Strongly \((p, q)\)-Summable Sequences

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Abstract. In this paper we provide a detailed study of the Banach space of strongly \((p, q)\)-summable sequences. We prove that this space is a topological dual of a class of mixed \((s, p)\)-summable sequences, showing in this way new properties of this space. We apply these results to obtain the characterization of the adjoints of \((r, p, q)\)-summing operators.

1. Introduction

In 1973, the Banach space of strongly \(p\)-summable sequences was defined by Cohen [4]. He used this space to study and characterize the class of strongly \(p\)-summing operators. After this, in 1976, Apiola studied the duality relations between the space of strongly \(p\)-summable sequences, the absolutely \(p\)-summable sequences and weakly \(p\)-summable sequences (see [1, Section 2]) and applied these relations to characterize the adjoints of absolutely \((p, q)\)-summing and Cohen \((p, q)\)-nuclear operators. The 1982 paper by Roshdi Khalil [7] is another cornerstone in this line of thought. He introduced there the Banach space of strongly \((p, q)\)-summable sequences, extending the space of strongly \(p\)-summable sequences in a natural way, and found his dual. In 2002, Arregui and Blasco published the paper [2], describing some properties of this space but under the name of \((p, q)\)-summing sequences. In the famous book [9] we find another interesting sequence space: the space of mixed \((s, p)\)-summable sequences (see also [8]).

In this work, we continue the study of the Banach space of strongly \((p, q)\)-summable sequences. We shall begin by showing that this space coincides with the one of \((p, q)\)-summing sequences (presented by Arregui and Blasco). We investigate the duality between the space of strongly \((p, q)\)-summable sequences and the space of mixed \((s, p)\)-summable sequences, obtaining in this way some relevant properties of this space. Also, we give an application to \((r, p, q)\)-summing operators introduced by Pietsch in [9].

The paper is organized as follows. After this introduction, in Section 2 we recall some notation and basic facts on some classes of vector-valued sequences. In Section 3 we focus in the study of strongly \((p, q)\)-summable sequences, and we show our main result: the space of mixed \((s, p)\)-summable sequences is a predual of the space of strongly \((q^*, s^*)\)-summable sequences. Also, we compare this space with the
spaces of absolutely \( p \)-summable sequences and strongly \( p \)-summable sequences, and we prove an inclusion theorem. Finally, Section 4 is devoted to characterize operators that belong to the space of \((r, p, q)\)-summing operators by defining the associated operator between adequate sequence spaces.

2. Notation and preliminaries

Throughout this paper we use standard Banach space notation. Let \( X \) be a Banach space over the scalar field \( \mathbb{K} \) (either \( \mathbb{R} \) or \( \mathbb{C} \)), \( B_X \) is the closed unit ball of \( X \) and \( X^* \) is the topological dual of \( X \). Let \( 1 \leq p \leq \infty \), we write \( p' \) for the real number satisfying \( 1/p + 1/p' = 1 \). The symbol \( X^N \) will denote the sequences with values in \( X \).

Let \( \ell_p(X) \) the Banach space of all absolutely \( p \)-summable sequences \( (x_n)_n \) in \( X \) with the norm

\[
\| (x_n)_n \|_{\ell_p(X)} = \left( \sum_{n \geq 1} \| x_n \|^p \right)^{1/p},
\]

and we have the isometric isomorphism identification \( \ell_p(X)^* = \ell_p(X^*) \).

We denote by \( \ell_{p,\omega}(X) \) the Banach space of all weakly \( p \)-summable sequences \( (x_n)_n \) in \( X \) with the norm

\[
\| (x_n)_n \|_{\ell_{p,\omega}(X)} = \sup_{\| x^* \| \leq 1} \left( \sum_{n \geq 1} | x^*(x_n) |^p \right)^{1/p}.
\]

If \( p = \infty \) we are restricted to the case of bounded sequences and in \( \ell_{p,\omega}(X) \) we use the sup norm. If we take \( X = \mathbb{K} \), then the spaces \( \ell_p(X) \) and \( \ell_{p,\omega}(X) \) coincides and we denote \( \ell_p(X) \) by \( \ell_p \). If \( 1 \leq p \leq s \leq \infty \), we consider the real number \( r \) satisfying \( 1/r + 1/s = 1/p \).

A sequence \( (x_n)_n \in X^N \) is said to be mixed \((s, p)\)-summing if there exists a sequence \( \tau = (\tau_n)_n \in \ell_r \), and a sequence \( x^0 = (x^0_n)_n \in \ell_{s,\omega}(X) \) such that for all \( n \in \mathbb{N} \) we have

\[
x_n = \tau_n \cdot x^0_n.
\]

We denote by \( \ell_{m(s, p)}(X) \) the Banach space of all mixed \((s, p)\)-summing sequences of elements of \( X \) with the norm

\[
\| (x_n)_n \|_{\ell_{m(s, p)}(X)} = \inf \| (\tau_n)_n \|_{\ell_r} \left\| (x^0_n)_n \right\|_{\ell_{s,\omega}(X)}
\]

where the infimum is taken over all possible representations of \( x \) in the form (1).

Note that if \( 1 \leq p, r_1, r_2 \leq \infty \) such that \( s_1 \leq s_2 \), then

\[
\ell_{m(s_1, p)}(X) \subset \ell_{m(s_2, p)}(X),
\]

with \( \| (x_n)_n \|_{\ell_{m(s_2, p)}(X)} \leq \| (x_n)_n \|_{\ell_{m(s_1, p)}(X)} \) for all \( (x_n)_n \in \ell_{m(s_1, p)}(X) \).

If \( s = p \) we have

\[
\ell_{m(p, p)}(X) = \ell_{p,\omega}(X),
\]

with \( \| (x_n)_n \|_{\ell_{p,\omega}(X)} = \| (x_n)_n \|_{\ell_{p,\omega}(X)} \) and for \( s = +\infty \) we obtain

\[
\ell_{m(\infty, p)}(X) = \ell_{p}(X),
\]

with \( \| (x_n)_n \|_{\ell_{m(\infty, p)}(X)} = \| (x_n)_n \|_{\ell_{p}(X)} \).

The space of strongly \( p \)-summing sequences \((1 < p < +\infty)\) was introduced by Cohen in [4] in order to give a characterization of the class of strongly \( p \)-summing linear operators.
A sequence \((x_n)_n \in X^N\) is strongly \(p\)-summable if the series \(\sum_{n=1}^{\infty} x_n^p(x_n)\) converges for all \((x_n)_n \in \ell_{p,\omega}(X')\).

We denote by \(\ell_p(X)\) the space of strongly \(p\)-summable sequences in \(X\) which is a Banach space (see [5, Proposition 2.1.8]) with the norm

\[
\| (x_n)_n \|_{\ell_p(X)} := \sup \left\{ \| (x_n)_n \|_{\ell_{p,\omega}(X')} : \sum_{n=1}^{\infty} x_n^p(x_n) \leq 1 \right\}.
\]  

If \(p = 1\) we have \(\ell_1(X) = \ell_1(X)\) with \(\| \cdot \|_{\ell_1(X)} = \| \cdot \|_{\ell_1(X)}\).

The relationships between the various sequence spaces are given by

\[
\ell_p(X) \subset \ell_p(X) \subset \ell_{m(p,\,q)}(X) \subset \ell_{p,\omega}(X),
\]

with

\[
\| (x_n)_n \|_{\ell_{m(p,\,q)}(X)} \leq \| (x_n)_n \|_{\ell_{p,\omega}(X)} \leq \| (x_n)_n \|_{\ell_p(X)},
\]

for all \((x_n)_n \in \ell_p(X)\).

Further, Apiola, in [1], shows the duality identifications

\[
\ell_p(X)^* = \ell_{p,\omega}(X^*) \quad \text{and} \quad \ell_{p,\omega}(X)^* = \ell_p(X^*).
\]

### 3. Strongly \((p, q)\)-summable sequences

Roshdi Khalil in [7] introduced the Banach space of strongly \((p, q)\)-summable sequences, \(\ell_{p,\omega}(X)\) \((1 \leq p, q \leq +\infty)\), naturally extending the space of strongly \(p\)-summable sequences which described as follows.

A sequence \((x_n)_n \in X\) is strongly \((p, q)\)-summable if \(\sum_{n} |x_n^p(x_n)|^q < +\infty\) for all \((x_n)_n \in \ell_{p,\omega}(X')\). The norm of \((x_n)_n\) is given by

\[
\| (x_n)_n \|_{\ell_{p,\omega}(X)} := \sup \left\{ \| (x_n)_n \|_{\ell_{p,\omega}(X')} : \sum_{n=1}^{\infty} x_n^p(x_n) \leq 1 \right\}.
\]

For \(p = 1\) we have

\[
\ell_{1,\omega}(X) \equiv \ell_{\omega}(X),
\]  

with \(\| \cdot \|_{\ell_{1,\omega}(X)} = \| \cdot \|_{\ell_{\omega}(X)}\).

Arregui and Blasco in [2] introduced and studied the Banach space, \(\ell_{p,\omega}(X)\), of \((p, q)\)-summing sequences \((1 \leq p, q < +\infty)\), to be the space of all sequence in \(X\) such that for some constant \(C \geq 0\) we have

\[
\left( \sum_{n=1}^{\infty} |x_n^p(x_n)|^q \right)^{\frac{1}{q}} \leq C \sup_{x \in X} \left( \sum_{n=1}^{\infty} |x_n^p(x)|^q \right)^{\frac{1}{q}}.
\]

The smallest constant \(C\) such that the above inequality holds is the norm of \((x_n)_n \in \ell_{p,\omega}(X)\), and is denoted by \(\pi_{p,\omega}(\ell_{p,\omega}(X))\).

In the following proposition we show that the spaces \(\ell_{p,\omega}(X)\) and \(\ell_{p,\omega}(X)\) coincide. The proof is straightforward using the closed graph theorem and will be omitted.
Proposition 3.1. The sequence \((x_n)_n \in X^\mathbb{N}\) is \((p,q^*)\)-summing sequence if and only if it is strongly \((p,q)\)-summable sequence. Moreover, we have
\[
\|(x_n)_n\|_{\ell_{p,q}(X)} = \tau_{p,q}(x_n)_n.
\]

The following theorem asserts that the topological dual of \(\ell_{p,q}(X)\) is the product space \(\ell_{p^*} \cdot \ell_{q^*}(X^*)\), i.e. the set of all elements of the form \(x \cdot y\) such that \(x \in \ell_{p^*}\) and \(y \in \ell_{p^*}(X^*)\) (see [7, Theorem 1.3]). Pietsch in [9, Page 225] mentioned that this set is exactly the Banach space \(\ell_{m(p,q)}(X^*)\) such that \(p = 1 + \frac{1}{q}\).

Theorem 3.2. Let \(1 \leq p,q,s \leq +\infty\) such that \(\frac{1}{s} = \frac{1}{p} + \frac{1}{q}\). The space \(\ell_{m(p,q)}(X^*)\) is isometrically isomorphic to \((\ell_{p,q}(X))^*\) through the mapping \(\psi\) given by
\[
\psi((x_n)_n)((x_n)_n) = \sum_{n \geq 1} x_n^*(x_n),
\]
for every \((x_n)_n \in \ell_{m(p,q)}(X^*)\) and \((x_n)_n \in \ell_{p,q}(X)\).

Remark 3.3. The duality identification \((\ell_{p,q}(X))^* \equiv \ell_{m(p,q)}(X^*)\) yields a new formula for the norm \(\|(x_n)_n\|_{\ell_{p,q}(X)}\),
\[
\|(x_n)_n\|_{\ell_{p,q}(X)} = \sup_{\|(x_n)_n\|_{\ell_{m(p,q)}(X^*)} \leq 1} \left| \sum_{n \geq 1} x_n^*(x_n) \right|.
\]
(7)

Consequently, we obtain a special case of the strongly \((p,q)\)-summable sequences.

Corollary 3.4. If \(q = 1\) then \(\ell_{p,1}(X) = \ell_p(X)\) with \(\|(x_n)_n\|_{\ell_p(X)} = \|(x_n)_n\|_{\ell_{p,1}(X)}\).

Proof. For all \((x_n)_n \in \ell_p(X)\), by (4) we have
\[
\|(x_n)_n\|_{\ell_{p,1}(X)} = \sup_{\|(x_n)_n\|_{\ell_{m(p,q)}(X^*)} \leq 1} \left| \sum_{n \geq 1} x_n^*(x_n) \right| = \sup_{\|(x_n)_n\|_{\ell_{p,q}(X)} \leq 1} \left| \sum_{n \geq 1} x_n^*(x_n) \right| = \|(x_n)_n\|_{\ell_p(X)} < \infty.
\]
\[\square\]

We can use (2) and (7) to establish useful inclusion relations between \(\ell_{p,q}(X)\).

Proposition 3.5. Let \(1 \leq p_1,p_2,q_1,q_2,s \leq +\infty\) such that \(1 + \frac{1}{s} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}\). If \(q_1 \leq q_2\) then \(p_2 \leq p_1\) and we have \(\ell_{p_1,q_2}(X) \subset \ell_{p_2,q_1}(X)\). In this case we have \(\|(x_n)_n\|_{\ell_{p_1,q_2}(X)} \leq \|(x_n)_n\|_{\ell_{p_2,q_1}(X)}\), for all \((x_n)_n \in \ell_{p_2,q_2}(X)\).

In the following proposition we prove a relationship between the space of absolutely \(p\)-summable sequences, strongly \(p\)-summable sequences and strongly \((p,q)\)-summable sequences.

Proposition 3.6. Let \(1 \leq p,q \leq +\infty\), we have the inclusions \(\ell_p(X) \subset \ell_{p,q}(X)\) and \(\ell_q(X) \subset \ell_{p,q}(X)\). In addition \(\|\|_{\ell_{p,q}(X)} \leq \|\|_{\ell_p(X)}\) and \(\|\|_{\ell_{p,q}(X)} \leq \|\|_{\ell_q(X)}\).

Proof. If \((x_n)_n \in \ell_p(X)\) we have
\[
\|(x_n)_n\|_{\ell_{p,q}(X)} \leq \sup_{\|(x_n)_n\|_{\ell_{m(p,q)}(X^*)} \leq 1} \|(x_n)_n\|_{\ell_{p,q}(X)} = \|(x_n)_n\|_{\ell_p(X)} < \infty.
\]
Similarly, if \((x_n)_n \in \ell_q(X)\),
\[
\| (x_n)_n \|_{\ell_q(X)} \leq \sup_{\| x_n \|_{\ell_p(X)} \leq 1} \| (x_n^*_n)_n \|_{\ell_1}
= \| (x_n)_n \|_{\ell_q(X)} = \| (x_n)_n \|_{\ell_q(X)} < \infty.
\]

Lemma 3.7. Let \((x_n)_n \in \ell_{pq}(X)\). Then,
\[
\| (x_n)_n \|_{\ell_{pq}(X)} = \sup_{\| a \|_{\ell_p} \leq 1} \| (a_n x_n)_n \|_{\ell_q(X)}. \tag{8}
\]

Proof. Let \((x_n)_n \in \ell_{pq}(X)\), by using the duality between the spaces \(\ell_p\) and \(\ell_{p'}\) we obtain
\[
\| (x_n)_n \|_{\ell_{pq}(X)} = \sup_{\| x_n \|_{\ell_{p'}(X)} \leq 1} \| (x_n^*_n)_n \|_{\ell_p}
= \sup_{\| x_n \|_{\ell_{p'}(X)} \leq 1} \sup_{\| a \|_{\ell_p} \leq 1} \left| \sum_{n \geq 1} a_n x_n^*(x_n) \right|
= \sup_{\| a \|_{\ell_p} \leq 1} \| (a_n x_n)_n \|_{\ell_q(X)}.
\]

Lemma 3.8. [3, Page 526]. For all \((x_n^*_n)_n \in \ell_p(X^*)\) we have
\[
\| (x_n^*_n)_n \|_{\ell_p(X^*)} = \sup_{\| (x_n^*_n)_n \|_{\ell_{p'}(X^*)} \leq 1} \| (x_n^*_n(x_n))_n \|_{\ell_1}. \tag{9}
\]

Proposition 3.9. For each \((x_n^*_n)_n \in \ell_{pq}(X^*)\), we have
\[
\| (x_n^*_n)_n \|_{\ell_{pq}(X^*)} = \sup_{\| (x_n)_n \|_{\ell_{p'}(X)} \leq 1} \| (x_n^*_n(x_n))_n \|_{\ell_p}. \tag{10}
\]

Proof. Let \((x_n^*_n)_n \in \ell_{pq}(X^*)\). By (8) and (9) we get
\[
\| (x_n^*_n)_n \|_{\ell_{pq}(X^*)} = \sup_{\| (x_n)_n \|_{\ell_{p'}(X)} \leq 1} \| (a_n x_n^*_n)_n \|_{\ell_q(X)}
= \sup_{\| (x_n)_n \|_{\ell_{p'}(X)} \leq 1} \sup_{\| a \|_{\ell_p} \leq 1} \| (a_n x_n^*_n(x_n))_n \|_{\ell_1}
= \sup_{\| (x_n)_n \|_{\ell_{p'}(X)} \leq 1} \sup_{\| a \|_{\ell_p} \leq 1} \| (a_n x_n^*_n(x_n))_n \|_{\ell_1}
= \sup_{\| (x_n)_n \|_{\ell_{p'}(X)} \leq 1} \| (x_n^*_n(x_n))_n \|_{\ell_p}.
\]
Theorem 3.10. If $1 \leq p, q, s \leq +\infty$ such that $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$ then we have the isometric isomorphic identification

$$\left(\ell_{m(s,p)}(X)\right)^* \equiv \ell_{q,s} \langle X^* \rangle.$$

through the mapping $T : \ell_{q,s} \langle X^* \rangle \rightarrow \left(\ell_{m(s,p)}(X)\right)^*$ defined by

$$T((x^*_n)_n)((x_n)_n) = \sum_{n \geq 1} x^*_n(x_n),$$

for all $(x^*_n)_n \in \ell_{q,s} \langle X^* \rangle$ and $(x_n)_n \in \ell_{m(s,p)}(X)$.

Proof. First note that $\frac{1}{q} + \frac{1}{p} = 1 + \frac{1}{s}$. It is easy to see that the correspondence $T$ is linear. We take $(x^*_n)_n \in \ell_{q,s} \langle X^* \rangle$ and let $(x_n)_n = (\tau_n x^*_n)_n \in \ell_{m(s,p)}(X)$ where $(\tau_n)_n \in \ell_q$ and $(x^*_n)_n \in \ell_{s,\omega}(X)$. Hence, by Hölder’s inequality it follows that

$$\left| \sum_{n \geq 1} x^*_n(x_n) \right| \leq \sum_{n \geq 1} |\tau_n| |x^*_n(x_n)| \leq \|\tau_n\|_{\ell_q} \left\| (x^*_n(x_n))_n \right\|_{\ell_s} \leq \|\tau_n\|_{\ell_q} \left( \sum_{n \geq 1} \left\| (x^*_n(x_n))_n \right\|_{\ell_s} \right) = \|\tau_n\|_{\ell_q} \left( \sum_{n \geq 1} \left\| (x^*_n(x_n))_n \right\|_{\ell_{q,s}} \right).$$

Since this holds for all possible factorization of the form $x_n = \tau_n x^*_n$, it follows that

$$|T((x^*_n)_n)((x_n)_n)| \leq \|x_n\|_{\ell_{s,\omega}(X)} \left( \left\| (x^*_n(x_n))_n \right\|_{\ell_{q,s}} \right).$$

Since $(x_n)_n$ is arbitrary it follows that

$$\left\| T((x^*_n)_n) \right\| \leq \left\| (x^*_n)_n \right\|_{\ell_{q,s}(X^*)}.$$

This implies that $T$ is well-defined and continuous. Now consider the linear operator $S : \left(\ell_{m(s,p)}(X)\right)^* \rightarrow \ell_{q,s} \langle X^* \rangle$ given by $S(g) = (g \circ \varphi_n)_n$ where $g \in \left(\ell_{m(s,p)}(X)\right)^*$ and $\varphi_n : X \rightarrow \ell_{m(s,p)}(X)$ is the linear operator defined by $\varphi_n(x) = (0, \ldots, 0, x, 0, \ldots)$ with $x$ placed in the $n$-th position. Using (10) and the duality between $\ell_q$ and $\ell_q^*$ we obtain

$$\left\| (g \circ \varphi_n)_n \right\|_{\ell_{q,s}(X^*)} = \sup_{\|x_n\|_{\ell_{s,\omega}(X)} \leq 1} \left\| (g \circ \varphi_n(x_n))_n \right\|_{\ell_q} = \sup_{\|x_n\|_{\ell_{s,\omega}(X)} \leq 1} \sup_{\|x_n\|_{\ell_q} \leq 1} \left| g \left( (\alpha_n(x_n))_n \right) \right| = \sup_{\|x_n\|_{\ell_{s,\omega}(X)} \leq 1} \sup_{\|x_n\|_{\ell_q} \leq 1} \left| g \left( (\alpha_n(x_n))_n \right) \right| \left( \sup_{\|x_n\|_{\ell_{s,\omega}(X)} \leq 1} \left\| (\alpha_n(x_n))_n \right\|_{\ell_{m(s,p)}(X)} \right) \leq \left\| g \right\| \sup_{\|x_n\|_{\ell_{s,\omega}(X)} \leq 1} \left\| (\alpha_n(x_n))_n \right\|_{\ell_{m(s,p)}(X)} \leq \left\| g \right\| \sup_{\|x_n\|_{\ell_{s,\omega}(X)} \leq 1} \left\| (\alpha_n(x_n))_n \right\|_{\ell_{s,\omega}(X)} \leq \left\| g \right\| < \infty.$$
This means that \((g \circ q_n)_n \in \ell_{p',s'}(X^*)\) and we can conclude that \(S\) is well-defined, continuous and \(\|S\| \leq 1\). On the other hand, a straightforward calculation shows that \(S\) and \(T\) are inverses. Finally, if \((x^*_n)_n \in \ell_{p',s'}(X^*)\) then

\[
\|T((x^*_n)_n)\| \leq \|x^*_n\|_{\ell_{p',s'}(X^*)} = \|S \circ T((x^*_n)_n)\|_{\ell_{p',s'}(X^*)} \leq \|T((x^*_n)_n)\|.
\]

\(\square\)

According to the above theorem and Hahn-Banach theorem, we have the following result.

**Corollary 3.11.** Let \(1 \leq p, q, s \leq +\infty\) such that \(\frac{1}{q} + \frac{1}{s} = \frac{1}{p}\). For every \((x_n)_n \in \ell_{m(s,p)}(X)\) we have,

\[
\|x_n\|_{\ell_{m(s,p)}(X)} = \sup_{\|x_n\|_{\ell_{p',s'}(X^*)} \leq 1} \left| \sum_{n \geq 1} x^*_n(x_n) \right|.
\]

A direct consequence of Theorem 3.2 and Theorem 3.10 is the following.

**Corollary 3.12.** We have the two isometric isomorphism identifications

(i) \(\ell_{p,q}(X)^{**} \equiv \ell_{p,q}(X^*)\).

(ii) \(\ell_{m(s,p)}(X)^{**} \equiv \ell_{m(s,p)}(X^*)\).

Using the principle of local reflexivity and previous corollary we obtain the following results.

**Proposition 3.13.** Let \(X\) be a Banach space and \(1 \leq p, q, s \leq +\infty\).

1. If \(\frac{1}{p} = \frac{1}{q} + \frac{1}{s}\) and \((x^*_n)_n \in \ell_{m(s,p)}(X^*)\) then

\[
\|x_n\|_{\ell_{m(s,p)}(X)} = \sup_{\|x_n\|_{\ell_{p',s'}(X^*)} \leq 1} \left| \sum_{n \geq 1} x^*_n(x_n) \right|.
\]

2. If \(\frac{1}{p} = \frac{1}{q} + \frac{1}{s}\) and \((x^*_n)_n \in \ell_{p,q}(X)\) then

\[
\|x_n\|_{\ell_{p,q}(X^*)} = \sup_{\|x_n\|_{\ell_{p',s'}(X^*)} \leq 1} \left| \sum_{n \geq 1} x^*_n(x_n) \right|.
\]

**Proof.** 1) Let \((x^*_n)_n \in \ell_{m(s,p)}(X)\). Since \(\ell_{p',s'}(X) \subseteq \ell_{p',s'}(X^*) \equiv (\ell_{q,s'}(X))^\ast\), we have

\[
\|x_n\|_{\ell_{m(s,p)}(X)} = \sup_{\|x_n\|_{\ell_{p',s'}(X^*)} \leq 1} \left| \sum_{n \geq 1} x^*_n(x_n) \right| \geq \sup_{\|x_n\|_{\ell_{q,s'}(X)} \leq 1} \left| \sum_{n \geq 1} x^*_n(x_n) \right|.
\]

For the reverse inequality, let \(E\) be the linear space spanned by the finite set \(\{x_1^*, ..., x_N^*\} \subset X^\ast\). By the principle of local reflexivity for each \(\varepsilon > 0\) there exists a bounded linear operator \(u : E \rightarrow X\) such that \(\|u\| \leq 1\) and \(\|x^*_j(x') - x^*_j(u(x'_j))\| \leq \frac{\varepsilon}{N}\) for all \(x'_j \in X^\ast, j = 1, ..., N\). Then

\[
\sum_{j \in N} \left| x^*_j(x'_j) \right| \leq \varepsilon + \sum_{j \in N} \left| x^*_j(u(x'_j)) \right| \\
\leq \varepsilon + \|x^*_n\|_{\ell_{p',s'}(X^*)} \sup_{\|x_n\|_{\ell_{q,s'}(X)} \leq 1} \left| \sum_{n \geq 1} x^*_n(x_n) \right|.
\]
Since this holds for every $N \in \mathbb{N}$ and $\varepsilon > 0$ it follows that
\[
\|(x_n^m)_{n}\|_{\ell_{p,(r,s)}(X)} = \sup_{\|(x_n^m)_{m}\|_{\ell_{p,(r,s)}(X)} \leq 1} \sum_{n \geq 1} |x_n^m(x_n)| \leq \sup_{\|(x_n^m)_{m}\|_{\ell_{p,(r,s)}(X)} \leq 1} \sum_{n \geq 1} |x_n^m(x_n)|.
\]

Part (2) is proved in a similar way. □

Remark 3.14. If we apply Theorem 3.2 and Theorem 3.10 for some extreme cases of parameters $p$, $q$, and $s$, we obtain the well-known duality identifications for the sequence spaces $\ell_q(X)$, $\ell_p(X)$ and $\ell_{p,\omega}(X)$.

(i) In the Theorem 3.2 if we take $p = 1$, then by (3) and (6) we obtain
\[
(\ell_q(X))^* \equiv (\ell_{1,q}(X))^* \equiv \ell_{m(q,r')}(X)^* \equiv \ell_{q,\omega}(X^*).
\]

(ii) In the Theorem 3.2 if we take $p = s$, then by (4) and Corollary 3.4 we obtain
\[
(\ell_q(X))^* \equiv (\ell_{p,1}(X))^* \equiv \ell_{m(\infty,p')}(X)^* \equiv \ell_{p'}(X^*).
\]

(iii) In the Theorem 3.10 if we take $s = p$, then we obtain
\[
(\ell_{p,\omega}(X))^* \equiv (\ell_{m(p,p)}(X))^* \equiv \ell_{1,p'}(X^*) \equiv \ell_p(X^*).
\]

In the following proposition we give the relation between the space of the strongly $(q,s)$-summable sequences and the spaces of the absolutely (strongly) $p$-summable sequences.

Proposition 3.15. Let $1 \leq p, q, s \leq \infty$ such that $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{s}$, then
\[
\ell_p(X) \subset \ell_{q,s}(X) \subset \ell_p(X).
\]

In this case we have
\[
\|(x_n^m)_{n}\|_{\ell_{q,s}(X)} \leq \|(x_n^m)_{n}\|_{\ell_p(X)} \leq \|(x_n^m)_{n}\|_{\ell_p(X)},
\]
for each $(x_n^m)_{n} \in \ell_p(X)$.

Proof. Since $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$ we get $\ell_p(X^*) \subset \ell_{m(p,q')}(X^*) \subset \ell_{p,\omega}(X^*)$. Let $(x_n^m)_{n} \in \ell_p(X)$. From the duality between $\ell_p(X)$ and $\ell_p(X^*)$ and equality (7), we obtain
\[
\|(x_n^m)_{n}\|_{\ell_p(X)} = \sup_{\|(x_n^m)_{m}\|_{\ell_{p,(r,s)}(X)} \leq 1} \sum_{n \geq 1} |x_n^m(x_n)| \leq \sup_{\|(x_n^m)_{m}\|_{\ell_{m(p,q')}}(X^*) \leq 1} \sum_{n \geq 1} |x_n^m(x_n)| \leq \|(x_n^m)_{n}\|_{\ell_{p,\omega}(X^*)} \leq \|(x_n^m)_{n}\|_{\ell_p(X)} < \infty.
\]

□

Regarding Proposition 3.15, let us show with an example the difference between $\ell_{q,s}(X)$ and $\ell_p(X)$.
Example 3.16. Let \((e_n)_n\) the unit vector basis of \(\ell_2\). The sequence \((x_n)_n\) defined by \(x_n = \frac{1}{\sqrt{n}}e_n\) belongs to \(\ell_\infty(\ell_2)\) but it is not in \(\ell_{2,2}(\ell_2)\). In order to see this, \(\| (x_n)_n \|_{\ell_\infty(\ell_2)} = \sup_n \frac{1}{\sqrt{n}} = 1\). On the other hand, since

\[
\| (e_n)_n \|_{\ell_{2,2}(\ell_2)} = \| (e_n)_n \|_{\ell_\infty(\ell_2)} = 1,
\]

we have that

\[
\| (x_n)_n \|_{\ell_{2,2}(\ell_2)} \geq \| (e_n^*(x_n))_n \|_{\ell_2} = \left( \sum_{n \geq 1} \frac{1}{n} \right)^{\frac{1}{2}} = +\infty.
\]

4. Applications to \((r,p,q)\)-summing operators

Let \(X \subset X^N\) and \(Y \subset Y^N\) be spaces of vector valued sequences in \(X\) and \(Y\) respectively. A linear continuous operator \(T \in \mathcal{L}(X, Y)\), between Banach spaces, induces a linear operator \(\hat{T}\) mapping \(X\) into \(Y^N\) in the following way: \(\hat{T}(x_n)_n = (T(x_n))_n\) for all \((x_n)_n \in X\). In the sequel, if \(\hat{T}(X) \subset Y\), we say that \(T\) transfers \(X\) into \(Y\).

Throughout this section, let \(1 \leq p, q, r \leq \infty\) such that \(\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}\). The definition of \((r,p,q)\)-summing operators is due to Pietsch [9, Section 17.1].

Definition 4.1. An operator \(T \in \mathcal{L}(X, Y)\) is \((r,p,q)\)-summing, in symbols \(T \in \Pi_{r,p,q}(X, Y)\), if there is \(C > 0\) such that

\[
\left\| (y_j^*(T(x_i)))_{1 \leq i \leq N} \right\|_{\ell_r} \leq C \left\| (x_i)_{1 \leq i \leq N} \right\|_{\ell_{p,q}(X)} \left\| (y_j^*)_{1 \leq j \leq N} \right\|_{\ell_{r,p,q}(Y)},
\]

for all \(n \in \mathbb{N}\), \((x_i)_{1 \leq i \leq n} \subset X\) and \((y_j^*)_{1 \leq j \leq n} \subset Y^*\).

This is equivalent to say that \(T\) induces a bounded bilinear map

\[
\hat{T} : \ell_{p,q}(X) \times \ell_{r,p,q}(Y) \to \ell_r, \quad \hat{T} \left((x_n)_n, (y_n)_n\right) = (\langle x_n, y_n \rangle)_n,
\]

(see [6, Page 196]). Note that \(\Pi_{r,p,q}(X, Y)\) is a Banach space equipped with the norm \(\pi_{r,p,q}(T)\) which is the smallest constant \(C\) satisfying the defining inequality or \(\pi_{r,p,q}(T) = \| T \|\).

As in the case of \(p\)-summing operators, the natural way of presenting the summability properties of \((r,p,q)\)-summing operators is by defining the corresponding operator \(\hat{T}\) between \(\ell_{p,q}(X)\) and \(\ell_{r,p,q}(Y)\).

Proposition 4.2. The operator \(T \in \mathcal{L}(X, Y)\) is \((r,p,q)\)-summing if and only if \(T\) transfers \(\ell_{p,q}(X)\) into \(\ell_{r,p,q}(Y)\).

Proof. Indeed, starting from (11) and pass to the limit for \(n\) tending to \(\infty\) we obtain

\[
\| T(x_n)_n \|_{\ell_{r,p}} \leq \pi_{r,p,q}(T) \| (x_n)_n \|_{\ell_{p,q}(X)},
\]

for all \((x_n)_n \in \ell_{p,q}(X)\). Then it follows that \(\hat{T} : \ell_{p,q}(X) \to \ell_{r,p,q}(Y)\) is well-defined and \(\hat{T} \left(\ell_{p,q}(X)\right) \subset \ell_{r,p,q}(Y)\).

In addition \(\hat{T}\) is continuous with norm \(\leq \pi_{r,p,q}(T)\). Suppose conversely that \(T\) transfers \(\ell_{p,q}(X)\) into \(\ell_{r,p,q}(Y)\), an appeal to the closed graph theorem shows that \(\hat{T}\) is continuous and

\[
\| (T(x_i))_{1 \leq i \leq N} \|_{\ell_{r,p,q}(Y)} \leq \| T \| \| (x_i)_{1 \leq i \leq N} \|_{\ell_{p,q}(X)},
\]

and so \(T \in \Pi_{r,p,q}(X, Y)\) with \(\pi_{r,p,q}(T) \leq \| T \|\). □
In the next result we give a new characterization of the \((r, p, q)-\text{summing operators}\) by using the Banach spaces of strongly \(q^*-\text{summing}\) and mixed \((p, s)\)-\text{summable sequences}\) obtaining in this way another corresponding operator \(\tilde{T}\) of the \((r, p, q)\)-\text{summing operator}\ \(T\).

**Theorem 4.3.** Let \(p, q, r, s \geq 1\) such that \(\frac{1}{r} = \frac{1}{s} + \frac{1}{p} + \frac{1}{q}\). The operator \(T \in \mathcal{L}(X, Y)\) is \((r, p, q)\)-\text{summing} if and only if there is a constant \(C > 0\) such that for any \(x_1, \ldots, x_n \in X\) we have

\[
\|T(x_i)\|_{1 \leq \infty} \leq C \|x_i\|_{1 \leq \infty} \|\alpha_x\|_{\ell_p(\mathcal{X})}.
\]

**Proof.** Suppose that \(T \in \Pi_{r,p,q}(X, Y)\). Let \(\left\langle y_i^* \right\rangle_{1 \leq \infty} \subset Y^*, \left\langle x_i \right\rangle_{1 \leq \infty} \subset X\) and \(\varepsilon > 0\). Choose \((\alpha_i)_{1 \leq \infty} \subset \mathcal{K}\) and \((z_i)_{1 \leq \infty} \subset X\) such that \(x_i = \alpha_i z_i, i = 1, \ldots, n\) and \(\| (\alpha_i)_{1 \leq \infty} \|_{\ell_p(\mathcal{X})} \| (z_i)_{1 \leq \infty} \|_{\ell_{p,q}(X)} \leq (1 + \varepsilon) \| (x_i)_{1 \leq \infty} \|_{\ell_{p,q}(X)}\). By Hölder’s inequality we get

\[
\sum_{1 \leq \infty} y_i^* (T(x_i)) = \sum_{1 \leq \infty} \alpha_i y_i^* (T(z_i)) \leq \| (\alpha_i)_{1 \leq \infty} \|_{\ell_p(\mathcal{X})} \| (y_i^*)_{1 \leq \infty} \|_{\ell_{p,q}(X)} \| (z_i)_{1 \leq \infty} \|_{\ell_{p,q}(X)} \| (y_i)_{1 \leq \infty} \|_{\ell_{p,q}(Y)}.
\]

By taking the supremum over all \((y_i)_{1 \leq \infty}\) such that \(\| (y_i)_{1 \leq \infty} \|_{\ell_{p,q}(Y)} \leq 1\) we obtain

\[
\|T(x_i)\|_{1 \leq \infty} \leq \pi_{r,p,q}(T)(1 + \varepsilon) \| (x_i)_{1 \leq \infty} \|_{\ell_{p,q}(X)}.
\]

Since this holds for every \(\varepsilon > 0\), we obtain (13).

Suppose conversely that the operator \(T\) satisfies the condition (13). For all \((y_i)_{1 \leq \infty} \subset Y^*, (x_i)_{1 \leq \infty} \subset X\) and \((\alpha_i)_{1 \leq \infty} \subset \mathcal{K}\) we have

\[
\sum_{1 \leq \infty} \alpha_i y_i^* (T(x_i)) = \sum_{1 \leq \infty} y_i^* (T(\alpha_i x_i)) \leq \| (y_i)_{1 \leq \infty} \|_{\ell_{p,q}(Y)} \| (T(\alpha_i x_i))_{1 \leq \infty} \|_{\ell_{p,q}(X)} \| (\alpha_i x_i)_{1 \leq \infty} \|_{\ell_{p,q}(X)} \| (\alpha_i x_i)_{1 \leq \infty} \|_{\ell_{p,q}(X)} \| (x_i)_{1 \leq \infty} \|_{\ell_{p,q}(X)} \| (y_i)_{1 \leq \infty} \|_{\ell_{p,q}(Y)}.
\]

Taking the supremum over all \((\alpha_i)_{1 \leq \infty} \subset \mathcal{K}\) such that \(\| (\alpha_i)_{1 \leq \infty} \|_{\ell_p(\mathcal{X})} \leq 1\) we get

\[
\| (y_i^* (T(x_i))_{1 \leq \infty} \|_{\ell_p(\mathcal{X})} \| (y_i^*)_{1 \leq \infty} \|_{\ell_{p,q}(Y)}.
\]

\(\square\)

The next corollary and its proof are similar to Proposition 4.2 except that (13) is used instead of (12).

**Corollary 4.4.** \(T \in \Pi_{r,p,q}(X, Y)\) if and only if \(T\) transfers \(\ell_{m,p,q}(X)\) into \(\ell_{p,q}(Y)\). In addition we have \(\pi_{r,p,q}(T) = \|T\|\).

Although the following result is essentially already known (it was proved by Pietsch, see [9, Theorem 17.1.5]), we write a new direct proof that highlights the role of the dual space of \(\ell_{m,(p,q)}(X)\) and \(\ell_{p,q}(X)\).
By using the above corollary, Proposition 4.2, the identifications \((\ell_m(p,s)(X))^\ast \equiv \ell_{q,r}(X^\ast)\) and \((\ell_r(q)\langle Y \rangle)^\ast \equiv \ell_{q,\omega}(Y^\ast)\) and taking into account that the adjoint of the operator \(\tilde{T} : \ell_m(p,s)(X) \rightarrow \ell_r(q)\langle Y \rangle\) can be identified with the operator

\[
\tilde{T}^\ast : \ell_{q,\omega}(Y^\ast) \rightarrow \ell_{r,p}(X^\ast); \quad \tilde{T}^\ast((y^\ast)_i) = (T^\ast(y^\ast)_i),
\]

we have the following.

**Theorem 4.5.** The operator \(T\) belongs to \(\Pi_{r,p,q}(X,Y)\) if and only if \(T^\ast\) belongs to \(\Pi_{r,q,p}(Y^\ast,X^\ast)\). Furthermore, \(\pi_{r,p,q}(T) = \pi_{r,q,p}(T^\ast)\).

It is easy to prove the following result.

**Corollary 4.6.** The operator \(T\) belongs to \(\Pi_{r,p,q}(X,Y)\) if and only if its bi-adjoint \(T^{{*{*}}}\) belongs to \(\Pi_{r,p,q}(X^{{*{*}}},Y^{{*{*}}})\). In addition, \(\pi_{r,p,q}(T) = \pi_{r,p,q}(T^{{*{*}}}^\ast)\).

**References**