Filomat 34:11 (2020), 3639–3654
https://doi.org/10.2298/FIL2011639Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# A New Young Type Inequality Involving Heinz Mean

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**Abstract.** In this paper, we gave a new Young type inequality and the relevant Heinz mean inequality. Furthermore, we also improved some inequalities with Kantorovich constant or Specht's ratio. Meanwhile, on the base of our scalars results, we obtain some new corresponding operator inequalities and matrix versions including Hilbert-Schmidt norm, unitarily invariant norm and related trace versions, which can be regarded as the application of our scalar results.

## 1. Introduction

It has been universally acknowledged that the classical Young inequality for scalars says that if $a, b \geq 0$ and $v \in [0,1]$, then

$$a^v b^{1-v} \leq va + (1-v)b. \tag{1.1}$$

with equality if and only if $a = b$. Simple as it is, what the inequality (1.1) conveys to us is not only interesting in itself but also significant in operator theory. Refining this inequality has taken great attention of a considerable number of researchers in this field, as a consequence, where adding a positive term or multiplying a coefficient which is greater or equal to the number 1 to the left side is possible. If $v = \frac{1}{2}$, particularly, we can get the arithmetic-geometric mean inequality

$$\sqrt{ab} \leq \frac{a+b}{2}$$

Among the first refinements and reverses in [9, 10] of this inequality were proved by Kittaneh and Manasrah, which can be stated in the following form

$$a^v b^{1-v} + r_0(\sqrt{a} - \sqrt{b})^2 \leq va + (1-v)b \leq a^v b^{1-v} + R_0(\sqrt{a} - \sqrt{b})^2. \tag{1.2}$$

2010 *Mathematics Subject Classification*. Primary 47A63 ; Secondary 46B10
*Keywords*. Kantorovich constant; Specht's ratio; Young type inequality; Heinz mean; Hilbert-Schmidt norm
Received: 18 November 2019; Revised: 13 March 2020; Accepted: 13 July 2020
Communicated by Dragan S. Djordjević
Research supported National Natural Science Foundation of China with grant (no. 11271112; 11771126,11701154) and Program for Graduate Innovative Research of Henan Normal University(No.YL201919).
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where $r_0 = \min\{v, 1-v\}$, $R_0 = \max\{v, 1-v\}$, $a, b \geq 0, v \in [0,1]$. The other improvements and reverses in [16] were presented by Zhao and Wu as follows:

$$\begin{aligned} &a^v b^{1-v} + v(\sqrt{a} - \sqrt{b})^2 + r_0(\sqrt[4]{ab} - \sqrt{b})^2 \leq va + (1-v)b \\ &\leq a^v b^{1-v} + (1-v)(\sqrt{a} - \sqrt{b})^2 - r_0(\sqrt[4]{ab} - \sqrt{a})^2, \quad 0 \leq v \leq \frac{1}{2} \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &a^v b^{1-v} + (1-v)(\sqrt{a} - \sqrt{b})^2 + r_0(\sqrt[4]{ab} - \sqrt{a})^2 \leq va + (1-v)b \\ &\leq a^v b^{1-v} + v(\sqrt{a} - \sqrt{b})^2 - r_0(\sqrt[4]{ab} - \sqrt{b})^2. \qquad \frac{1}{2} \leq v \leq 1 \end{aligned} \tag{1.4}$$

where $r = min\{v, 1-v\}, r_0 = min\{2r, 1-2r\}$.

Also, Furuichi and Tominaga obtained respectively the other interesting multiplicative refinement and reverse with Specht's ratio in [5, 14]

$$S((\frac{b}{a})^r)a^v b^{1-v} \leq va + (1-v)b \leq S(\frac{b}{a})a^v b^{1-v} \tag{1.5}$$

for $a, b > 0$, $v \in [0,1]$ and $r = min\{v, 1-v\}$, where the Specht's ratio [5, 6, 14] was defined by

$$S(h) = \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h \in (0,1) \cup (1, \infty), \\ 1 & \text{if } h = 1. \end{cases} \tag{1.6}$$

The function $S(\cdot)$ has the following properties:

i) $S(\cdot)$ is decreasing on $(0,1)$ and increasing on $(1, \infty)$;

ii) $\lim_{h \to 1} S(h) = 1$, $S(h) = S(\frac{1}{h}) > 1$ for any $h > 0, h \neq 1$.

In addition, the inequality (1.1) was refined and reversed respectively with Kantorovich constant by Zuo and Liao [12, 17] as follows:

$$K^r(\frac{b}{a}, 2)a^v b^{1-v} \leq va + (1-v)b \leq K^R(\frac{b}{a}, 2)a^v b^{1-v}. \tag{1.7}$$

for $a, b > 0$, $v \in [0,1]$, where $r = min\{v, 1-v\}$, $R = max\{v, 1-v\}$, $K(h, 2)$ is the Kantorovich constant defined by

$$K(h, 2) = \frac{(h+1)^2}{4h}, h > 0. \tag{1.8}$$

Likely, the function $K(\cdot, 2)$ has the following properties:

i) $K(\cdot, 2)$ is decreasing on $(0,1)$ and increasing on $(1, \infty)$;

ii) $K(h, 2) = K(\frac{1}{h}, 2) \geq 1$ for any $h > 0$.

It is obvious that a common fact that it refers to (1.5) and (1.7) which are the refinements and reverses of (1.1) is multiplying one coefficient.

Several years later, Hu in [8] gave the following Young type inequality

$$(v^2 a)^v b^{1-v} + v^2(\sqrt{a} - \sqrt{b})^2 \leq v^2 a + (1-v)^2 b, \qquad 0 \leq v \leq \frac{1}{2} \tag{1.9}$$

$$a^v[(1-v)^2 b]^{1-v} + (1-v)^2(\sqrt{a} - \sqrt{b})^2 \leq v^2 a + (1-v)^2 b. \quad \frac{1}{2} \leq v \leq 1 \tag{1.10}$$

for $a, b \geq 0, v \in [0,1]$.

Afterwards, Burqan and Khandaqji in [2] presented the reverse of (1.9) and (1.10), which can be stated in the following form

$$v^2a + (1-v)^2b \le a^v[(1-v)^2b]^{1-v} + (1-v)^2(\sqrt{a} - \sqrt{b})^2, \quad 0 \le v \le \frac{1}{2} \tag{1.11}$$

$$v^2a + (1-v)^2b \le (v^2a)^v b^{1-v} + v^2(\sqrt{a} - \sqrt{b})^2. \qquad \frac{1}{2} \le v \le 1 \tag{1.12}$$

In a recent work, Ghazanfari, Malekinejad and Talebi in [7] gave a new inequality, which can be stated that if $a, b \ge 0$ and $v \in (0, 1]$, then

$$(1 - v^2 + v^3)a + (1 - v^2)b \le v^{v-2}a^v b^{1-v} + (\sqrt{a} - \sqrt{b})^2. \tag{1.13}$$

Throughout the paper, $M_n$ denotes the space of all $n \times n$ complex matrices. $M_n^+$ denotes the set of all positive semidefinite matrices in $M_n$, $X \ge Y$ for $X, Y \in M_n$ means that $X$ and $Y$ are Hermitians and $X - Y \in M_n^+$. The set of all strictly positive definite matrices in $M_n$ is denoted by $M_n^{++}$. The unitarily invariance of the $\|.\|$ on $M_n$ means that $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all the unitary matrices $U, V \in M_n$. For $A = [a_{ij}] \in M_n$, the Hilbert-Schmidt (or Frobenius) norm and the trace norm of $A$ are defined by

$$\|A\|_2 = \sqrt{\sum_{j=1}^{n} s_j^2(A)}, \qquad \|A\|_1 = \sum_{j=1}^{n} s_j(A),$$

respectively, where $s_1(A) \ge s_2(A) \ge ... \ge s_n(A)$ are the singular values of $A$, that is, the eigenvalues of the positive matrix $|A| = (A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. Moreover, it is well known that $\|.\|_2$ is unitarily invariant.

What is more, let $B(H)$ denotes the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $H$. In the case of dim $H = n$, we identify $B(H)$ with the matrix Algebra of all $n \times n$ matrices with entries in the complex field. A operator $A \in B(H)$ is called positive, if

$$\langle Ax, x\rangle \ge 0$$

for all $x \in H$, and we write $A \ge 0$. The set of all positive operators on a complex Hilbert space $H$ is denoted by $B^+(H)$. Also, the set of all positive invertible operators on a complex Hilbert space H is denoted with $B^{++}(H)$. If $A \in B^{++}(H)$, we write $A > 0$.

For the notations adopted in this paper, moreover, we defined $v$−weighted arithmetic mean, geometric mean for scalars and operators

$$a\nabla_v b = (1-v)a + vb, \quad a\sharp_v b = a^{1-v}b^v;$$
$$A\nabla_v B = (1-v)A + vB, \quad A\sharp_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}}$$

for $a, b \ge 0$, $v \in [0, 1]$ and $A, B \in B^{++}(H)$. In particular, denoted by $a\nabla b$ and $a\sharp b$, $A\nabla B$ and $A\sharp B$ for brevity respectively when $v = \frac{1}{2}$.

Also, Heinz means for scalars and operators introduced in [3] were defined as

$$H_v(a, b) = \frac{a\sharp_v b + a\sharp_{1-v} b}{2}, \quad a, b \ge 0, \quad v \in [0, 1]$$
$$H_v(A, B) = \frac{A\sharp_v B + A\sharp_{1-v} B}{2}, \quad A, B \in B^{++}(H), \quad v \in [0, 1]$$

which can be interpolated between arithmetic and geometric means. Put another way,

$$a\sharp b \le H_v(a, b) \le a\nabla b, \quad A\sharp B \le H_v(A, B) \le A\nabla B$$

were called Heinz means inequalities.

This paper is organized in the following way: In Section 2, we first give the reverse of a new Young type inequality for scalar, and then we refine this inequality with Specht's ratio and Kantorovich constant. Subsequently, we obtain the new corresponding Heinz mean inequalities. In Section 2, on the basis of our main scalar results, we obtained the related Heinz operator mean version, including the refinement of (3.1). In the last Section, as an application, we establish some relevant inequalities including Heinz mean or Young type of matrix version for Hilbert-Schmidt norm, unitarily invariant norm, and trace versions based on the conclusion of part one.

## 2. Reverse of Young Type and Heinz mean Inequality with Kantorovich Constant or Specht's ratio

First of all, we prove the reverse of a new Young type inequality, then we get the new Heinz mean inequality and give some discussions for the new Young type inequality (2.2). Because this part is mainly researched Heinz mean, it does not take (2.2), (2.4), (2.8) as a theorem respectively to prove. However, these conclusions are also our main results. For convenience, let $\mathbb{R}$ denote the field of real numbers.

**Theorem 2.1** Suppose that $a, b \geq 0$, $N_1, N_2 \in \mathbb{R}$ and $v \in (0,1]$, then

$$\left[v^{N_1+1}(v-1) - v^{N_2+2}\right](a\nabla b) + 2(a\sharp b) \leq v^{-(1-v)N_1-vN_2-1}H_v(a,b). \tag{2.1}$$

*Proof.* Now, we prove the following new Young type inequality which can be regard as one of our main results in the first place.

$$(1 - v^{N_1+1} + v^{N_1+2})a + (1 - v^{N_2+2})b \leq v^{-(1-v)N_1-vN_2-1}a^v b^{1-v} + (\sqrt{a} - \sqrt{b})^2. \tag{2.2}$$

By (1.1) and simple calculation, then we have

$$\begin{aligned}
&(\sqrt{a} - \sqrt{b})^2 - (1 - v^{N_1+1} + v^{N_1+2})a - (1 - v^{N_2+2})b + v^{-(1-v)N_1-vN_2-1}a^v b^{1-v}\\
&= (1-v)v^{N_1+1}a + v^{N_2+2}b - 2\sqrt{ab} + v^{-(1-v)N_1-vN_2-1}a^v b^{1-v}\\
&\geq (v^{N_1+1}a)^{1-v}(v^{N_2+1}b)^v - 2\sqrt{ab} + v^{-(1-v)N_1-vN_2-1}a^v b^{1-v}\\
&= v^{(1-v)N_1+vN_2+1}a^{1-v}b^v - 2\sqrt{ab} + v^{-(1-v)N_1-vN_2-1}a^v b^{1-v}\\
&= (v^{\frac{(1-v)N_1+vN_2+1}{2}}a^{\frac{1-v}{2}}b^{\frac{v}{2}} - v^{\frac{-(1-v)N_1-vN_2-1}{2}}a^{\frac{v}{2}}b^{\frac{1-v}{2}})^2\\
&\geq 0
\end{aligned}$$

So (2.2) holds.

And then, if we replace $a$ by $b$, $b$ by $a$ and add the resulting inequality to (2.2), we get the desired result (2.1). This completes the proof. □

**Remark 2.2** For one thing, it's obvious that (1.13) is a special case of inequality (2.2) for $N_1 = 1, N_2 = 0$, which implies that (2.2) is a generalization of (1.13). And for any $N_1, N_2 \geq 0$, it's not difficult to find that both the left hand side and the right hand side in inequality (2.2) are greater than or equal to the corresponding sides in the second inequalities of (1.3) and (1.4) respectively, which indicates that (2.2) can be regarded as a new Young type inequality.

By the similar way in Theorem 2.1, we obtain Young type inequality with Specht's ratio or Kantorovich constant which is the refinement of (2.2) and get the new relevant Heniz mean inequality.

**Theorem 2.3** For $a, b > 0$, $v \in (0,1]$ and $N_1, N_2 \in \mathbb{R}$, then

$$\begin{aligned}
&S^{-1}(h^r)H_v(a,b)\\
&\geq \frac{1}{2}\left[(v-1)v^{(2-v)N_1+vN_2+2} - v^{(1+v)N_1+(1-v)N_2+3}\right]a\\
&+ \frac{1}{2}\left[(v-1)v^{vN_1+(2-v)N_2+2} - v^{(1-v)N_1+(1+v)N_2+3}\right]b\\
&+ (v^{(1-v)N_1+vN_2+1} + v^{vN_1+(1-v)N_2+1})(a\sharp b),
\end{aligned} \tag{2.3}$$

where $r = min\{v, 1-v\}, h = \frac{v^{N_1-N_2}a}{b} \neq 1$ and $S(.)$ is the Specht's ratio.

*Proof.* First of all, we prove

$$(1 - v^{N_1+1} + v^{N_1+2})a + (1 - v^{N_2+2})b \leq S^{-1}(h^r)v^{-(1-v)N_1-vN_2-1}a^v b^{1-v} + (\sqrt{a} - \sqrt{b})^2. \tag{2.4}$$

According to the first inequality of (1.5), we have

$$\begin{aligned}
&S^{-1}(h^r)v^{-(1-v)N_1-vN_2-1}a^v b^{1-v} + (\sqrt{a} - \sqrt{b})^2 - (1 - v^{N_1+1} + v^{N_1+2})a - (1 - v^{N_2+2})b\\
&= S^{-1}(h^r)v^{-(1-v)N_1-vN_2-1}a^v b^{1-v} + v^{N_1+1}(1-v)a + v^{N_2+2}b - 2\sqrt{ab}\\
&\geq S^{-1}(h^r)v^{-(1-v)N_1-vN_2-1}a^v b^{1-v} + S(h^r)(v^{N_1+1}a)^{1-v}(v^{N_2+1}b)^v - 2\sqrt{ab}\\
&= S^{-1}(h^r)v^{-(1-v)N_1-vN_2-1}a^v b^{1-v} - 2\sqrt{ab} + S(h^r)v^{(1-v)N_1+vN_2+1}a^{1-v}b^v\\
&= \left(S^{-\frac{1}{2}}(h^r)v^{\frac{-(1-v)N_1-vN_2-1}{2}}a^{\frac{v}{2}}b^{\frac{1-v}{2}} - S^{\frac{1}{2}}(h^r)v^{\frac{(1-v)N_1+vN_2+1}{2}}a^{\frac{1-v}{2}}b^{\frac{v}{2}}\right)^2\\
&\geq 0
\end{aligned}$$

So we get the required inequality (2.4).
Now, by exchanging $a$ for $b$, $N_1$ for $N_2$ in (2.4) and applying the property $S(h) = S(h^{-1})$, we have

$$(1 - v^{N_2+1} + v^{N_2+2})b + (1 - v^{N_1+2})a \leq S^{-1}(h^r)v^{-(1-v)N_2-vN_1-1}a^{1-v}b^v + (\sqrt{a} - \sqrt{b})^2.$$

That is

$$v^{(1-v)N_2+vN_1+1}[v^{N_2+1}(v-1)b - v^{N_1+2}a] \leq S^{-1}(h^r)a^{1-v}b^v - 2v^{(1-v)N_2+vN_1+1}\sqrt{ab}. \tag{2.5}$$

Similarly, the inequality (2.4) can be restated as

$$v^{(1-v)N_1+vN_2+1}[v^{N_1+1}(v-1)a - v^{N_2+2}b] \leq S^{-1}(h^r)a^v b^{1-v} - 2v^{(1-v)N_1+vN_2+1}\sqrt{ab}.$$

Finally, if we add (2.5) to the above inequality, we get

$$\begin{aligned}
&S^{-1}(h^r)(a^v b^{1-v} + a^{1-v}b^v)\\
&\geq \left[(v-1)v^{(2-v)N_1+vN_2+2} - v^{(1+v)N_1+(1-v)N_2+3}\right]a + \left[(v-1)v^{vN_1+(2-v)N_2+2}\right.\\
&\left.- v^{(1-v)N_1+(1+v)N_2+3}\right]b + 2(v^{(1-v)N_1+vN_2+1} + v^{vN_1+(1-v)N_2+1})\sqrt{ab}
\end{aligned}$$

which is equivalent to the desired result (2.3). This completes the proof. □

The Corollary 2.4 is a special case of Theorem 2.3.

**Corollary 2.4** Let all assumptions of Theorem 2.3 be satisfied and $N_1 = N_2 = N \in \mathbb{R}$, then

$$S^{-1}(h^r)H_v(a,b) + v^{2(N+1)}(a\nabla b) \geq 2v^{N+1}(a\sharp b). \tag{2.6}$$

**Theorem 2.5** Let $a, b > 0, v \in (0,1]$ and $N_1, N_2 \in \mathbb{R}$, then

$$\begin{aligned}
&K^{-r}(h,2)H_v(a,b)\\
&\geq \frac{1}{2}\left[(v-1)v^{(2-v)N_1+vN_2+2} - v^{(1+v)N_1+(1-v)N_2+3}\right]a\\
&+ \frac{1}{2}\left[(v-1)v^{vN_1+(2-v)N_2+2} - v^{(1-v)N_1+(1+v)N_2+3}\right]b\\
&+ (v^{(1-v)N_1+vN_2+1} + v^{vN_1+(1-v)N_2+1})(a\sharp b),
\end{aligned} \tag{2.7}$$

where $r = min\{v, 1-v\}, h = \frac{v^{N_1-N_2}a}{b}$ and $K(\cdot, 2)$ is Kantorovich constants.

*Proof.* By the first inequality of (1.7), we have
$K^{-r}(h,2)v^{-(1-v)N_1-vN_2-1}a^v b^{1-v} + (\sqrt{a}-\sqrt{b})^2 - (1-v^{N_1+1}+v^{N_1+2})a - (1-v^{N_2+2})b$
$= K^{-r}(h,2)v^{-(1-v)N_1-vN_2-1}a^v b^{1-v} + v^{N_1+1}(1-v)a + v^{N_2+2}b - 2\sqrt{ab}$
$\geq K^{-r}(h,2)v^{-(1-v)N_1-vN_2-1}a^v b^{1-v} + K^r(h,2)(v^{N_1+1}a)^{1-v}(v^{N_2+1}b)^v - 2\sqrt{ab}$
$= K^{-r}(h,2)v^{-(1-v)N_1-vN_2-1}a^v b^{1-v} - 2\sqrt{ab} + K^r(h,2)v^{(1-v)N_1+vN_2+1}a^{1-v}b^v$
$= \left(K^{-\frac{r}{2}}(h,2)v^{\frac{-(1-v)N_1-vN_2-1}{2}}a^{\frac{v}{2}}b^{\frac{1-v}{2}} - K^{\frac{r}{2}}(h,2)v^{\frac{(1-v)N_1+vN_2+1}{2}}a^{\frac{1-v}{2}}b^{\frac{v}{2}}\right)^2$
$\geq 0$
That is

$$(1-v^{N_1+1}+v^{N_1+2})a + (1-v^{N_2+2})b \leq K^{-r}(h,2)v^{-(1-v)N_1-vN_2-1}a^v b^{1-v} + (\sqrt{a}-\sqrt{b})^2. \tag{2.8}$$

Now, if we interchange $a$ for $b$ and $N_1$ for $N_2$ in (2.8), and then adding the resulting inequality to (2.8), we get the required result (2.7). □

**Corollary 2.6** Let all assumptions of Theorem 2.5 be satisfied and $N_1 = N_2 = N \in \mathbb{R}$, then

$$K^{-r}(h,2)H_v(a,b) + v^{2(N+1)}(a\nabla b) \geq 2v^{N+1}(a\sharp b). \tag{2.9}$$

By replacing $a$ by $a^2$ and $b$ by $b^2$ in inequalities (2.2), (2.4) and (2.8) respectively, we obtain the following corollary.

**Corollary 2.7** With the assumptions of Theorem 2.5 we have

$$(1-v^{N_1+1}+v^{N_1+2})a^2 + (1-v^{N_2+2})b^2 \leq v^{-(1-v)N_1-vN_2-1}(a^v b^{1-v})^2 + (a-b)^2. \tag{2.10}$$

$$(1-v^{N_1+1}+v^{N_1+2})a^2 + (1-v^{N_2+2})b^2 \leq S^{-1}(t^r)v^{-(1-v)N_1-vN_2-1}(a^v b^{1-v})^2 + (a-b)^2 \tag{2.11}$$

and

$$(1-v^{N_1+1}+v^{N_1+2})a^2 + (1-v^{N_2+2})b^2 \leq K^{-r}(t,2)v^{-(1-v)N_1-vN_2-1}(a^v b^{1-v})^2 + (a-b)^2. \tag{2.12}$$

where $r = min\{v, 1-v\}, t = \frac{v^{N_1-N_2}a^2}{b^2} \neq 1$.

To achieve our further results, we need the following lemma.

**Lemma 2.8** Let $\phi$ be a strictly increasing convex function defined on an interval $D$. If $x, y, z$ and $w$ are points in $D$ such that

$$z - w \leq x - y,$$

where $w \leq z \leq x$ and $y \leq x$, then

$$(0 \leq) \quad \phi(z) - \phi(w) \leq \phi(x) - \phi(y).$$

We can see [13] for more details.

**Proposition 2.9** Let $\phi : (0,+\infty) \longrightarrow \mathbb{R}$ be a strictly increasing convex function, if $a, b > 0$, and $v \in (0,1]$, $N_1, N_2 \in \mathbb{R}$, then

$$\begin{aligned}&\phi\Big((1+(1-v)v^{(2-v)N_1+vN_2+2}+v^{(1-v)N_1+(1+v)N_2+3})(a\nabla b)\Big)\\ &-\phi(2v^{(1-v)N_1+vN_2+1}(a\sharp b)) \geq \phi(a\nabla b) - \phi(H_v(a,b)).\end{aligned} \tag{2.13}$$

*Proof.* Let $x = [1+(1-v)v^{(2-v)N_1+vN_2+2}+v^{(1-v)N_1+(1+v)N_2+3}](a\nabla b)$, $y = 2v^{(1-v)N_1+vN_2+1}(a\sharp b)$, $z = a\nabla b$, $w = H_v(a,b)$. Now, we just need to prove $z - w \leq x - y$ and $w \leq z \leq x$.

According to (2.1), we have

$$z - w \leq x - y.$$

And it's clear that the second inequalities hold by applying Heinz mean inequality for scalars. So by Lemma 2.8, we have the intended result (2.13). This completes the proof. □

**Corollary 2.10** Let $\phi(x) = x^p, p \in \mathbb{R}, p \geq 1$, if $a, b > 0$, $v \in (0, 1]$ and $N_1, N_2 \in \mathbb{R}$, then we have

$$\begin{aligned}&\Big(1 + (1-v)v^{(2-v)N_1+vN_2+2} + v^{(1-v)N_1+(1+v)N_2+3}\Big)^p (a+b)^p \\ &- \Big(4v^{(1-v)N_1+vN_2+1}\Big)^p (\sqrt{ab})^p \geq (a+b)^p - (a^v b^{1-v} + a^{1-v} b^v)^p. \end{aligned} \tag{2.14}$$

## 3. Reversed versions for Heinz operator mean

In this section, we will give some new reversed versions for Heinz operator mean by the monotonic property of operator functions. However, we need to recall the following lemma that is necessary to obtain our main results.

**Lemma 3.1** ([4]) For $X \in B(H)$ be self-adjoint and $f, g$ be continuous real functions such that $f(t) \geq g(t)$ for all $t \in Sp(X)$ (the Spectrum of $X$). Then $f(X) \geq g(X)$.

According to (2.1), now, we get the new Heinz operator inequalities as follows.

**Theorem 3.2** Let $A, B \in B^{++}(H)$ and $N_1, N_2 \in \mathbb{R}$, if $v \in (0, 1]$, then

$$\Big[v^{N_1+1}(v-1) - v^{N_2+2}\Big](A\nabla B) + 2(A\sharp B) \leq v^{-(1-v)N_1 - vN_2 - 1} H_v(A, B). \tag{3.1}$$

*Proof.* Let $a = 1, b = t > 0$ in (2.1), then the inequality becomes

$$\Big[v^{N_1+1}(v-1) - v^{N_2+2}\Big](1+t) + 4\sqrt{t} \leq v^{-(1-v)N_1 - vN_2 - 1}(t^v + t^{1-v}). \tag{3.2}$$

For the operator $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ has a positive spectrum, $I$ be the identity operator, then according to Lemma 3.1 and insert $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (3.2), we have

$$\begin{aligned}&\Big[v^{N_1+1}(v-1) - v^{N_2+2}\Big](I + A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) + 4(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} \\ &\leq v^{-(1-v)N_1 - vN_2 - 1}\Big((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-v}\Big). \end{aligned} \tag{3.3}$$

Finally, if we multiply the inequality (3.3) by $A^{\frac{1}{2}}$ on both sides, we get

$$\Big[v^{N_1+1}(v-1) - v^{N_2+2}\Big](A + B) + 4(A\sharp B) \leq v^{-(1-v)N_1 - vN_2 - 1}(A\sharp_v B + A\sharp_{1-v} B) \tag{3.4}$$

which implies the required result (3.1). □

On the basis of (2.3), in addition, we obtain the following theorem whose proof method is similar to the way presented by Theorem 3.2 if we take Specht's ratio into account.

**Theorem 3.3** Suppose that $A, B \in B^{++}(H)$, $v \in (0, 1]$ and positive real numbers $m, m', M, M'$ satisfying either of the following conditions
*i)* $0 < v^{N_1 - N_2} m' I \leq A \leq v^{N_1 - N_2} m I < MI \leq B \leq M' I$;

*ii)* $0 < v^{N_2-N_1}m'I \le B \le v^{N_2-N_1}mI < MI \le A \le M'I$.
with $h = \frac{M}{m} > 1$ and $h' = \frac{M'}{m'}$, then

$$\begin{aligned}
&S^{-1}(h^r)H_v(A,B)\\
&\ge \frac{1}{2}\Big[(v-1)v^{vN_1+(2-v)N_2+2} - v^{(1-v)N_1+(1+v)N_2+3}\Big]A\\
&+ \frac{1}{2}\Big[(v-1)v^{(2-v)N_1+vN_2+2} - v^{(1+v)N_1+(1-v)N_2+3}\Big]B\\
&+ (v^{(1-v)N_1+vN_2+1} + v^{vN_1+(1-v)N_2+1})(A\sharp B). \qquad (3.5)
\end{aligned}$$

In particular, if $N_1 = N_2 = N$, then

$$S^{-1}(h^r)H_v(A,B) + v^{2(N+1)}(A\nabla B) \ge 2v^{N+1}(A\sharp B). \qquad (3.6)$$

where $r = \min\{v, 1-v\}$, $N_1, N_2 \in \mathbb{R}$ and $S(\cdot)$ is the Specht's ratio.

*Proof.* Let $b = 1, a = x > 0$ in (2.3), we have

$$\begin{aligned}
&S^{-1}((v^{N_1-N_2}x)^r)(x^v + x^{1-v})\\
&\ge \Big[(v-1)v^{(2-v)N_1+vN_2+2} - v^{(1+v)N_1+(1-v)N_2+3}\Big]x + 2(v^{(1-v)N_1+vN_2+1}\\
&+ v^{vN_1+(1-v)N_2+1})\sqrt{x} + (v-1)v^{vN_1+(2-v)N_2+2} - v^{(1-v)N_1+(1+v)N_2+3} \qquad (3.7)
\end{aligned}$$

for any $x > 0$.
For the positive operator $X$ such that $0 < \alpha I \le X \le \beta I$. Therefore, by Lemma 3.1, we have

$$\begin{aligned}
&\Big(\min_{\alpha\le x\le\beta} S\big((v^{N_1-N_2}x)^r\big)\Big)^{-1}(X^v + X^{1-v})\\
&\ge \Big[(v-1)v^{(2-v)N_1+vN_2+2} - v^{(1+v)N_1+(1-v)N_2+3}\Big]X + \Big[(v-1)v^{vN_1+(2-v)N_2+2}\\
&- v^{(1-v)N_1+(1+v)N_2+3}\Big]I + 2(v^{(1-v)N_1+vN_2+1} + v^{vN_1+(1-v)N_2+1})X^{\frac{1}{2}}.
\end{aligned}$$

Under the first condition, we get $I < \frac{h}{v^{N_1-N_2}}I = \frac{M}{v^{N_1-N_2}m}I \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le \frac{M'}{v^{N_1-N_2}m'}I = \frac{h'}{v^{N_1-N_2}}I$, then $Sp(X) \subseteq [\frac{h}{v^{N_1-N_2}}, \frac{h'}{v^{N_1-N_2}}] \subseteq (1, +\infty)$ and $v^{N_1-N_2}Sp(X) \subseteq [h, h'] \subseteq (1, +\infty)$. Let $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, therefore

$$\begin{aligned}
&\Big(\min_{\frac{h}{v^{N_1-N_2}}\le x\le\frac{h'}{v^{N_1-N_2}}} S\big((v^{N_1-N_2}x)^r\big)\Big)^{-1}\Big((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-v}\Big)\\
&\ge \Big[(v-1)v^{(2-v)N_1+vN_2+2} - v^{(1+v)N_1+(1-v)N_2+3}\Big]A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + \Big[(v-1)v^{vN_1+(2-v)N_2+2}\\
&- v^{(1-v)N_1+(1+v)N_2+3}\Big]I + 2(v^{(1-v)N_1+vN_2+1} + v^{vN_1+(1-v)N_2+1})(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}.
\end{aligned}$$

Since S(x) is an increasing function for $x > 1$, so we have

$$\begin{aligned}
&S^{-1}(h^r)\Big((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-v}\Big)\\
&\ge \Big[(v-1)v^{(2-v)N_1+vN_2+2} - v^{(1+v)N_1+(1-v)N_2+3}\Big]A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + \Big[(v-1)v^{vN_1+(2-v)N_2+2}\\
&- v^{(1-v)N_1+(1+v)N_2+3}\Big]I + 2(v^{(1-v)N_1+vN_2+1} + v^{vN_1+(1-v)N_2+1})(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}. \qquad (3.8)
\end{aligned}$$

Similarly, under the second condition, we also have $\frac{v^{N_2-N_1}}{h'}I = \frac{v^{N_2-N_1}m'}{M'}I \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le \frac{v^{N_2-N_1}m}{M}I = \frac{v^{N_2-N_1}}{h}I < I$, then $Sp(X) \subseteq [\frac{v^{N_2-N_1}}{h'}, \frac{v^{N_2-N_1}}{h}] \subseteq (0,1)$ and $v^{N_1-N_2}Sp(X) \subseteq [\frac{1}{h'}, \frac{1}{h}] \subseteq (0,1)$. Therefore

$$\begin{aligned}
&\Big(\min_{\frac{v^{N_2-N_1}}{h'} \le x \le \frac{v^{N_2-N_1}}{h}} S\big((v^{N_1-N_2}x)^r\big)\Big)^{-1}\Big((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-v}\Big)\\
&\ge \Big[(v-1)v^{(2-v)N_1+vN_2+2} - v^{(1+v)N_1+(1-v)N_2+3}\Big]A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + \Big[(v-1)v^{vN_1+(2-v)N_2+2}\\
&- v^{(1-v)N_1+(1+v)N_2+3}\Big]I + 2(v^{(1-v)N_1+vN_2+1} + v^{vN_1+(1-v)N_2+1})(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}.
\end{aligned}$$

Since $S(x)$ is a decreasing function on $(0,1)$, so we have

$$\begin{aligned}
&S^{-1}\Big(\big(\frac{1}{h}\big)^r\Big)\Big((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-v}\Big)\\
&\ge \Big[(v-1)v^{(2-v)N_1+vN_2+2} - v^{(1+v)N_1+(1-v)N_2+3}\Big]A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + \Big[(v-1)v^{vN_1+(2-v)N_2+2}\\
&- v^{(1-v)N_1+(1+v)N_2+3}\Big]I + 2(v^{(1-v)N_1+vN_2+1} + v^{vN_1+(1-v)N_2+1})(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}.
\end{aligned} \tag{3.9}$$

And then, by the property $S(x) = S(\frac{1}{x})$ for $x > 0$, we have that the inequality (3.9) is same to the inequality (3.8).
Finally, if we multiply $A^{\frac{1}{2}}$ from the both sides to the inequalities (3.8) or (3.9), then we get the required inequality (3.5). □

Also, using the same technique presented by Theorem 3.3 and (2.7), we get the following result with Kantorovich constants. The reader can refer to Theorem 3.3 to get the proof of the following theorem.

**Theorem 3.4** For $A, B \in B^{++}(H)$ and $I < \frac{h}{v^{N_1-N_2}}I \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le \frac{h'}{v^{N_1-N_2}}I$ or $0 < \frac{v^{N_2-N_1}}{h'}I \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le \frac{v^{N_2-N_1}}{h}I < I$ with $h > 1$, then

$$\begin{aligned}
&K^{-r}(h,2)H_v(A,B)\\
&\ge \frac{1}{2}\Big[(v-1)v^{vN_1+(2-v)N_2+2} - v^{(1-v)N_1+(1+v)N_2+3}\Big]A\\
&+ \frac{1}{2}\Big[(v-1)v^{(2-v)N_1+vN_2+2} - v^{(1+v)N_1+(1-v)N_2+3}\Big]B\\
&+ (v^{(1-v)N_1+vN_2+1} + v^{vN_1+(1-v)N_2+1})(A\sharp B).
\end{aligned} \tag{3.10}$$

In particular, if $N_1 = N_2 = N$, then

$$K^{-r}(h,2)H_v(A,B) + v^{2(N+1)}(A\nabla B) \ge 2v^{N+1}(A\sharp B). \tag{3.11}$$

for all $v \in (0,1]$, where $r = \min\{v, 1-v\}$, $N_1, N_2 \in \mathbb{R}$ and $K(\cdot, 2)$ is Kantorovich constants .

In order to achieve our further results, furthermore, we need the following lemma [18].

**Lemma 3.5** If $A, B$ are positive operators on a Hilbert space and $\tau, \omega \in [0,1]$, then

$$A\nabla_\tau(A\sharp_\omega B) = A\nabla_{\tau\omega}B - \tau(A\nabla_\omega B - A\sharp_\omega B).$$
$$A\sharp_\tau(A\sharp_\omega B) = A\sharp_{\tau\omega}B$$

*Proof.*

$$\begin{aligned}
A\nabla_\tau(A\sharp_\omega B) &= (1-\tau)A + \tau(A\sharp_\omega B)\\
&= A - \tau A + \tau\omega B - \tau\omega B + \tau\omega A - \tau\omega A + \tau(A\sharp_\omega B)\\
&= \tau\omega B + (1-\tau\omega)A - \tau[(1-\omega)A + \omega B - A\sharp_\omega B]\\
&= A\nabla_{\tau\omega}B - \tau(A\nabla_\omega B - A\sharp_\omega B).
\end{aligned}$$

$$
\begin{aligned}
& A\sharp_\tau(A\sharp_\omega B)\\
& = A\sharp_\tau(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\omega}A^{\frac{1}{2}})\\
& = A^{\frac{1}{2}}[A^{-\frac{1}{2}}(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\omega}A^{\frac{1}{2}})A^{-\frac{1}{2}}]^{\tau}A^{\frac{1}{2}}\\
& = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\tau\omega}A^{\frac{1}{2}}\\
& = A\sharp_{\tau\omega}B
\end{aligned}
$$

□

At the end of this section, we establish the following theorem by applying Lemma 3.5 and (3.1).

**Theorem 3.6** For $A, B \in B^{++}(H)$, $N_1, N_2 \in \mathbb{R}$ and $v \in (0,1)$.
i)If $v \in (0, \frac{1}{2}]$, then

$$
\begin{aligned}
&[(2r)^{N_1+1}(2r-1)-(2r)^{N_2+2}][A\nabla_{\frac{1}{4}}B-\frac{1}{2}(A\nabla B-A\sharp B)]+2(A\sharp_{\frac{1}{4}}B)\\
&\qquad \leq \frac{1}{2}(2r)^{-(1-2r)N_1-2rN_2-1}(A\sharp_v B+A\sharp_{\frac{1}{2}-v}B).
\end{aligned}
\tag{3.12}
$$

ii) If $v \in [\frac{1}{2}, 1)$, then

$$
\begin{aligned}
&[(2r)^{N_1+1}(2r-1)-(2r)^{N_2+2}][A\nabla_{\frac{3}{4}}B-\frac{1}{2}(A\nabla B-A\sharp B)]+2(A\sharp_{\frac{3}{4}}B)\\
&\qquad \leq \frac{1}{2}(2r)^{-(1-2r)N_1-2rN_2-1}(A\sharp_v B+A\sharp_{\frac{3}{2}-v}B),
\end{aligned}
\tag{3.13}
$$

where $r = \min\{v, 1-v\}$.

*Proof.* i) If $v \in (0, \frac{1}{2}]$, then $2v \in (0,1]$. By substituting $B$ by $A\sharp B$ and $v$ by $2v$ in (3.1), now, then it follows that

$$
\begin{aligned}
&[(2v)^{N_1+1}(2v-1)-(2v)^{N_2+2}](A\nabla(A\sharp B))+2(A\sharp(A\sharp B))\\
&\leq \frac{1}{2}(2v)^{-(1-2v)N_1-2vN_2-1}(A\sharp_{2v}(A\sharp B)+A\sharp_{1-2v}(A\sharp B)).
\end{aligned}
$$

Then by Lemma 3.5, we have
$A\nabla(A\sharp B) = A\nabla_{\frac{1}{4}}B - \frac{1}{2}(A\nabla B - A\sharp B)$, $\quad A\sharp(A\sharp B) = A\sharp_{\frac{1}{4}}B$.
Therefore, we get

$$
\begin{aligned}
&[(2r)^{N_1+1}(2r-1)-(2r)^{N_2+2}][A\nabla_{\frac{1}{4}}B-\frac{1}{2}(A\nabla B-A\sharp B)]+2(A\sharp_{\frac{1}{4}}B)\\
&\leq \frac{1}{2}(2r)^{-(1-2r)N_1-2rN_2-1}(A\sharp_v B+A\sharp_{\frac{1}{2}-v}B).
\end{aligned}
$$

So (3.12) holds.
ii)If $v \in [\frac{1}{2}, 1)$, then $1-v \in (0, \frac{1}{2}]$ and $r = 1-v$.
Exchanging $A$ for $B$ and $v$ for $1-v$ in the above inequality, we have

$$
\begin{aligned}
&[(2r)^{N_1+1}(2r-1)-(2r)^{N_2+2}][B\nabla_{\frac{1}{4}}A-\frac{1}{2}(B\nabla A-B\sharp A)]+2(B\sharp_{\frac{1}{4}}A)\\
&\leq \frac{1}{2}(2r)^{-(1-2r)N_1-2rN_2-1}(B\sharp_{1-v}A+B\sharp_{v-\frac{1}{2}}A).
\end{aligned}
$$

It is shown in [4] that $B\sharp_{1-v}A = A\sharp_v B$, thus the above inequality becomes

$$[(2r)^{N_1+1}(2r-1)-(2r)^{N_2+2}][A\nabla_{\frac{3}{4}}B-\frac{1}{2}(A\nabla B-A\sharp B)]+2(A\sharp_{\frac{3}{4}}B)$$
$$\leq \frac{1}{2}(2r)^{-(1-2r)N_1-2rN_2-1}(A\sharp_v B+A\sharp_{\frac{3}{2}-v}B).$$

So (3.13) holds, this completes the proof. □

**Corollary 3.7** Let $A, B \in B^{++}(H)$, then

$$\Big[(2r)^{N_1+1}(2r-1)-(2r)^{N_2+2}\Big]\Big[A\nabla_{\frac{2[2v]+1}{4}}B-\frac{1}{2}(A\nabla B-A\sharp B)\Big]+2(A\sharp_{\frac{2[2v]+1}{4}}B)$$
$$\leq \frac{1}{2}(2r)^{-(1-2r)N_1-2rN_2-1}\Big(A\sharp_v B+A\sharp_{\frac{2[2v]-2v+1}{2}}B\Big). \tag{3.14}$$

for all $v \in (0,1)$, where $r=\min\{v,1-v\}$, $N_1, N_2 \in \mathbb{R}$ and $[x]$ is the greatest integer less than or equal to $x$.

**Theorem 3.8** Assume that $A, B \in B^{++}(H)$, $v \in (0,1)$, $N \in \mathbb{R}$ and positive real numbers $m, m', M, M'$ satisfy either $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$ or $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$.
i) If $v \in (0, \frac{1}{2}]$, then

$$\frac{1}{2}K^{-R}(\sqrt{h},2)(A\sharp_v B+A\sharp_{\frac{1}{2}-v}B)+(2r)^{2(N+1)}[A\nabla_{\frac{1}{4}}B-\frac{1}{2}(A\nabla B-A\sharp B)]$$
$$\geq 2^{N+2}r^{N+1}(A\sharp_{\frac{1}{4}}B). \tag{3.15}$$

ii) If $v \in [\frac{1}{2}, 1)$, then

$$\frac{1}{2}K^{-R}(\sqrt{h},2)(A\sharp_v B+A\sharp_{\frac{3}{2}-v}B)+(2r)^{2(N+1)}[A\nabla_{\frac{3}{4}}B-\frac{1}{2}(A\nabla B-A\sharp B)]$$
$$\geq 2^{N+2}r^{N+1}(A\sharp_{\frac{3}{4}}B), \tag{3.16}$$

where $r=\min\{v,1-v\}$, $R=\min\{2r,1-2r\}$, $h=\frac{M}{m}>1$, $h'=\frac{M'}{m'}$ and $K(\cdot,2)$ is Kantorovich constants.

*Proof.* In the case of $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$, we have $I < hI \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq h'I$, $I < \sqrt{h}I \leq A^{-\frac{1}{2}}(A\sharp B)A^{-\frac{1}{2}} \leq \sqrt{h'}I$.

In the case of $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$, we have $0 < \frac{1}{h'}I \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \frac{1}{h}I < I$, $\frac{1}{\sqrt{h'}}I \leq A^{-\frac{1}{2}}(A\sharp B)A^{-\frac{1}{2}} \leq \frac{1}{\sqrt{h}}I < I$.
And then, using the same method presented in Theorem 3.6 and (3.11), we can get the desired inequalities (3.15) and (3.16). This completes the proof. □

**Corollary 3.9** Let all assumptions of Theorem 3.8 be satisfied, then

$$\frac{1}{2}K^{-R}(\sqrt{h},2)(A\sharp_v B+A\sharp_{\frac{2[2v]-2v+1}{2}}B)+(2r)^{2(N+1)}[A\nabla_{\frac{2[2v]+1}{4}}B-\frac{1}{2}(A\nabla B-A\sharp B)]$$
$$\geq 2^{N+2}r^{N+1}(A\sharp_{\frac{2[2v]+1}{4}}B). \tag{3.17}$$

## 4. The reverse of matrix inequalities for Heinz mean and Young type

In this section, we establish some interesting matrix versions related to Heinz and Young type for Hilbert-Schmidt norm, unitarily invariant norm, trace norm and trace. To do these, we need the following lemmas. However, it is worth mentioning that the second lemma is a Heinz-Kato type inequality for unitarily invariant norms.

**Lemma 4.1** ([1]) Let $A, B \in M_n$, then

$$\sum_{j=1}^{n} s_j(AB) \leq \sum_{j=1}^{n} s_j(A)s_j(B).$$

**Lemma 4.2.** ( [11]) Suppose $A, B, X \in M_n$ such that $A, B$ are positive semidefinite matrices. If $0 \leq v \leq 1$, then

$$|||A^v XB^{1-v}||| \leq |||AX|||^v |||XB|||^{1-v}.$$

In particular,

$$tr|A^v B^{1-v}| \leq (trA)^v (trB)^{1-v}.$$

We first establish the following Heinz matrix mean version for Hilbert-Schmidt norm, now, whose proof is depended on the spectral theorem and (2.1).

**Theorem 4.3** Assume that $A, B, X \in M_n$ such that $A, B \in M_n^+$, if $v \in (0, 1]$, then

$$\begin{aligned}
&\|4(A^{\frac{1}{2}} XB^{\frac{1}{2}})\|_2 \\
&\leq \|v^{-(1-v)N_1 - vN_2 - 1}(A^v XB^{1-v} + A^{1-v} XB^v) - ((v-1)v^{N_1+1} - v^{N_2+2})(AX + XB)\|_2 \\
&\leq v^{-(1-v)N_1 - vN_2 - 1}\|A^v XB^{1-v} + A^{1-v} XB^v\|_2 + [(1-v)v^{N_1+1} + v^{N_2+2}]\|AX + XB\|_2,
\end{aligned} \tag{4.1}$$

where $N_1, N_2 \in \mathbb{R}$, $I$ represents an identity matrix.

*Proof.* Since $A, B \in M_n^+$, it follows by spectral theorem that there exist unitary matrices $U, V \in M_n$ such that

$$A = U\Lambda_1 U^*, B = V\Lambda_2 V^*,$$

where $\Lambda_1 = diag(\lambda_1, \lambda_2, \cdots, \lambda_n), \Lambda_2 = diag(\mu_1, \mu_2, \cdots, \mu_n), \lambda_m, \mu_l \geq 0, m, l = 1, 2, \cdots, n.$
For our computations, let $Y = U^* XV = [y_{ml}]$, then we have

$$A^v XB^{1-v} + A^{1-v} XB^v = U(\Lambda_1^v Y\Lambda_2^{1-v} + \Lambda_1^{1-v} Y\Lambda_2^v)V^* = U[(\lambda_m^v \mu_l^{1-v} + \lambda_m^{1-v} \mu_l^v)y_{ml}]V^*$$

$$AX + XB = U[(\lambda_m + \mu_l)y_{ml}]V^*, \quad A^{\frac{1}{2}} XB^{\frac{1}{2}} = U[(\lambda_m \mu_l)^{\frac{1}{2}} y_{ml}]V^*$$

and

$$\begin{aligned}
&v^{-(1-v)N_1 - vN_2 - 1}(A^v XB^{1-v} + A^{1-v} XB^v) - [(v-1)v^{N_1+1} - v^{N_2+2}](AX + XB) \\
&= U\Big[\frac{\Lambda_1^v Y\Lambda_2^{1-v} + \Lambda_1^{1-v} Y\Lambda_2^v}{v^{(1-v)N_1 + vN_2 + 1}} - \Big((v-1)v^{N_1+1} - v^{N_2+2}\Big)(\Lambda_1 Y + Y\Lambda_2)\Big]V^* \\
&= U\Big[\Big(\frac{\lambda_m^v \mu_l^{1-v} + \lambda_m^{1-v} \mu_l^v}{v^{(1-v)N_1 + vN_2 + 1}} - \Big((v-1)v^{N_1+1} - v^{N_2+2}\Big)(\lambda_m + \mu_l)\Big)y_{ml}\Big]V^*
\end{aligned}$$

Now, by (2.1) and the unitarily invariant of the Hilbert-Schmidt norm, then we have

$$\begin{aligned}
&\|4(A^{\frac{1}{2}} XB^{\frac{1}{2}})\|_2^2 \\
&= \sum_{m,l=1}^{n} \Big[4(\lambda_m \mu_l)^{\frac{1}{2}}\Big]^2 |y_{ml}|^2 \\
&\leq \sum_{m,l=1}^{n} \Big[v^{-(1-v)N_1 - vN_2 - 1}(\lambda_m^v \mu_l^{1-v} + \lambda_m^{1-v} \mu_l^v) - \Big((v-1)v^{N_1+1} - v^{N_2+2}\Big)(\lambda_m + \mu_l)\Big]^2 |y_{ml}|^2 \\
&= \|v^{-(1-v)N_1 - vN_2 - 1}(A^v XB^{1-v} + A^{1-v} XB^v) - \Big((v-1)v^{N_1+1} - v^{N_2+2}\Big)(AX + XB)\|_2^2
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|4(A^{\frac{1}{2}}XB^{\frac{1}{2}})\|_2 \\
&\leq \|v^{-(1-v)N_1-vN_2-1}(A^vXB^{1-v}+A^{1-v}XB^v)-\Big((v-1)v^{N_1+1}-v^{N_2+2}\Big)(AX+XB)\|_2 \\
&\leq v^{-(1-v)N_1-vN_2-1}\|A^vXB^{1-v}+A^{1-v}XB^v\|_2+\Big[(1-v)v^{N_1+1}+v^{N_2+2}\Big](AX+XB)\|_2.
\end{aligned}$$

Here we complete the proof. □

Similarly, we obtain the following results with Specht's ratio and Kantorovich constants by applying (2.12) and (2.11).

**Theorem 4.4** Suppose that $A, B, X \in M_n$ such that $A, B \in M_n^{++}$ and satisfy $0 < \sqrt{M}I \leq A \leq \sqrt{M'}I, 0 < \sqrt{v^{N_1-N_2}m'}I \leq B \leq \sqrt{v^{N_1-N_2}m}I$ or $0 < \sqrt{m'}I \leq A \leq \sqrt{m}I, 0 < \sqrt{\frac{M}{v^{N_2-N_1}}}I \leq B \leq \sqrt{\frac{M'}{v^{N_2-N_1}}}I$, if $v \in (0,1]$, then

*i)* 
$$\begin{aligned}
&(1-v^{N_1+1}+v^{N_1+2})\|AX\|_2^2+(1-v^{N_2+2})\|XB\|_2^2 \\
&\leq K^{-r}(h,2)v^{-(1-v)N_1-vN_2-1}\|A^vXB^{1-v}\|_2^2+\|AX-XB\|_2^2
\end{aligned} \tag{4.2}$$

*ii)* 
$$\begin{aligned}
&(1-v^{N_1+1}+v^{N_1+2})\|AX\|_2^2+(1-v^{N_2+2})\|XB\|_2^2 \\
&\leq S^{-1}(h^r)v^{-(1-v)N_1-vN_2-1}\|A^vXB^{1-v}\|_2^2+\|AX-XB\|_2^2,
\end{aligned} \tag{4.3}$$

where $h = \frac{M}{m}$ with $m < M$, $h' = \frac{M'}{m'}$, $r = \min\{v, 1-v\}$, $N_1, N_2 \in \mathbb{R}$, $I$ represents an identity matrix and $m, m', M, M' \in \mathbb{R}$.

*Proof.* i) Since $A, B \in M_n^{++}$, it follows by spectral theorem that there exist unitary matrices $S, T \in M_n$ such that

$$A = S\Gamma_1 S^*, B = T\Gamma_2 T^*,$$

where $\Gamma_1 = diag(\xi_1, \xi_2, \cdots, \xi_n), \Gamma_2 = diag(\sigma_1, \sigma_2, \cdots, \sigma_n), \xi_i, \sigma_j > 0, i, j = 1, 2, \cdots, n.$
For our computations, let $Y = S^*XT = [y_{ij}]$, then we have

$$A^vXB^{1-v} = S(\Gamma_1^v Y \Gamma_2^{1-v})T^* = S[(\xi_i^v\sigma_j^{1-v})y_{ij}]T^*$$

$$AX - XB = S[(\xi_i - \sigma_j)y_{ij}]T^*, \quad AX = S(\xi_i y_{ij})T^* \quad and \quad XB = S(\sigma_j y_{ij})T^*.$$

By the unitarily invariant of Hilbert-Schmidt norm and (2.12) , now, we have

$$\begin{aligned}
& (1-v^{N_1+1}+v^{N_1+2})\|AX\|_2^2+(1-v^{N_2+2})\|XB\|_2^2 \\
= & (1-v^{N_1+1}+v^{N_1+2})\sum_{i,j=1}^{n}\xi_i^2|y_{ij}|^2+(1-v^{N_2+2})\sum_{i,j=1}^{n}\sigma_j^2|y_{ij}|^2 \\
= & \sum_{i,j=1}^{n}\Big[(1-v^{N_1+1}+v^{N_1+2})\xi_i^2+(1-v^{N_2+2})\sigma_j^2\Big]|y_{ij}|^2 \\
\leq & \sum_{i,j=1}^{n}\Big[\Big(\min K(v^{N_1-N_2}t_{ij},2)\Big)^{-r}v^{-(1-v)N_1-vN_2-1}(\xi_i^v\sigma_j^{1-v})^2 \\
+ & (\xi_i-\sigma_j)^2\Big]|y_{ij}|^2,
\end{aligned}$$

where $t_{ij} = \frac{\xi_i^2}{\sigma_j^2}, \quad 1 \leq i, j \leq n.$

By the case of $0 < \sqrt{M}I \leq A \leq \sqrt{M'}I, 0 < \sqrt{v^{N_1-N_2}m'}I \leq B \leq \sqrt{v^{N_1-N_2}m}I$, we have $t_{ij} = \frac{\xi_i^2}{\sigma_j^2} \in [\frac{h}{v^{N_1-N_2}}, \frac{h'}{v^{N_1-N_2}}]$ and $v^{N_1-N_2}t_{ij} \in [h, h'] \subseteq (1, +\infty)$. So by the properties of the Kantorovich constant, we get $\min K(v^{N_1-N_2}t_{ij}, 2) = K(h, 2)$. Similarly, by the case $0 < \sqrt{m'}I \leq A \leq \sqrt{m}I, 0 < \sqrt{\frac{M}{v^{N_2-N_1}}}I \leq B \leq \sqrt{\frac{M'}{v^{N_2-N_1}}}I$, we have $t_{ij} = \frac{\xi_i^2}{\sigma_j^2} \in [\frac{v^{N_2-N_1}}{h'}, \frac{v^{N_2-N_1}}{h}]$ and $v^{N_1-N_2}t_{ij} \in [\frac{1}{h'}, \frac{1}{h}] \subseteq (0, 1)$. By the properties of the Kantorovich constant, accordingly, we also get $\min K(v^{N_1-N_2}t_{ij}, 2) = K(h, 2)$.

Therefore,

$$
\begin{aligned}
& (1-v^{N_1+1}+v^{N_1+2})\|AX\|_2^2+(1-v^{N_2+2})\|XB\|_2^2 \\
\leq\ & \sum_{i,j=1}^{n}\Big[\Big(\min K(v^{N_1-N_2}t_{ij},2)\Big)^{-r}v^{-(1-v)N_1-vN_2-1}(\xi_i^v\sigma_j^{1-v})^2 \\
+\ & (\xi_i-\sigma_j)^2\Big]|y_{ij}|^2 \\
=\ & K^{-r}(h,2)v^{-(1-v)N_1-vN_2-1}\sum_{i,j=1}^{n}(\xi_i^v\sigma_j^{1-v})^2|y_{ij}|^2+\sum_{i,j=1}^{n}(\xi_i-\sigma_j)^2|y_{ij}|^2 \\
=\ & K^{-r}(h,2)v^{-(1-v)N_1-vN_2-1}\|A^vXB^{1-v}\|_2^2+\|AX-XB\|_2^2.
\end{aligned}
$$

So (4.2) holds.
ii) Using the same method in (2.11), we can get (4.3). Thus we omit it.
Here we complete the proof.

Besides, by (2.8) and (2.12), we obtain some inequalities with Kantorovich constants for trace norm. As with the Theorem 4.5, we get the similar conclusion with Specht's ratio which refers to Corollary 4.6, thus we omit the details.

**Theorem 4.5** Let $A, B \in M_n$ such that $A, B \in M_n^{++}$ and satisfy $v \in (0, 1]$, then

$$
\begin{aligned}
& i)\, tr\Big[(1-v^{N_1+1}+v^{N_1+2})A+(1-v^{N_2+2})B\Big] \\
& \leq K_1^{-r}v^{-(1-v)N_1-vN_2-1}\|A^v\|_2\|B^{1-v}\|_2+\|A\|_1+\|B\|_1-2\|A^{\frac{1}{2}}B^{\frac{1}{2}}\|_1
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
& ii)\ tr\Big[(1-v^{N_1+1}+v^{N_1+2})A^2+(1-v^{N_2+2})B^2\Big] \\
& \leq K_2^{-r}v^{-(1-v)N_1-vN_2-1}\|A^v\|_2^2\ \ \|B^{1-v}\|_2^2+\|A\|_2^2+\|B\|_2^2-2\|AB\|_1,
\end{aligned}
\tag{4.5}
$$

where $r = \min\{v, 1-v\}$, $N_1, N_2 \in \mathbb{R}$, $K_1 = \min\{K(\frac{v^{N_1-N_2}s_j(A)}{s_j(B)}, 2), 1 \leq j \leq n\}$, $K_2 = \min\{K(\frac{v^{N_1-N_2}s_j^2(A)}{s_j^2(B)}, 2), 1 \leq j \leq n\}$ and $K$ is the the Kantorovich constant.

*Proof.* i) By (2.8), Lemma 4.1 and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
& tr\Big[(1-v^{N_1+1}+v^{N_1+2})A+(1-v^{N_2+2})B\Big] \\
=\ & \sum_{j=1}^{n}\Big[(1-v^{N_1+1}+v^{N_1+2})s_j(A)+(1-v^{N_2+2})s_j(B)\Big] \\
\leq\ & \sum_{j=1}^{n}\Big[\Big(\min K(\tfrac{v^{N_1-N_2}s_j(A)}{s_j(B)},2)\Big)^{-r}v^{-(1-v)N_1-vN_2-1}s_j^v(A)s_j^{1-v}(B) \\
+\ & \Big(\sqrt{s_j(A)}-\sqrt{s_j(B)}\Big)^2\Big] \\
=\ & K_1^{-r}v^{-(1-v)N_1-vN_2-1}\sum_{j=1}^{n}s_j(A^v)s_j(B^{1-v})+\sum_{j=1}^{n}s_j(A)+\sum_{j=1}^{n}s_j(B) \\
-\ & 2\sum_{j=1}^{n}\sqrt{s_j(A)s_j(B)} \\
\leq\ & K_1^{-r}v^{-(1-v)N_1-vN_2-1}\Big(\sum_{j=1}^{n}s_j^2(A^v)\Big)^{\frac{1}{2}}\Big(s_j^2(B^{1-v})\Big)^{\frac{1}{2}}+\sum_{j=1}^{n}s_j(A)+\sum_{j=1}^{n}s_j(B) \\
-\ & 2\sum_{j=1}^{n}s_j(A^{\frac{1}{2}}B^{\frac{1}{2}}) \\
=\ & K_1^{-r}v^{-(1-v)N_1-vN_2-1}\|A^v\|_2\ \|B^{1-v}\|_2+\|A\|_1+\|B\|_1-2\|A^{\frac{1}{2}}B^{\frac{1}{2}}\|_1.
\end{aligned}
$$

So (4.4) holds.
ii) The proof of (4.5) is similar to the one presented in *i*) by applying the inequality (2.12), thus we omit it. □

**Corollary 4.6** Let all assumptions of Theorem 4.5 be satisfied, then

$$\begin{aligned} i)\ & tr\Big[(1-v^{N_1+1}+v^{N_1+2})A+(1-v^{N_2+2})B\Big] \\ & \le S_1^{-1}v^{-(1-v)N_1-vN_2-1}\|A^v\|_2\|B^{1-v}\|_2+\|A\|_1+\|B\|_1-2\|A^{\frac{1}{2}}B^{\frac{1}{2}}\|_1 \end{aligned} \tag{4.6}$$

$$\begin{aligned} ii)\ & tr\Big[(1-v^{N_1+1}+v^{N_1+2})A^2+(1-v^{N_2+2})B^2\Big] \\ & \le S_2^{-1}v^{-(1-v)N_1-vN_2-1}\|A^v\|_2^2\ \ \|B^{1-v}\|_2^2+\|A\|_2^2+\|B\|_2^2-2\|AB\|_1, \end{aligned} \tag{4.7}$$

where $N_1, N_2 \in \mathbb{R}$, $S_1 = \min\{S\Big(\Big(\frac{v^{N_1-N_2}s_j(A)}{s_j(B)}\Big)^r\Big), 1 \le j \le n\}$, $S_2 = \min\{S\Big(\Big(\frac{v^{N_1-N_2}s_j^2(A)}{s_j^2(B)}\Big)^r\Big)$ $, 1 \le j \le n\}$, $r = \min\{v, 1-v\}$ and $S(\cdot)$ is the Specht's ratio.

By Lemma 4.2 and (2.8), finally, we have the following conclusions with Kantorovich constants for unitarily invariant norm and trace. The reader can refer to Theorem 4.5 or [15] to get the process of proof. Just like Corollary 4.7, of course, the reader can come to the similar conclusion with Specht's ratio. Thus we omit them.

**Corollary 4.7** For $A, B \in M_n$ with $A, B \in M_n^{++}$ and satisfy $v \in (0, 1]$, then

$$\begin{aligned} i)\ & (1-v^{N_1+1}+v^{N_1+2})|||AX|||+(1-v^{N_2+2})|||XB||| \le K^{-r}(v^{N_1-N_2}h_1, 2) \\ & v^{-(1-v)N_1-vN_2-1}|||AX|||^v\ |||XB|||^{1-v}+|||AX|||+|||XB|||-2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \end{aligned} \tag{4.8}$$

$$\begin{aligned} ii)\ & tr\Big[(1-v^{N_1+1}+v^{N_1+2})A+(1-v^{N_2+2})B\Big] \le K^{-r}(\ v^{N_1-N_2}h_2, 2) \\ & v^{-(1-v)N_1-vN_2-1}(trA)^v(trB)^{1-v}+trA+trB-2tr|A^{\frac{1}{2}}B^{\frac{1}{2}}|, \end{aligned} \tag{4.9}$$

where $r = \min\{v, 1-v\}$, $N_1, N_2 \in \mathbb{R}$, $h_1 = \frac{|||AX|||}{|||XB|||}$, $h_2 = \frac{tr(A)}{tr(B)}$.
□

**Acknowledgement.** The authors would like to thank the anonymous referee for valuable comments that have been implemented in the final version of the paper.

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