$ is a uniformity for a (nonempty) set $X$, we denote by $\tau_\mathcal{N}$ the uniform topology defined by $\mathcal{N}$. A base $\{ U_\alpha : \alpha \in \mathbb{N}^\mathbb{N} \}$ of $\mathcal{N}$ is called a $\Theta$-base if $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$. There is no loss of generality by assuming that each $U_\alpha$ is a symmetric vicinity. On the other hand, if $\{ U_\lambda : \lambda \in \Lambda \}$ is the family of all admissible uniformities for a completely regular space $(X, \tau)$, the smallest uniformity $\mathcal{U}_{\text{ad}}$ that makes all $\tau$-continuous functions $f : X \to \mathbb{R}$ uniformly continuous, is called the Nachbin uniform structure of $X$, [61].
Theorem 4.5 (Ferrando, [21, Theorem 1]). $C_b(X)$ has a resolution consisting of equicontinuous sets if and only if there exists an admissible uniformity for $X$, larger than or equal to the Nachbin uniformity, with a $\Theta$-base.

Proof. Assume $\mathcal{N}$ is a uniformity for $X$ which contains the Nachbin uniform structure and let $\{U_\alpha : \alpha \in \mathbb{N}^N\}$ be a $\Theta$-base of $\mathcal{N}$. If $\{a_n\}_{n=1}^\infty$ is a sequence in $\mathbb{N}^N$, encode $\{a_n\}_{n=1}^\infty$ in $\alpha$ as indicated in the proof of Theorem 4.2 and define

$$P_\alpha = \left\{ f \in C(X) : \sup_{(x,y) \in U_\alpha} |f(x) - f(y)| \leq \frac{1}{n} \forall n \in \mathbb{N} \right\}.$$ 

We claim that $\{P_\alpha : \alpha \in \mathbb{N}^N\}$ is a resolution for $C_b(X)$ consisting of equicontinuous sets. In fact, since if $\alpha \leq \beta$ then $a_n \leq b_n$ for every $n \in \mathbb{N}$, clearly $P_\alpha \subseteq P_\beta$. On the other hand, if $f \in C(X)$, since $\mathcal{N}$ is larger than the Nachbin uniformity, $f$ is $\mathcal{N}$-uniformly continuous on $X$. Bearing in mind that $\{U_\alpha : \alpha \in \mathbb{N}^N\}$ is a $\Theta$-base of $\mathcal{N}$, for each $n \in \mathbb{N}$ there exists $a_n \in \mathbb{N}^N$ such that $|f(x) - f(y)| \leq 1/n$ whenever $(x,y) \in U_{a_n}$, which shows that $f \in P_\alpha$ for $\alpha$ defined as above. Finally, let us see that each set $P_\alpha$ is equicontinuous. Indeed, given $\epsilon > 0$ take $n \in \mathbb{N}$ such that $1/n < \epsilon$. According to the definition of $P_\alpha$, there is $a_n \in \mathbb{N}^N$, which we extract from $\alpha$ as explained earlier, such that $|f(x) - f(y)| < \epsilon$ whenever $(x,y) \in U_{a_n}$ and this happens for every $f \in P_\alpha$, which shows that $P_\alpha$ is uniformly equicontinuous, hence equicontinuous.

For the converse, suppose that $\{P_\alpha : \alpha \in \mathbb{N}^N\}$ is a resolution of $C_b(X)$ consisting of equicontinuous sets. For each $\alpha \in \mathbb{N}^N$ define

$$V_\alpha = \{(x,y) \in X \times X : \sup_{f \in P_\alpha} |f(x) - f(y)| < \alpha \downarrow 1\}.$$ 

If $\alpha \leq \beta$ then $P_\alpha \subseteq P_\beta$, which implies that $V_\beta \subseteq V_\alpha$. Let us see that $\{V_\alpha : \alpha \in \mathbb{N}^N\}$ is a base of some uniformity $\mathcal{N}$ for $X$. First observe that the diagonal $\Delta(X) = \{(x,x) : x \in X\}$ is contained in each $V_\alpha$, so no $V_\alpha$ is empty. On the other hand, clearly $\{V_\alpha : \alpha \in \mathbb{N}^N\}$ is a filter-base with $V_\alpha^{-1} = V_\alpha$. In addition, if $\beta \in \mathbb{N}^N$ satisfies that $\beta \geq \alpha$ with $\beta(1) \geq 2\alpha(1)$ we claim that $V_\beta \circ V_\beta \subseteq V_\alpha$. Indeed, if $(x,y) \in V_\beta \circ V_\beta$ there is $z \in X$ with $(x,z),(z,y) \in V_\beta$. Hence $|f(x) - f(z)| < \beta(1)^{-1}$ and $|f(z) - f(y)| < \beta(1)^{-1}$ for every $f \in P_\beta$. So, $|f(x) - f(y)| < 2\beta(1)^{-1} \leq \alpha(1)^{-1}$ for all $f \in P_\alpha \subseteq P_\beta$, which shows that $(x,y) \in V_\alpha$.

Let us check that $\mathcal{N}$ is an admissible uniformity for $X$, i.e., that $\tau_{\mathcal{N}}$ coincides with the original topology of $X$. Since $X$ is completely regular, it suffices to show that $X$ and $(X,\tau_{\mathcal{N}})$ have the same continuous functions. Take $f \in C(X)$, pick an arbitrary point $x_0 \in X$ and choose $\epsilon > 0$. Then select $\alpha \in \mathbb{N}^N$ such that $f \in P_\alpha$ and $\alpha(1)^{-1} < \epsilon$. Clearly

$$V_\alpha(x_0) = \{y \in X : (x_0,y) \in V_\alpha\}$$

is a $\tau_{\mathcal{N}}$-neighborhood of $x_0$, and since $|f(x) - f(y)| < \alpha(1)^{-1} < \epsilon$ for every $(x,y) \in V_\alpha$, we have in particular that $|f(x_0) - f(y)| < \epsilon$ for all $y \in V_\alpha(x_0)$. This shows that $f$ is continuous at $x_0$ under $\tau_{\mathcal{N}}$. Assume conversely that $f \in C(X,\tau_{\mathcal{N}})$ and fix $x_0 \in X$ and $\epsilon > 0$. Then there is $\alpha \in \mathbb{N}^N$ with $|f(x_0) - f(y)| < \epsilon$ for every $y \in V_\alpha(x_0)$. But, since $P_\alpha$ is equicontinuous at $x_0$, there exists a neighborhood $V$ of $x_0$ of the original topology of $X$ such that $\sup_{n \in \mathbb{N}} \|h(y) - h(x_0)\| < \alpha(1)^{-1}$ for every $y \in V$. Hence if $x \in V$ then $\sup_{n \in \mathbb{N}} \|h(x) - h(x_0)\| < \alpha(1)^{-1}$, which according to the definition of $V_\alpha$ means that $x \in V_\alpha(x_0)$. This shows that $V \subseteq V_\alpha(x_0)$ and thus $|f(x_0) - f(y)| < \epsilon$ for all $y \in V$. So $f$ is continuous at $x_0$ under the original topology of $X$ and $f \in C(X)$.

Let us finally check that the uniformity $\mathcal{N}$ generated by the base $\{V_\alpha : \alpha \in \mathbb{N}^N\}$ is larger than the Nachbin uniformity. We have to prove that every real-valued continuous function on $X$ is $\mathcal{N}$-uniformly continuous. Now, given $f \in C(X)$ and $\epsilon > 0$, taking advantage of the fact that $\{P_\alpha : \alpha \in \mathbb{N}^N\}$ is a resolution of $C(X)$, we can choose $\gamma \in \mathbb{N}^N$ such that $\gamma(1)^{-1} < \epsilon$ and $f \in P_\gamma$. Consequently, for each $(x,y) \in V_\gamma$, it happens that $|f(x) - f(y)| < \gamma(1)^{-1} < \epsilon$, which shows that $f$ is $\mathcal{N}$-uniformly continuous, as stated.

Corollary 4.6. Let $X$ be a $k_\mathbb{R}$-space. If $C_b(X)$ is K-analytic then there exists an admissible uniformity for $X$, larger than or equal to the Nachbin uniformity, with a $\Theta$-base.
Theorem 4.7 (Ferrando-Gabriyelyan-Kakol, [27, Theorem 1.8]). \( C_k(X) \) has a resolution consisting of weakly compact sets that swallows the weakly compact sets if and only if \( X \) is countable and discrete.

Proof. First we claim that if \( C_k(X) \) has a resolution \( \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \) consisting of weakly compact sets that swallows the weakly compact sets in \( C_k(X) \), each compact set in \( X \) is finite. As \( C_p(X) \) admits a resolution of compact sets, it is \( K \)-analytic by Theorem 3.10, so \( C_p(C_k(X)) \) is angelic by Lemma 3.1. Hence, each compact set of \( X \) is Fréchet-Urysohn. If there exists an infinite compact set \( K \) in \( X \), then \( K \) contains an infinite convergent sequence that, together with its limit, is homeomorphic to a metrizable compact subset of \( \beta X \). Thus, there is a continuous linear extender map \( \psi : C_p(Q) \to C_p(\beta X) \), [5]. If \( S : C_p(\beta X) \to C_p(X) \) is the restriction map \( Sg = g|_X \), the mapping \( \psi = S \circ \phi \) is a continuous linear extender, i.e., \( \psi(f)|_Q = f \) for every \( f \in C(Q) \). This ensures that the linear map \( \psi : C(Q) \to C_k(X) \) is weakly compact set with \( X \) equipped with its weak topology, has closed graph. Since \( C(Q) \) is a Banach space and \( C_k(X) \) has a resolution of compact sets, the closed graph theorem [31, Theorem 1] ensures that \( \psi : C(Q) \to C_k(X) \) (weak) is continuous, so weakly continuous.

A routine procedure shows that the family \( \{\psi^{-1}(A_\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\} \) is a resolution for the Banach space \( C(Q) \) consisting of weakly compact sets. If \( P \) is a compact set under the weak topology of \( C_k(Q) \), then \( \psi(P) \) is a compact set in \( C_k(X) \) (weak). Hence, there is a \( \gamma \in \mathbb{N}^\mathbb{N} \) such that \( \psi(P) \subseteq A_\gamma \), so that \( P \subseteq \psi^{-1}(A_\gamma) \). This means that \( \{\psi^{-1}(A_\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\} \) swallows the weakly compact sets of \( C(Q) \) (weak). It is shown in [55] that for compact \( Q \), if the Banach space \( C(Q) \) has a resolution of weakly compact sets that swallows the weakly compact sets, then \( Q \) is finite. Thus \( Q \) must be a finite set, a contradiction.

Finally, since each compact set in \( X \) is finite, one has \( C_k(X) = C_k(X) \) (weak) = \( C_p(X) \). So \( X \) must be countable and discrete by Theorem 3.15.

A Fréchet space \( E \) is called a Strongly Weakly Countably Generated (briefly a SWCG) space if every bounded set in \( (E', \mu(E', E)) \) is metrizable. Equivalently, \( E \) is a SWCG space if given a base of closed absolutely convex neighborhoods of zero \( \{U_n : n \in \mathbb{N}\} \) with \( 2U_{n+1} \subseteq U_n \) for each \( n \in \mathbb{N} \) there exists an absolutely convex weakly compact set \( K \subseteq E \) such that for every weakly compact set \( L \subseteq E \) and every \( n \in \mathbb{N} \) there is \( \alpha(n) \in \mathbb{N} \) with \( L \subseteq \alpha(n)K + U_n \) [30, Theorem 9]. A Fréchet space \( E \) is called Strongly Weakly K-Analytic (briefly SWKA) space if \( (E, \sigma(E, E')) \) admits a compact resolution that swallows the \( \sigma(E, E') \)-compact sets.

If \( E \) is a Fréchet space with a base of closed absolutely convex neighborhoods of zero \( \{U_n : n \in \mathbb{N}\} \) such that \( 2U_{n+1} \subseteq U_n \) for each \( n \in \mathbb{N} \), a resolution \( \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \) for \( E \) is called weakly compactly generated if there exists an absolutely convex weakly compact set \( K \) such that

\[
A_\alpha = \bigcap_{n=1}^{\infty} (\alpha(n)K + U_n)
\]

for every \( \alpha \in \mathbb{N}^\mathbb{N} \). Clearly \( A_\alpha \subseteq A_\beta \) whenever \( \alpha \leq \beta \), and the condition imposed to the base implies that each \( A_\alpha \) is closed. Hence \( \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \) is a weakly compact resolution for \( E \), as follows from [30, Claim 6].

Theorem 4.8. A Fréchet space \( E \) is SWCG if and only if \( E \) has a weakly compactly generated resolution that swallows the weakly compact sets.

Proof. Assume that \( E \) is a SWCG space, and let \( \{U_n : n \in \mathbb{N}\} \) be a base of closed absolutely convex neighborhoods of the origin such that \( 2U_{n+1} \subseteq U_n \) for every \( n \in \mathbb{N} \). For every \( \alpha \in \mathbb{N}^\mathbb{N} \) set \( A_\alpha := \bigcap_{n=1}^{\infty} (\alpha(n)K + U_n) \), where \( K \) is the absolute convex weakly compact set mentioned after the definition of SWCG space. Clearly \( \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \) is a weakly compactly generated resolution for \( E \). If \( L \subseteq E \) is a weakly compact set in \( E \), for each \( n \in \mathbb{N} \) there exists \( \gamma(n) \in \mathbb{N} \) such that \( L \subseteq \gamma(n)K + U_n \), so that \( L \subseteq A_\gamma \). Hence \( \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \) swallows the weakly compact sets of \( E \).

Assume conversely that \( E \) contains a weakly compactly generated resolution \( \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \) that swallows the weakly compact sets. Then there exists a weakly compact absolutely convex set \( Q \) such that \( A_\alpha = \bigcap_{n=1}^{\infty} (\alpha(n)Q + U_n) \) for every \( \alpha \in \mathbb{N}^\mathbb{N} \). If \( L \) is any weakly compact set in \( E \) there is \( \gamma \in \mathbb{N}^\mathbb{N} \) such that \( L \subseteq A_\gamma \), hence for each \( n \in \mathbb{N} \) one gets \( L \subseteq \gamma(n)Q + U_n \). So \( E \) is a SWCG space.
Theorem 4.9 (Ferrando-Kąkol, [30, Theorem 22]). If $C_p(X)$ is a Fréchet space, the following statements are equivalent

1. $C_p(X)$ is a SWCG space.
2. $C_p(X)$ is a SWKA space.
3. $X$ is countable and discrete.

Proof. Clearly 1 $\Rightarrow$ 2. Equivalence 2 $\Leftrightarrow$ 3 is consequence of Theorem 4.7. If $X$ is countable and discrete then $C_p(X) = R^X$ is reflexive, so 3 $\Rightarrow$ 1. \qed

5. Closure-preserving coverings for $C_p (X)$

A closure-preserving covering of $C_p(X)$ is a generalization of a locally finite covering. A covering $\mathcal{F}$ of a space $X$ is called closure-preserving if

$$\bigcup \{F : F \in \mathcal{G}\} = \bigcup \{\overline{F} : F \in \mathcal{G}\}$$

for any $\mathcal{G} \subseteq \mathcal{F}$.

Theorem 5.1 (Guerrero, [40, Corollary 2.7]). $C_p(X)$ admits a closure-preserving covering by closed $\sigma$-countably compact sets if and only if $X$ is finite.

Proof. First let us suppose that $\mathcal{F}$ is a closure-preserving covering of $C_p (X)$ by closed $\sigma$-compact subspaces. Note that $X$ must be pseudocompact. Otherwise $C_p(X)$ has a closed homeomorphic copy of $R^N$ and hence $C_p(N)$ has a closure-preserving covering $\mathcal{G}$ by closed $\sigma$-compact subspaces. As the space $C_p(N)$ is separable, there exists a countable subfamily $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\bigcup \mathcal{G}_0 = \bigcup \mathcal{G}_0 = C_p(N)$, which means that $C_p(N)$ is covered by a countable family of compact sets. Thus $N$ should be finite by Velichko’s theorem, a contradiction.

If $f \in C(X)$ we claim that $f(X)$ is finite. Indeed, if $Y := f(X)$ since $Y$ is a separable metric space the space $C_p(Y)$ is separable. On the other hand, since $X$ is pseudocompact and $Y$ is second countable $f$ is an $R$-quotient map [72, S.154, Fact 3], so the pullback $f^* : C_p(Y) \to C_p(X)$ defined by $f^*(g) = g \circ f$ embeds $C_p(Y)$ in $C_p(X)$ as a closed subspace [1, 0.4.10 Proposition]. Therefore, $C_p(Y)$ is covered by a closure-preserving family $\mathcal{M}$ of closed $\sigma$-compact subspaces and there exists a countable subfamily $\mathcal{N}$ of $\mathcal{M}$ such that $\bigcup \mathcal{N} = \bigcup \overline{\mathcal{N}} = C_p(Y)$. Again Velichko’s theorem implies that $Y$ must be finite.

Since $f(X)$ is finite for every $f \in C(X)$, the space $X$ must be finite. If not there is a countable discrete subspace $D = \{x_n : n \in N\}$ in $X$ and a countable family of open sets $\{U_n : n \in N\}$ such that $U_n \cap D = \{x_n\}$ and $\overline{U_i} \cap \overline{U_j} = \emptyset$ if $i \neq j$. So, for each $n \in N$ there is $f_n \in C(X)$ with $0 \leq f_n \leq 1$ such that $f_n(x_n) = 1$ and $f_n(x) = 0$ if $x \in X \setminus U_n$. Then clearly $f = \sum_{n=1}^{\infty} f_n \in C(X)$ but $f(X) \supseteq D$, which is infinite, a contradiction.

If the closure-preserving covering consists of closed $\sigma$-countably compact sets instead of closed $\sigma$-compact sets, we get the same conclusion by using the Tkachuk-Shakhmatov theorem instead of Velichko’s theorem.

Conversely, $R^n$ can always be covered by a countable family of compact balls. \qed

Theorem 5.2 (Guerrero, [40, Corollary 2.8]). If $C_p(X)$ admits a closure-preserving covering by countably compact sets then $X$ is finite.

Proof. Let $\mathcal{F}$ be a closure-preserving cover of $C_p (X)$ by countably compact sets. If $X$ is not pseudocompact, there is a sequence $\{F_n : n \in N\}$ in $\mathcal{F}$ with $\bigcup_{n=1}^{\infty} F_n \cap R^N = R^N$. So, $R^N$ is covered by a countable family of pseudocompact sets. In this case Theorem 2.6 forces $N$ to be pseudocompact, a contradiction. So, $X$ is pseudocompact.

Then [1, 3.4.23 Theorem] shows that each member of $\mathcal{F}$ is a compact set. Hence, $\mathcal{F}$ is a closure-preserving cover of $C_p(X)$ by compact sets, and the conclusion follows from the preceding theorem. \qed
Lemma 5.3 (Guerrero, [40, Lemma 2.10]). Let $X$ be an infinite compact space. If $C_p(X)$ admits a closure-preserving covering by subspaces of density less than or equal to an infinite cardinal $\kappa$ then $w(X) \leq \kappa$.

Proof. We shall restrict ourselves to the case $\kappa = \aleph_0$, what will be used later. So, assume $C_p(X)$ admits a closure-preserving covering $\mathcal{F}$ by closed separable subspaces. Proceed by contradiction by supposing $w(X) > \aleph_0$. It suffices to consider the case $w(X) = \aleph_1$.

Since $d(C_p(X)) = \omega w(X) = \aleph_1$ [58], there is $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $|\mathcal{F}_0| \leq \aleph_1$ and $C_p(X) = \bigcup \mathcal{F}_0$. This covering can be rewritten as $\{F_\alpha : 0 \leq \alpha < \omega_1\}$ and if we define $G_\alpha = \bigcup \{F_\beta : 0 \leq \beta < \alpha\}$ for every $0 \leq \alpha < \omega_1$, clearly $\mathcal{G} = \{G_\alpha : 0 \leq \alpha < \omega_1\}$ is an increasing closure-preserving covering of $C_p(X)$ by separable subspaces which swallows the separable sets in $C_p(X)$.

As $X$ is embeddable in $[0, 1]^{\omega_1}$, let us consider the natural projections $\pi_\alpha : X \rightarrow [0, 1]^{\omega_1}$ for $0 \leq \alpha < \omega_1$. For each $\alpha < \omega_1$ define $Z_\alpha = \pi_\alpha(X)$ and set $M_\alpha := \pi_\alpha^* (C(Z_\alpha))$, where $\pi_\alpha^* : C_p(Z_\alpha) \rightarrow C_p(X)$ is the pullback of $\pi_\alpha$, defined as usual by $\pi_\alpha^* (f) = h \circ \pi_\alpha$. It can be easily seen that the family $M = \{M_\alpha : 0 \leq \alpha < \omega_1\}$ is another increasing covering of $C_p(X)$ such that $d(M_\alpha) = \omega(Z_\alpha) \leq \aleph_0$. So, for each $\alpha < \omega_1$ there exists $\alpha \leq \beta < \omega_1$ with $M_\beta \subseteq C_p$. Conversely, for each $\beta < \omega_1$ there exists $\beta \leq \gamma < \omega_1$ with $C_p \subseteq M_\gamma$.

Note that $X$ cannot be embedded in $[0, 1]^{\omega_1}$ for any $\alpha < \omega_1$, otherwise if $X \rightarrow [0, 1]^{\omega_1}$ then $\omega(X) = \omega_1 = \aleph_0$, a contradiction. This entails that for each $\alpha < \omega_1$ there exists $\beta < \omega_1$ with $\beta \geq \alpha$ such that both $\mathcal{G}_\alpha \subseteq M_\beta$ and the natural projection $\pi_{\alpha, \beta} : Z_\beta \rightarrow Z_\alpha$ is not injective. So we may get an increasing sequence of countable ordinals $\{\alpha_n : n \in \mathbb{N}\}$ such that $G_{\alpha_{n+1}} \subseteq M_{\alpha_n} \subseteq G_{\alpha_{n+1}}$ and the projection $\pi_{\alpha_{n+1}, \alpha_n} : Z_{\alpha_{n+1}} \rightarrow Z_{\alpha_n}$ is not injective. Let $\gamma := \sup \{\alpha_n : n \in \mathbb{N}\}$ and for each $n$ choose two different points $x_n, y_n \in Z_{\alpha_{n+1}} \subseteq Z_\gamma$ with $\pi_{\alpha_{n+1}, \alpha_n}(x_n) = \pi_{\alpha_{n+1}, \alpha_n}(y_n) = \pi_{\alpha_{n+1}, \gamma}(y_n)$.

According to [40, Lemma 2.9] there is $g \in C(Z_\gamma)$ whose restriction to $\{x_n, y_n : n \in \mathbb{N}\}$ is injective, so that $g(x_n) \neq g(y_n)$ for every $n \in \mathbb{N}$. This means that $\sup g \not\in Z_{\alpha_n}$ for all $n \in \mathbb{N}$, in other words, $g$ does not belong to $C(Z_{\alpha_n})$ for any $n \in \mathbb{N}$. Hence, the function $f = \pi_{\gamma, \alpha_n}^* (g) \in \pi_{\gamma, \alpha_n}^* (C(Z_\gamma)) = M_\gamma$ does not belong to $M_{\alpha_n}$ for any $n \in \mathbb{N}$. Thus $f \not\in \bigcup_{n=1}^{\infty} M_{\alpha_n} = G_\gamma$, the latter equality because both $G$ and $M$ are increasing, $G_{\alpha_{n+1}} \subseteq M_{\alpha_n} \subseteq G_{\alpha_{n+1}}$, and $\mathcal{G} = C(X)$.

On the other hand, let a finite subset $A$ of $X$ and $\epsilon > 0$ be given. Let

$$U_f = \{h \in M_f : |h(x) - f(x)| < \epsilon, x \in A\}$$

be a neighborhood of $f$ in the relative topology of $M_f$. If $\pi_{\alpha_n}(x) = \pi_{\gamma}(y)$ for $x, y \in A$ then $f(x) = f(y)$, so we may assume $\pi_{\alpha_n}(x) \neq \pi_{\gamma}(y)$ for each pair $x, y \in A$. In this case there is $l \in \mathbb{N}$ such that $\pi_{\alpha_n, \gamma}^* l$ is one-to-one on $\pi_{\gamma}(A)$. Hence $\pi_{\alpha_n} = \pi_{\alpha_n, \gamma}^* \circ \pi_{\gamma}$ is one-to-one on $A$. So, we can choose $\varphi \in C(Z_{\alpha_n})$ such that $\varphi(\pi_{\alpha_n}(x)) = f(x)$ for each $x \in A$.

Since $h := \varphi \circ \pi_{\alpha_n} = \pi_{\gamma, \alpha_n}^* (\varphi) \in M_{\gamma} \subseteq M_f$, clearly $h \in U_f$. So, $f \in M_f$ and consequently $f \in M_f \subseteq G_{\alpha_{n+1}} \subseteq G_{\alpha_{n+1}} \subseteq G_\gamma$, a contradiction. □

Theorem 5.4 (Guerrero, [40, Corollary 2.13]). Let $X$ be an infinite compact space. $C_p(X)$ admits a closure-preserving covering by separable subspaces if and only if $X$ is metrizable.

Proof. If $X$ is a compact metrizable space, then $C_p(X)$ is separable. Let $D$ be a countable dense subspace of $C_p(X)$. For every $f \in C(X)$ put $D_f := D \cup \{f\}$. Then clearly $\mathcal{F} = \{D_f : f \in C(X)\}$ is a closure-preserving covering of $C_p(X)$ by separable subspaces. Conversely, if the space $C_p(X)$ admits a closure-preserving covering by separable subspaces, Lemma 5.3 with $\kappa = \aleph_0$ yields $w(X) \leq \aleph_0$. Since $X$ is compact, this implies that $X$ must be metrizable. □

Theorem 5.5 (Guerrero, [40, Corollary 2.14]). Let $X$ be an infinite compact space. $C_p(X)$ admits a closure-preserving covering by second countable subspaces if and only if $X$ is countable.

Proof. If $C_p(X)$ admits a closure-preserving covering by second countable subspaces, then $C_p(X)$ admits a closure-preserving cover by separable subspaces. Hence $X$ is metrizable by the previous theorem and,
consequently, \( C_p(X) \) is separable. This clearly implies that \( C_p(X) \) has indeed a countable covering by second countable subspaces, so we may apply [70, Corollary 1.7] to guarantee that \( X \) is countable. 

For the following lemma, given a function \( f \in C^b(X) \) and a number \( \epsilon > 0 \) let
\[
I(f, \epsilon) = \left\{ g \in C^b(X) : \|f - g\|_\infty \leq \epsilon \right\}.
\]

Lemma 5.6 (Guerrero-Tkachuk [43, Proposition 2.1 (a)]). If \( \mathcal{F} \) is a closure-preserving covering of \( C_p(X) \) by closed subspaces, there exist \( F \in \mathcal{F} \) and \( f \in C^b(X) \) such that \( I(f, \epsilon) \subseteq F \) for some \( \epsilon > 0 \).

Proof. We claim that the family \( \left\{ F \cap C^b(X) : F \in \mathcal{F} \right\} \) is also a closure-preserving covering by closed subspaces of the Banach space \( C^b(X) \) equipped with the supremum-norm \( \|\cdot\|_\infty \). Indeed, since the Banach topology \( \tau_u \) is stronger than the pointwise topology, denoting \( C^b(X) \) by \( G \), if \( \mathcal{F}' \subseteq \mathcal{F} \) one has
\[
\bigcup_{F \in \mathcal{F}'} F \cap G = G \cap \left( \bigcup_{F \in \mathcal{F}'} F \right) = G \cap \left( \bigcup_{F \in \mathcal{F}'} F \right) = \bigcup_{F \in \mathcal{F}'} F \cap G.
\]

By [68, Theorem 2.5] there exist \( F \in \mathcal{F} \) and \( f \in C^b(X) \) for which there is an open ball
\[
B(f, \delta) = \left\{ g \in C^b(X) : \|f - g\|_\infty < \delta \right\}
\]
centered at \( f \) in the Čech-complete space \( (C^b(X), \|\cdot\|_\infty) \), such that \( B(f, \delta) \subseteq F \cap G \). Hence, if \( \epsilon = \delta/2 \) we get \( I(f, \epsilon) \subseteq F \). 

Theorem 5.7 (Guerrero-Tkachuk [43, Corollary 2.5]). If \( \mathcal{P} \) is a hereditary topological property and \( C_p(X) \) has a closure-preserving cover \( \mathcal{F} \) by closed subspaces such that each \( F \in \mathcal{F} \) has property \( \mathcal{P} \), both \( C_p(X, [0, 1]) \) and \( C_p(X) \) have property \( \mathcal{P} \).

Proof. Under these hypotheses we claim that some \( F \in \mathcal{F} \) contains a homeomorphic copy of \( C_p(X) \). By Lemma 5.6 there exist \( F \in \mathcal{F} \) and \( f \in C^b(X) \) such that \( I(f, \epsilon) \subseteq F \) for some \( \epsilon > 0 \). Then the map \( \varphi : C_p(X, [0, 1]) \to C^b(X) \) defined by
\[
\varphi(g) = 2\epsilon \left( g - \frac{1}{2} \right) + f
\]
is a homeomorphism such that \( \varphi(C(X, [0, 1])) = I(f, \epsilon) \). Since \( F \) has the hereditary property \( \mathcal{P} \), the set \( I(f, \epsilon) \) also has property \( \mathcal{P} \) and consequently \( C_p(X, [0, 1]) \) has property \( \mathcal{P} \). But \( C_p(X, [0, 1]) \) contains \( C_p(X, (0, 1)) \), which is homeomorphic to \( C_p(X) \). So \( C_p(X) \) also has property \( \mathcal{P} \). 

Theorem 5.8 (Guerrero-Tkachuk [43, Theorem 2.7]). Let \( \mathcal{P} \) be a closed hereditary topological property. If \( C_p(X) \) has a closure-preserving cover \( \mathcal{F} \) by closed subspaces such that each \( F \in \mathcal{F} \) has property \( \mathcal{P} \), then \( C_p(X, [0, 1]) \) has property \( \mathcal{P} \).

Proof. Again Lemma 5.6 provides \( F \in \mathcal{F} \) and \( f \in C^b(X) \) such that \( I(f, \epsilon) \subseteq F \) for some \( \epsilon > 0 \). By the proof of Theorem 5.7 the subspace \( I(f, \epsilon) \) of \( C^b(X) \) is homeomorphic to \( C_p(X, [0, 1]) \) and closed in \( F \), so \( C_p(X, [0, 1]) \) has property \( \mathcal{P} \).

Remark 5.9. Applications of the preceding results. Theorem 5.7 applies for instance to the Fréchet-Urysohn property and metrizability. Theorem 5.8 applies to \( K \)-analyticity, Lindelöf \( \Sigma \)-property and normality. Concerning realcompactness, if \( C_p(X) \) has a closure-preserving cover \( \mathcal{F} \) by closed subspaces such that each \( F \in \mathcal{F} \) is realcompact, Theorem 5.8 ensures that the space \( C_p(X, [0, 1]) \) is realcompact. Since \( C(X, (0, 1)) \) can be obtained from \( C_p(X, [0, 1]) \) by throwing out a union of \( G_\beta \)-subsets of \( C_p(X, [0, 1]) \), it turns out that \( C(X, (0, 1)) \) is realcompact (see [72, Problem 408]). Hence \( C_p(X) \) is realcompact.
Corollary 5.10 (Guerrero-Tkachuk [43, Proposition 2.20]). If \( X \) is a Lindelöf \( \Sigma \)-space and \( C_p(X) \) has a closure-preserving cover by closed Lindelöf \( \Sigma \)-subspaces then \( C_p(X) \) is a Lindelöf \( \Sigma \)-space.

Proof. According to Theorem 5.8, \( C_p(X, [0,1]) \) must be a Lindelöf \( \Sigma \)-space. So [1, IV.9.17 Theorem] ensures that \( C_p(X) \) is a Lindelöf \( \Sigma \)-space. \( \square \)

6. Domination by a second countable space

Given a Tychonoff space \( M \), a family of sets \( \mathcal{A} \) of another Tychonoff space \( X \) is said to be \( M \)-ordered (or ordered by \( M \)) if \( \mathcal{A} = \{ A_k : k \in \mathcal{K}(M) \} \), where \( \mathcal{K}(M) \) denotes the family of all compact sets in \( M \), and \( P \subseteq Q \) in \( M \) implies \( A_P \subseteq A_Q \). The space \( X \) is said to be \( M \)-dominated (or dominated by the space \( M \)) if \( X \) has an \( M \)-ordered covering \( \mathcal{A} \) consisting of compact sets (an \( M \)-ordered compact covering).

Theorem 6.1 (Cascales-Orihuela-Tkachuk, [14, 2.1(a) Theorem]). Every Lindelöf \( \Sigma \)-space is dominated by a second countable space.

Proof. An equivalent definition of Lindelöf \( \Sigma \)-space says that \( X \) is a Lindelöf \( \Sigma \)-space if and only if there exists a second countable space \( M \) and a compact-valued \( \omega \)-map \( T : M \rightarrow \mathcal{K}(X) \) such that \( \bigcup \{ T(x) : x \in M \} = X \). If \( K \) is a compact set in \( M \), define \( A_K = \bigcup \{ T(x) : x \in K \} \). Clearly \( \mathcal{A} = \{ A_K : K \in \mathcal{K}(M) \} \) is an \( M \)-ordered cover consisting of compact sets. \( \square \)

The class of spaces dominated by second countable spaces has good stability properties [14, 2.1 Theorem].

Theorem 6.2 (Cascales-Orihuela-Tkachuk, [14, 2.2 Proposition]). The following relations are equivalent for a Tychonoff space \( X \).

1. \( X \) has a resolution consisting of compact sets.
2. \( X \) is \( \mathbb{N}^n \)-dominated.
3. \( X \) is dominated by a Polish space.

Proof. 1 \( \Rightarrow \) 2. Let \( \{ A_i : i \in \mathbb{N}^n \} \) be a resolution for \( X \) of compact sets. If \( P \in \mathcal{K}(\mathbb{N}^n) \), define \( a_P \in \mathbb{N}^n \) by \( a_P(i) = \max \pi_i(P) \), where \( \pi_i : \mathbb{N}^n \rightarrow \mathbb{N} \) is the canonical \( i \)-th projection. Clearly \( a_P \leq a_Q \) if \( P \subseteq Q \) and if we set \( A_P := A_{a_P} \), for every \( P \in \mathcal{K}(\mathbb{N}^n) \), then \( \mathcal{A} = \{ A_P : P \in \mathcal{K}(\mathbb{N}^n) \} \) is an \( \mathbb{N}^n \)-ordered family of compact sets which covers \( X \). The latter because if \( x \in X \) there is \( y \in \mathbb{N}^n \) with \( x \in A_y \), and the set \( Q_y := \{ x \in \mathbb{N}^n : a(l_i) \leq y(i) \; \forall i \in \mathbb{N} \} \) is compact in \( \mathbb{N}^n \) and verifies that \( a_{Q_y} = y \). So \( x \in A_{Q_y} \).

2 \( \Rightarrow \) 1. Let \( \{ A_P : P \in \mathcal{K}(\mathbb{N}^n) \} \) be an \( \mathbb{N}^n \)-ordered compact cover of \( X \). If \( y \in \mathbb{N}^n \) let \( Q_y \in \mathcal{K}(\mathbb{N}^n) \) be the previously defined set that verifies the equality \( a_{Q_y} = y \). Then the family \( \mathcal{A} = \{ A_y : y \in \mathbb{N}^n \} \) with \( A_y := A_{Q_y} \), verifies that \( A_y \subseteq A_z \) if \( y \subseteq z \). Moreover, \( \mathcal{A} \) covers \( X \). For if \( x \in X \) there is \( P \in \mathcal{K}(\mathbb{N}) \) with \( x \in A_P \). So, if \( \sigma(i) = \max \pi_i(P) \) for every \( i \in \mathbb{N} \) then \( A_P \subseteq A_{\sigma} \) and hence \( x \in A_{\sigma} \). Therefore \( \mathcal{A} \) is a resolution for \( X \) by compact sets.

2 \( \Rightarrow \) 3 is clear. Finally, if \( X \) is dominated by a Polish space \( M \), there is an \( M \)-ordered compact cover \( \mathcal{A} = \{ A_k : k \in \mathcal{K}(M) \} \). Since \( M \) is a Polish space, there is an open continuous map \( \varphi : \mathbb{N}^n \rightarrow M \) from \( \mathbb{N}^n \) onto \( M \). Consider the family \( \mathcal{F} = \{ A_{\varphi(P)} : a \in \mathbb{N}^n \} \). If \( x \in X \) there is a compact set \( K \) in \( M \) such that \( x \in A_K \) and there exists \( P \in \mathcal{K}(\mathbb{N}) \) such that \( \varphi(P) = K \) [18, 5.5.8]. If \( \sigma(i) = \max \pi_i(P) \) for every \( i \in \mathbb{N} \) then \( P \subseteq Q_{\sigma} \) and hence \( K = \varphi(P) \subseteq \varphi(Q_{\sigma}) \) so that \( x \in A_{\varphi(P)} \). Hence \( \mathcal{F} \) covers \( X \) and clearly \( \mathcal{F} \) is an \( \mathbb{N}^n \)-ordered compact covering for \( X \). So \( X \) is \( \mathbb{N}^n \)-dominated. This shows that 3 \( \Rightarrow \) 2. \( \square \)

Theorem 6.3 (Cascales-Orihuela-Tkachuk, [14, 2.4 Corollary]). \( C_p(X) \) is dominated by a Polish space if and only if it is \( K \)-analytic.

Proof. If \( C_p(X) \) is dominated by a Polish space, by the previous theorem \( C_p(X) \) has a resolution consisting of compact sets. So \( C_p(X) \) is \( K \)-analytic by Theorem 3.10. Conversely, if \( C_p(X) \) is \( K \)-analytic, it has a resolution of compact sets [67]. Thus, according to Theorem 6.2, \( C_p(X) \) is dominated by a Polish space. \( \square \)
Lemma 6.4. If $X$ is dominated by a second countable space, $X$ has a countable network modulo a covering by countably compact sets and $C_p(X)$ is Lindelöf $\Sigma$-framed in $\mathbb{R}^X$.

Proof. If $X$ is dominated by a second countable space $M$, the first statement of the consequent follows from [14, 2.6 Proposition], where one should notice that the fact that $M$ is second countable is critical. The second follows from [71, 2.7 Proposition].

Theorem 6.5 (Cascales-Orihuela-Tkachuk, [14, 2.15 Theorem]). $C_p(X)$ is dominated by a second countable space if and only if it is a Lindelöf $\Sigma$-space.

Proof. Sufficiency is Theorem 6.1. For the necessity assume that $\{F_K : K \in \mathcal{K}(M)\}$ is an $M$-ordered compact covering of $C_p(X)$. Apply Lemma 6.4 to show that $C_p(C_p(X))$ is Lindelöf $\Sigma$-framed in $\mathbb{R}^X$ and then Lemma 3.1 to get that $C_p(C_p(X))$ is a Lindelöf $\Sigma$-space. Then apply [1, IV.9.1 Theorem] to conclude that $\nu X$ is also a Lindelöf $\Sigma$-space, which guarantees that the space $C_p(\nu X)$ is angelic [25, Theorem 78].

If $T : C_p(\nu X) \to C_p(X)$ denotes the restriction map $Tf = f|_{\nu X}$, it can be easily seen that $\{G_K : K \in \mathcal{K}(M)\}$, where $G_K = T^{-1}(F_K)$ is an $M$-ordered compact covering of $C_p(\nu X)$. Since $C_p(\nu X)$ is dominated by a second countable space, Lemma 6.4 asserts that $C_p(\nu X)$ has a countable network modulo a covering by countably compact subsets. But $C_p(\nu X)$ angelicity ensures that $C_p(\nu X)$ has a countable network modulo a covering by compact sets. So, according to [1, IV.9.1 Proposition], $C_p(\nu X)$ is a Lindelöf $\Sigma$-space. Consequently $C_p(X)$, as a continuous image of a Lindelöf $\Sigma$-space, is also a Lindelöf $\Sigma$-space.

Domination of each subspace of $C_p(X)$ by a second countable space also leads to some interesting properties. We state the following theorem without proof (see [14] for details).

Theorem 6.6 (Cascales-Orihuela-Tkachuk, [14, 2.18 Proposition]). If every subspace of $C_p(X)$ is dominated by a second countable space, then $C_p(X)$ is cosmic.

A Tychonoff space $X$ is strongly dominated by $M$ if there exists an $M$-ordered compact covering $\mathcal{F}$ of $X$ that swallows the compact sets in $X$. Strong domination by second countable spaces has been extensively studied in [14, 41, 45, 74]. Under CH it is shown in [14, 3.10 Theorem] that, for compact $X$, if $C_p(X)$ is strongly dominated by a second countable space, $X$ must be countable. The CH is removed in [41], where it is proved that, assuming $C_p(X)$ is a strongly dominated by a second countable space, if $X$ is separable, scattered, second countable, compact or pseudocompact, then $X$ is countable. Theorem 6.8 below extends this result to all Tychonoff spaces.

Lemma 6.7 (Guerrero-Tkachuk, [45, Lemma 3.4.5]). Let $X$ be an uncountable Lindelöf $\Sigma$-space. Assume $C_p(X)$ is strongly dominated by a second countable space $M$, and let $\{F_K : K \in \mathcal{K}(M)\}$ be an $M$-ordered compact covering of $C_p(X)$ that swallows the compact sets in $C_p(X)$. Then there exists a family $Q = \{Q_K : K \in \mathcal{K}(M)\}$ of compact sets of $\mathbb{R}^X$ such that $Q_K \subseteq Q_L$ if $K \subseteq L$ and $\bigcup Q$ contains the linear subspace $\Sigma(X)$ of all countable supported functions of $\mathbb{R}^X$.

Proof. If $K \in \mathcal{K}(M)$, let $a_K(x) = \inf \{g(x) : g \in F_K\}$ and $b_K(x) = \sup \{g(x) : g \in F_K\}$. Letting

$$Q_K = \prod_{x \in X} [a_K(x), b_K(x)]$$

the family $Q = \{Q_K : K \in \mathcal{K}(M)\}$ consists of compact sets in $\mathbb{R}^X$ and clearly verifies that $Q_K \subseteq Q_L$ if $Q \subseteq L$. We claim that $\Sigma(X) \subseteq \bigcup Q$.

Choose $f \in \Sigma(X)$ and denote by $A = \{x_i : i \in \mathbb{N}\}$ the countable support of $f$. By Theorem 6.5 we know that $C_p(X)$ is a Lindelöf $\Sigma$-space, hence [64, Theorem 5.4] provides a retraction $r : X \to F$ such that $A \subseteq F$ and $|F| \leq \aleph_0$. If $F = \{y_n : n \in \mathbb{N}\}$, put $U_1 = F$ and $U_{n+1} = F \setminus \{y_1, \ldots, y_n\}$ for each $n \in \mathbb{N}$. Clearly, the family $\{U_n : n \in \mathbb{N}\}$ consists of $F$-open sets and is point-finite in $F$, i.e., each $x \in F$ belongs at most to finitely-many sets $U_n$. Moreover $y_n \in U_n$ for every $n \in \mathbb{N}$. Since $F$ is a retract of $X$, it follows that the family $\{V_n : n \in \mathbb{N}\}$, where $V_n := r^{-1}(U_n)$ for each $n \in \mathbb{N}$, consists of open sets in $X$, is point-finite in $X$ and verifies that $y_n \in V_n$.
for every \( n \in \mathbb{N} \). If \( x_i = y_n \), and we set \( W_i = V_n \), for each \( i \in \mathbb{N} \), the family \( \{W_i : i \in \mathbb{N}\} \) consists of open sets in \( X \), is point-finite in \( X \) and verifies that \( x_i \in W_i \) for every \( i \in \mathbb{N} \).

For each \( i \in \mathbb{N} \) choose \( f_i \in C(X) \) with \( 0 \leq f_i \leq 1 \) such that \( f_i(x_i) = 1 \) and define \( g_i = \left[ f(x_i) \right] \cdot f_i \) and \( h_i = -\left[ f(x_i) \right] \cdot f_i \). As the family \( \{W_i : i \in \mathbb{N}\} \) is point-finite, the set \( P = \{g_i, h_i : i \in \mathbb{N}\} \cup \{\} \), where here \( \cdot \) stands for the identically null function on \( X \), is compact in \( C_p(X) \). Consequently, there exists some \( K \in \mathcal{K}(M) \) such that \( P \subseteq F_K \), which means that \( g(x) \in [a_K(x), b_K(x)] \) for each \( g \in P \). Since \( f(x_i) \) coincides with \( g_i(x_i) \) or with \( h_i(x_i) \), clearly \( f(x_i) \in [a_K(x_i), b_K(x_i)] \) for each \( i \in \mathbb{N} \), whereas if \( x \notin A \) then \( f(x) = 0 \in [a_K(x), b_K(x)] \) since \( P \subseteq F_K \). Therefore \( f \in Q_K \) and the proof is over. \( \square \)

**Theorem 6.8 (Guerrero-Tkachuk, [45, Theorem 3.4]).** \( C_p(X) \) is strongly dominated by a second countable space if and only if \( X \) is countable.

**Proof.** Suppose that \( C_p(X) \) is strongly dominated by a second countable space and let \( \{F_K : K \in \mathcal{K}(M)\} \) be an \( M \)-ordered compact covering of \( C_p(X) \) that swallows the compact sets in \( C_p(X) \). Proceeding by contradiction, assume that \( X \) is uncountable. By [41, Theorem 3.10] there is no loss of generality if we assume that \( X \) is a Lindelöf \( \Sigma \)-space. So, according to Lemma 6.7, there exists a family \( Q = \{Q_K : K \in \mathcal{K}(M)\} \) of compact sets in \( \mathbb{R}^X \) such that \( Q_K \subseteq Q_L \), if \( Q \subseteq L \) and \( Y = \bigcup Q \) contains the linear subspace \( X \) (of countable supported functions of \( \mathbb{R}^X \)).

It is not hard to see that this implies that there exists a Lindelöf \( \Sigma \)-space \( Z \) such that \( Y \subseteq Z \subseteq \mathbb{R}^X \), so that \( \Sigma(X) \subseteq Z \). But \( \Sigma(X) \) is not Lindelöf \( \Sigma \)-framed in \( \mathbb{R}^X \) if \( X \) is uncountable [45, Proposition 3.1]. \( \square \)

In [45, Theorem 3.9] it is showed that, for compact \( X \), if \( C_p(X, [0, 1]) \) is strongly dominated by a second countable space, then \( X \) is countable. In [74] the requirement of compactness of \( X \) is relaxed by the following result, which we state without proof.

**Theorem 6.9 (Tkachuk, [74, 3.7 Theorem]).** Let \( X \) be a Lindelöf \( \Sigma \)-space. If \( C_p([0, 1]) \) is strongly dominated by a second countable space, then \( X \) is countable.

### 7. Some examples

**Example 7.1.** If \( \Omega \) is a nonempty open subset of \( \mathbb{R}^n \), both the space \( D(\Omega) \) of test functions equipped with its usual inductive limit topology and the space of distributions \( D'(\Omega) \) endowed with the Mackey topology \( \mu(D'(\Omega), D(\Omega)) \), which coincides with the strong topology \( \beta(D'(\Omega), D(\Omega)) \) (see [46, Chapter 4]), are analytic. The first statement is consequence of the fact that the inductive limit of a sequence of Fréchet-Montel spaces is analytic, the second follows from the fact that the strong dual of an inductive limit of a sequence of Fréchet-Montel locally convex spaces is also analytic (see [76, I.4.4.(21) and I.4.4.(23)]).

**Example 7.2.** The space \( C_p(Z) \) with \( Z \) being the set of all weak \( P \)-points of \( N^* \). If \( X \) is a Tychonoff space, a point \( x \in X \) is called a weak \( P \)-point of \( X \) if \( x \notin \overline{A} \) for any countable set \( A \subseteq X \setminus \{x\} \). Every \( P \)-point of \( X \) is a weak \( P \)-point of \( X \). The subspace \( Z \) of all weak \( P \)-points of the remainder \( N^* = \beta N \setminus N \) of the Stone-Čech compactification \( \beta N \) of \( N \) is dense in \( N^* \) [53], so it is infinite. But the space \( C_p(Z) \) is covered by a sequence of pseudocompact sets (i.e., \( C_p(Z) \) is \( \sigma \)-pseudocompact) [2, 6.4 Example]. By Theorem 2.6, the space \( Z \) is pseudocompact (see [2, 6.3 Proposition] for a direct proof of this property) and each countable subset of \( Z \) is closed, discrete and \( C^* \)-embedded in \( Z \). Note that \( C_p(Z) \) is not \( \sigma \)-compact, otherwise \( Z \) would be finite by Velichko’s theorem.

**Example 7.3.** The Sorgenfrey line \( \mathbb{S} \) is a (hereditarily) Lindelöf space which is not a Lindelöf \( \Sigma \)-space, since \( S \times S \) is not Lindelöf. Hence Lemma 3.1 prevents the space \( C_p(S) \) to have a resolution consisting of pointwise bounded sets.

**Example 7.4.** The space \( C_p(\mathbb{N}^\mathbb{N}) \) is not \( K \)-analytic-framed in \( \mathbb{R}^{\mathbb{N}^\mathbb{N}} \). By Corollary 3.8 the space \( C_p(\mathbb{N}^\mathbb{N}) \) is not analytic and does not admit a resolution of pointwise bounded sets. Hence, \( C_p(\mathbb{N}^\mathbb{N}) \) is not \( K \)-analytic-framed in \( \mathbb{R}^{\mathbb{N}^\mathbb{N}} \) because of Theorem 3.3.
Example 7.5. The spaces $C_k (\mathbb{R})$ and $C_k (\mathbb{Q})$. Both spaces have a resolution of compact sets that swallows the compact sets by virtue of Theorem 4.2. By Theorem 4.3 they also have a fundamental resolution of bounded sets, and according to Theorem 4.4 both spaces are analytic.

Example 7.6. The spaces $C_p (\mathbb{R})$ and $C_p (\mathbb{Q})$. Although both spaces have a resolution of compact sets, according to Theorem 3.15 they do not have a resolution of compact sets that swallows the compact sets. The space $C_p (\mathbb{Q})$ has a fundamental resolution of pointwise bounded sets but, as follows from Theorem 3.16, such a resolution lacks in $C_p (\mathbb{R})$.

Example 7.7. Let $\mathbb{N}$ be equipped with the discrete topology and choose $p \in \beta \mathbb{N} \setminus \mathbb{N}$. Then $X := \mathbb{N} \cup \{p\}$ with the relative topology of $\beta \mathbb{N}$ is a non discrete space with finite compact sets, hence hemicompact. So, $C_p (X)$ is analytic by Theorem 4.4, and $C_k (X)$ is a Fréchet space with a resolution of compact sets that swallows the compact sets by Theorem 4.2.

Example 7.8. The space $\omega_1$ of countable ordinals with the order topology. It is essentially well-known that if $\mathcal{N}_1 = \mathbf{b}$ (the dominating cardinal) the space $\omega_1$ has a resolution of compact sets that swallows the compact sets in $\omega_1$. However $\omega_1$ is not even a $\mu$-space, since $\omega_1$ is pseudocompact but not compact.

Example 7.9. The space $C_p (\omega_1)$. Clearly $C_p (\omega_1)$ is not analytic because of Theorem 3.7. Actually, $C_p (\omega_1)$ does not admit a resolution of compact sets, since every topological space with a resolution of compact sets has countable extent (closed discrete sets are countable) [49, Corollary 3.5] whereas the extent of $C_p (\omega_1)$ is uncountable. Consequently, $C_p (\omega_1)$ is not $K$-analytic although, as is well-known, it is a Lindelöf space. Note that $\omega_1$ is pseudocompact, hence projectively $\sigma$-compact.

Example 7.10. If $C_k (X)$ admits a resolution of convex compact sets that swallows the local null sequences, $X$ need not be countable or discrete. If $X$ is an infinite $\sigma$-compact metric space, then $C_k (X)$ has a resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of compact sets that swallows the compact sets (hence the local null sequences) of $C_k (X)$ by virtue of Theorem 4.2. But, if one looks at the proof of this theorem, the sets $A_\alpha$ are absolutely convex. So, $C_k (\mathbb{R})$ has a resolution of absolutely convex compact sets that swallows the local null sequences.

Example 7.11. A $K$-analytic not analytic $C_p$-space. Let $X$ be the Reznichenko compact space mentioned in [2, 7.14 Example]. This is a Talagrand compact space with a point $p$ such that $X = \beta Y$ with $Y = X \setminus \{p\}$. Hence $Y$ is a pseudocompact non realcompact space, so that $X = \nu Y$. Since $C_p (Y)$ is a continuous image of $C_p (X)$, the space $C_p (Y)$ is $K$-analytic. $C_p (Y)$ is not analytic by Theorem 3.7.

Example 7.12. The space $C_p (L (\mathbb{N}_1))$, where $L (\mathbb{N}_1)$ is the Lindelöfication of the discrete space of cardinality $\mathcal{N}_1$. Since $L (\mathbb{N}_1)$ is a $P$-space, $C_p (L (\mathbb{N}_1))$ is Baire. So, by Theorem 3.13, $C_p (L (\mathbb{N}_1))$ lacks a resolution of pointwise bounded sets. As $L (\mathbb{N}_1)$ is a Lindelöf $P$-space, it is projectively $\sigma$-compact [3, Proposition 2.2]. Hence, the converse of Corollary 3.5 fails.

Example 7.13. The space $C_p (L (\mathbb{N}_1), [0, 1])$. Under CH the space $C_p (L (\mathbb{N}_1), [0, 1])$ has a compact resolution [71, 2.10 Example]. Since $C_p (L (\mathbb{N}_1), [0, 1])$ is countably compact but not compact, $C (L (\mathbb{N}_1), [0, 1])$ is not a $\mu$-space, hence it is not $K$-analytic.

Example 7.14. $C_p (X)$ need not be Lindelöf if $C_p (X, [0, 1])$ is a Lindelöf $\Sigma$-space. If $X$ is a discrete space of cardinality $\mathcal{N}_1$ then $C_p (X, [0, 1]) = [0, 1]^{\omega_1}$ is compact, but $C_p (X) = \mathbb{R}^{\omega_1}$ is not Lindelöf. If both $\nu X$ and $C_p (X, [0, 1])$ are Lindelöf $\Sigma$-spaces, then $C_p (X)$ is a Lindelöf $\Sigma$-space (see [1, IV.9.17 Proposition] or [73, Problem 217]).

Example 7.15. Neither $\mathbb{R}^\mathbb{N}$ nor $C_p (0, 1)$ admits a closure-preserving covering by functionally bounded subspaces. Otherwise, since both spaces are separable and the closure of a functionally bounded set is also functionally bounded, both would admit a countable covering by (closed) functionally bounded subspaces. So, Theorem 2.6 would require $\mathbb{N}$ to be pseudocompact, which is not, and every countable set in $[0, 1]$ should be closed, which is neither the case since $[0, 1]$ is uncountable and separable.
Example 7.16 (Okunev, [59, Example 2.7]). There exists a $\sigma$-compact space $X$ such that $C_p(X)$ is not Lindelöf. If $Y$ is the subspace of $[0, 1]^{\omega_1}$ consisting of all function of finite support and $g \in [0, 1]^{\omega_1}$ is the constant function $g(t) = 1$ for every $t \in \omega_1$, define $X = Y \cup \{g\}$. Then $C_p(X)$ is such space. Note that $g \in Y$ but no countable subset of $Y$ contains $g$ in its closure, so that $X$ has uncountable tightness $t(X)$. Hence $C_p(X)$ is not a Lindelöf space because of Asanov’s theorem [1, I.4.1 Theorem]. Clearly $C_p(X)$ has a bounded resolution. So, according to Theorem 3.3, $C_p(X)$ is K-analytic-framed in $\mathbb{R}^2$. But $C_p(X)$ is not K-analytic because it is not Lindelöf.

Example 7.17 (Guerrero-Tkachuk, [43, Example 3.8]). There exists a $\sigma$-compact space $X$ such that $C_p(X)$ is not Lindelöf but it contains a dense $\sigma$-compact subspace $M$. Let $Z$ be the subspace of $[0, 1]^{\omega_1}$ consisting of those functions of finite support and define the function $g$ as in the previous example. The space $X = Z \cup \{g\}$ is as promised. For each $f \in C(X)$ put $M_f := M \cup \{f\}$. Then $\mathcal{F} = \{M_f : f \in C(X)\}$ is a closure-preserving cover of $C_p(X)$ by $\sigma$-compact subspaces. This shows that the closedness condition of the sets of the closure-preserving covering in the statement of Theorem 5.1 cannot be dropped.

8. Further research

If $\Delta = \{(x,x) : x \in X\}$ is the diagonal of $X \times X$, much research has been developed on the (strong) domination of the space $(X \times X) \setminus \Delta$ by a second countable space. We provide a brief account of this investigation, but first let us point out a couple of facts.

In first place, according to [18, Exercise 4.2.B] a compact space $X$ is metrizable if and only if $\Delta$ is a $G_\delta$-set in $X \times X$. On the other hand, for compact $X$, if $C_p(X,[0,1])$ is a Lindelöf $\Sigma$-space, clearly $C_p(X)$ is also a Lindelöf $\Sigma$-space. Hence Baturov’s theorem [1, III.6.1 Theorem] shows that for every subspace $Y$ of $C_p(X)$ the extent $\text{ext}(Y)$ of $Y$ equals the Lindelöf number $l(Y)$ of $Y$. As $X^{\omega_1}$ is embedded in $L_p(X)$, for each $n \in \mathbb{N}$, as a closed subspace [72, Problem 337], so in $C_p(X)$, clearly $X^{\omega_1} \setminus \Delta$ is embedded in $C_p(X)$. Consequently, one gets $l(X^{\omega_1} \setminus \Delta) = \text{ext}(X^{\omega_1} \setminus \Delta)$. On the other hand, since each space which is dominated by a second countable space has countable extent [14, 2.1 (h) Theorem], if $(X \times X) \setminus \Delta$ is dominated by a second countable space, it follows that $l((X^{\omega_1} \setminus \Delta) = \mathbb{N}$, i.e., $(X \times X) \setminus \Delta$ is a Lindelöf space. This implies that $\Delta$ is a $G_\delta$-set in $X \times X$, so $\Delta$ must be metrizable.

The first result on this subject is [13, Theorem 1], whence it follows that, for compact $X$, if the space $(X \times X) \setminus \Delta$ is strongly $\mathbb{N}^\omega$-dominated (equivalently, strongly dominated by a Polish space) then $X$ is metrizable. This extends to the following.

Theorem 8.1 (Cascales-Muñoz-Orihuela [12, Corollary 22]). For a compact space $X$ the following statements are equivalent.

1. $X$ is metrizable.
2. $(X \times X) \setminus \Delta$ is strongly dominated by a Polish space.
3. $(X \times X) \setminus \Delta$ is strongly dominated by a separable metric space.

For strong domination of a Tychonoff space by a second countable space, one has

Theorem 8.2 (Guerrero-Tkachuk, [45, Corollary 3.6]). If $(X \times X) \setminus \Delta$ is strongly dominated by a second countable space, then $X$ is cosmic.

Since each compact cosmic space is metrizable, one gets again

Corollary 8.3 (Cascales-Orihuela-Tkachuk, [14, 3.11 Theorem]). A compact space $X$ is metrizable if and only if $(X \times X) \setminus \Delta$ is strongly dominated by a second countable space.
In [74, Example 4.6] it is shown that under MA the space \((X \times X) \setminus \Delta\) with \(X\) non-metrizable, first countable, compact space, is strongly dominated by a countable space (with a unique non-isolated point). So, under MA, for compact \(X\) strong domination of \((X \times X) \setminus \Delta\) by a countable space does not imply the metrizability of \(X\).

In [16] it is shown that under CH a compact space \(X\) is metrizable whenever \((X \times X) \setminus \Delta\) is dominated by a Polish space. This result was extended to ZFC in [17] as follows.

**Theorem 8.4 (Dow-Hart, [17, Theorem 8]).** A compact space \(X\) is metrizable if and only if \((X \times X) \setminus \Delta\) is dominated by a Polish space.

Under CH one may change the Polish space domination of the previous theorem into second countable domination.

**Theorem 8.5 (Guerrero-Tkachuk, [44, 3.3 Corollary]).** Under CH a compact space \(X\) is metrizable if and only if \((X \times X) \setminus \Delta\) is dominated by a second countable space.

Recently, the following ZFC result has been published.

**Theorem 8.6 (Feng, [19, Theorem 5.3]).** Let \(X\) be a compact space. If \((X \times X) \setminus \Delta\) is dominated by the space \(Q\), then \(X\) is metrizable.

Since [44], when \((X \times X) \setminus \Delta\) is dominated by a space \(M\), it is usual to say that \(X\) has an \(M\)-diagonal. With this new terminology and since the space \(P\) of irrationals is homeomorphic to the Polish space \(\mathbb{N}^{\aleph_0}\), Theorems 8.4 and 8.6 can be stated as follows.

**Theorem 8.7.** Let \(X\) be a compact space. If \(X\) has either a \(P\)-diagonal or a \(Q\)-diagonal, then \(X\) is metrizable.

The following result is a proper extension of Theorem 8.1.

**Theorem 8.8 (Guerrero, [42, Theorem 2.3]).** If \(M\) is a separable metric space, every compact space with an \(M\)-diagonal is metrizable.

In [44, Theorem 3.4 (a)] it is shown that under CH if a Tychonoff space \(X\) has a second countable diagonal, then \(X\) is cosmic. The following result show that the preceding statement also holds in ZFC.

**Theorem 8.9 (Guerrero, [42, Corollary 2.4]).** For a Tychonoff space \(X\), if \((X \times X) \setminus \Delta\) is dominated by a second countable space, then \(X\) is cosmic.

Since, as mentioned earlier, each compact cosmic space is metrizable, this solves in the positive the following question originally posed by Cascales, Orihuela and Tkachuk in [14].

**Problem 8.10 (Guerrero-Tkachuk, [44, Question 4.1]).** Let \(X\) be a compact space. If \((X \times X) \setminus \Delta\) is dominated by a second countable space, is it true in ZFC that \(X\) metrizable?

It is proved in [70] that if \(C_p(X)\) is covered by a countable family of countably tight sets, then \(C_p(X)\) has countable tightness. In [78] is shown that a compact space with a closure-preserving covering by finite sets must be Eberlein compact. Related research about domination and strong domination of a space \(X\) by a locally compact second countable space \(M\), by an \(\omega\)-hyperbounded space \(M\) (i.e., an space in which the closure of each \(\sigma\)-compact subspace is compact) or by a \(\kappa\)-hemicompact space \(M\) (for a given infinite cardinal \(\kappa\)) can be found in [48].

On the other hand, the bidual \(M(X)\) of \(C_p(X)\) equipped with the relative topology of \(\mathbb{R}^X\) has recently deserved some attention in relation to the distinguished property of \(C_p(X)\) (see [33] and references therein), after the discovering that not always \(M(X)\) coincides with \(\mathbb{R}^X\) (in fact, it can be shown that \(M(X) = \mathbb{R}^X\) exactly when \(C_p(X)\) is distinguished, which is not always the case). Let us mention the following result (from which Theorem 3.16 is a straightforward consequence).

**Theorem 8.11 (Ferrando, [25, Theorem 28]).** The bidual of \(C_p(X)\) has a resolution consisting of pointwise bounded sets if and only if \(X\) is countable.
9. Some open problems

Theorem 2.4 asserts that if \( C_p(X) \) is covered by a sequence of relatively sequentially complete sets, then \( X \) is a \( P \)-space.

**Problem 9.1.** If \( C_k(X) \) is covered by a sequence of weakly relatively sequentially complete sets, is \( X \) a \( P \)-space?

Theorem 2.5 states that if \( C_p(X) \) is covered by a sequence of pointwise bounded relatively sequentially complete sets, then \( X \) is finite.

**Problem 9.2.** If \( C_k(X) \) is covered by a sequence of bounded weakly relatively sequentially complete sets, must \( X \) be finite?

By Corollary 3.5 if \( C_p(X) \) has a resolution consisting of pointwise bounded sets, then \( X \) is projectively \( \sigma \)-compact. On the other hand, according to [26, Theorem 3.1] the space \( X \) is \( \sigma \)-compact if and only if there exists a metrizable locally convex topology \( \tau \) on \( C(X) \) such that \( \tau_p \leq \tau \leq \tau_k \). If \( \tau \) is a metrizable locally convex topology on \( C(X) \) stronger than \( \tau_p \), certainly \( C_p(X) \) has a resolution consisting of pointwise bounded sets, but if \( X \) is a \( \mu \)-space the \( \tau_k \)-closures of a fundamental system of \( \tau \)-neighborhoods of the origin in \( C(X) \) define a metrizable locally convex topology \( \eta \) on \( C(X) \) coarser than \( \tau_k \) because of the Nachbin-Shirota theorem. In other words, if \( X \) is a \( \mu \)-space and there is a metrizable locally convex topology \( \tau \) on \( C(X) \) such that \( \tau_p \leq \tau \), there exists a metrizable locally convex topology \( \eta \) on \( C(X) \) such that \( \tau_p \leq \eta \leq \tau_k \). So, the following makes sense.

**Problem 9.3 (Kąkol).** Assume that \( X \) is a \( \mu \)-space. If \( C_p(X) \) has a resolution consisting of pointwise bounded sets, is there always a stronger metrizable locally convex topology \( \tau \) on \( C(X) \) such that \( \tau_p \leq \tau \leq \tau_k \)?

Observe that a positive answer to this question, also gives a positive answer to the following question.

**Problem 9.4.** Assume that \( X \) is a \( \mu \)-space. Is it true that \( C_p(X) \) has a resolution consisting of pointwise bounded sets if and only if \( X \) is \( \sigma \)-compact? Equivalently, is it true that \( C_p(X) \) is K-analytic-framed in \( \mathbb{R}^X \) if and only if \( X \) is \( \sigma \)-compact?

According to Theorem 3.16, the space \( C_p(X) \) has a resolution of pointwise bounded sets that swallows the pointwise bounded sets if and only if \( X \) is countable.

**Problem 9.5.** If \( C_p(X) \) has a resolution of pointwise bounded sets that swallows the compact sets, is \( X \) countable?

If \( X \) is first countable, Theorem 3.21 asserts that \( C_p(X) \) has a resolution of pointwise bounded sets that swallows the Cauchy sequences if and only if \( X \) is countable.

**Problem 9.6.** If \( C_p(X) \) has a resolution of pointwise bounded sets that swallows the Cauchy sequences, is \( X \) countable?

In Theorem 2.5 it is shown that \( C_p(X) \) is covered by a sequence of pointwise bounded relatively sequentially complete sets if and only if \( X \) is finite.

**Problem 9.7.** If \( C_p(X) \) has a resolution of pointwise bounded relatively sequentially complete sets that swallows the pointwise bounded relatively sequentially complete sets, is \( X \) countable?

Theorem 2.6 shows that if \( C_p(X) \) is covered by a sequence of functionally bounded sets, \( X \) is pseudo-compact and each countable set in \( X \) is closed, discrete and \( C^* \)-embdedded.

**Problem 9.8.** If \( C_p(X) \) has a resolution of functionally bounded sets that swallows the functionally bounded sets, is \( X \) countable and discrete?

If \( X \) is metrizable, according to Theorem 4.2 the space \( C_k(X) \) has a resolution of compact sets that swallows the compact sets if and only if \( X \) is \( \sigma \)-compact.
Problem 9.9. Characterize in terms of the topology of $X$ those spaces $C_k(X)$ which admit a resolution of compact sets that swallows the compact sets.

If $X$ is metrizable, Theorem 4.3 shows that $C_k(X)$ has a resolution of bounded sets that swallows the bounded sets if and only if $X$ is $\sigma$-compact, and in [23, Theorem 8] is proved that $C_k(X)$ has a resolution of bounded sets that swallows the bounded sets if and only if $X$ is a so-called $cn$-space [23, p. 3].

Problem 9.10. Is there a nicer characterization in terms of the topology of $X$ of those spaces $C_k(X)$ which admit a resolution of bounded sets that swallows the bounded sets.

By Theorem 6.9, if $X$ is a Lindelöf $\Sigma$-space and $C_p(X, [0, 1])$ is strongly dominated by a second countable space, then $X$ is countable. So $C_p(X)$ is metrizable and separable, i.e., $C_p(X)$ is cosmic. Consequently $C_p(X)$ is a Lindelöf $\Sigma$-space.

Problem 9.11 (Guerrero-Tkachuk, [45, Question 4.1]). Suppose that $C_p(X, [0, 1])$ is dominated by a second countable space. Must $C_p(X, [0, 1])$ be a Lindelöf $\Sigma$-space?

Problem 9.12 (Guerrero-Tkachuk, [45, Question 4.2]). If $X$ is metrizable and $C_p(X, [0, 1])$ is dominated by a second countable space, must $C_p(X, [0, 1])$ be a Lindelöf $\Sigma$-space?

Problem 9.13 (Guerrero-Tkachuk, [45, Question 4.3]). If $X$ is Lindelöf and $C_p(X, [0, 1])$ is dominated by a second countable space, must $C_p(X, [0, 1])$ be a Lindelöf $\Sigma$-space?

Recalling again Theorem 3.16, the following natural question makes sense.

Problem 9.14. Let $M$ be a second countable space. If $C_p(X)$ is covered by an $M$-ordered family $\mathcal{A} = \{A_K : K \in \mathcal{K}(M)\}$ consisting of pointwise bounded sets that swallows the pointwise bounded sets in $C_p(X)$, must $X$ be countable?

Problem 9.15 (Guerrero-Tkachuk, [44, 4.4 Question]). If $X$ is a compact space with a $\sigma$-compact diagonal, is $X$ metrizable?

Problem 9.16 (Guerrero-Tkachuk, [44, 4.5 Question]). If $X$ is a compact space with a Lindelöf $\Sigma$ diagonal, is $X$ metrizable?

Problem 9.17 (Tkachuk, [74, 5.6 Question]). Is it true in ZFC that for any compact first countable space $X$ there exists a countable space $M$ that strongly dominates $(X \times X) \setminus \Delta$?

Problem 9.18 (Guerrero, [42, Problem 4.1]). Let $X$ be a compact space. If $(X \times X) \setminus \Delta$ is dominated by a metric space, is $X$ metrizable?

In [25, Corollary 22] it is shown that for $X$ realcompact, the weak* bidual $M(X)$ of $C_p(X)$ is a Lindelöf $\Sigma$-space if and only if $X$ is countable.

Problem 9.19. May we drop the condition that $X$ is realcompact in the previous statement?