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Central Invariants and Enveloping Algebras of Braided Hom-Lie Algebras

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Abstract. Let (H, α) be a monoidal Hom-Hopf algebra and ${}^H_H\mathcal{HYD}$ the Hom-Yetter-Drinfeld category over (H, α). Then in this paper, we first introduce the definition of braided Hom-Lie algebras and show that each monoidal Hom-algebra in ${}^H_H\mathcal{HYD}$ gives rise to a braided Hom-Lie algebra. Second, we prove that if (A, β) is a sum of two H-commutative monoidal Hom-subalgebras, then the commutator Hom-ideal $[A, A]$ of A is nilpotent. Also, we study the central invariant of braided Hom-Lie algebras as a generalization of generalized Lie algebras. Finally, we obtain a construction of the enveloping algebras of braided Hom-Lie algebras and show that the enveloping algebras are H-cocommutative Hom-Hopf algebras.

1. Introduction

Hom-algebras were first introduced in the Lie algebra setting [14] with motivation from physics though its origin can be traced back in earlier literature such as [15]. In a Hom-Lie algebra, the Jacobi identity is replaced by the so called Hom-Jacobi identity via a homomorphism. In 2008, Makhlouf and Silvestrov [20] introduced the definition of Hom-associative algebras, where the associativity of a Hom-algebra is twisted by an endomorphism (here we call it the Hom-structure map). The definition of BiHom-Hopf algebras given in [12] is even more general, and involves four different structure maps, including Hom-bialgebras, Hom-Hopf algebras were developed in [9], [21], [22], [23]. Further research on Hom-Hopf algebras could be found in [5], [11], [17], [31], [33] and references cited therein.

In [4], Caenepeel and Goyvaerts studied Hom-Lie algebras and Hom-Hopf algebras from a categorical view point, they proved a (co)monoid in the Hom-category is a Hom-(co)algebra, and a bimonoid in the Hom-category is a monoidal Hom-bialgebra. Note that a monoidal Hom-Hopf algebra is a Hom-Hopf algebra if and only if the Hom-structure map is involutive. Later, Graziani et al. [12] defined BiHom-Hopf algebras using two commuting multiplicative linear maps α, β, unified Hom-Hopf algebras and monoidal Hom-Hopf algebras by setting $\alpha = \beta$ and $\alpha = \beta^{-1}$ respectively.

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Recently, the theory of Hom-Yetter-Drinfeld categories has attracted attention in mathematics and mathematical physics. In [19], Makhlouf and Panaite defined Yetter-Drinfeld modules over Hom-bialgebras and showed that Yetter-Drinfeld modules over a Hom-bialgebra with bijective structure map provide solutions of the Hom-Yang-Baxter equation. Also Liu and Shen [18], Chen and Zhang [7] studied Hom-Yetter-Drinfeld modules over monoidal Hom-bialgebras in a slightly different way to [19]. As a part of the theory of Hom-Yetter-Drinfeld categories, we [29] gave sufficient and necessary conditions for the Hom-Yetter-Drinfeld category ${}^H_H\mathcal{HYD}$ to be symmetric and pseudosymmetric respectively. With the symmetries of Hom-Yetter-Drinfeld categories, it is a natural question to ask whether we can extend the notion of monoidal Hom-Lie algebras to Hom-Yetter-Drinfeld categories. This becomes our first motivation of writing this paper.

It is well known that Lie algebras in braided monoidal categories is a very important part of Lie theories. As a generalization of Lie superalgebras [16] and Lie color algebras [25], Manin [24] studied Lie algebras in some symmetric categories from an algebraic point of view. Later, Cohen, Fishman and Westreich [8] studied Lie algebras in the category of modules over triangular Hopf algebras and proved Schur's double centralizer theorem, Fishman and Montgomery [10] did similar work in the category of comodules over cotriangular Hopf algebras. Later, Bahturin, Fishman and Montgomery [3] studied the structure of the generalized Lie algebras in the category of comodules.

Wang [27] studied the central invariant of ρ-Lie algebras in Yetter-Drinfeld categories. Wang [28] introduced the notion of generalized Lie algebras in Yetter-Drinfeld categories and extended the Kegel's theorem to generalized Lie algebras. Later, we [30] extended Wang's results in [28] to Hom-Lie algebras in Yetter-Drinfeld categories, which unifies the notions of Hom-Lie superalgebras in [1] and Hom-Lie color algebras in [32]. In the present paper, we will study monoidal Hom-Lie algebras in Hom-Yetter-Drinfeld categories, which is different from [30] in two aspects. First, Hom-Yetter-Drinfeld categories include Yetter-Drinfeld categories as a special case. Second, the main purpose of this paper is to study the central invariants and enveloping algebras of braided Hom-Lie algebras, which has not been involved in [30].

This paper is organized as follows. In Section 2, we recall some basic definitions about monoidal Hom-Hopf algebras and Hom-Yetter-Drinfeld modules.

In Section 3, we define braided Hom-Lie algebras and show that any monoidal Hom-algebra in ${}^H_H\mathcal{HYD}$ gives rise to a braided Hom-Lie algebra by the natural bracket product (see Proposition 3.2), and prove that if (A,β) is H-semisimple and a sum of two H-commutative monoidal Hom-subalgebras, then (A,β) is H-commutative (see Corollary 3.9). In Section 4, we consider the central invariant of braided Hom-Lie algebras (see Theorem 4.7). In Section 5, we construct the enveloping algebras of braided Hom-Lie algebras and present its Hopf structures. As an application, we study the enveloping algebra of $End(V)$ and construct a Radford's Hom-biproduct $(U(End(V))^{\times}_{\sharp}H, \delta\otimes id)$ (see Proposition 5.10).

2. Preliminary

In this section, we recall some basic definitions and results related to our paper. Throughout the paper, all algebraic systems are supposed to be over a field k of characteristic not 2. The reader is referred to Caenepeel and Goyvaerts [4] as general references about monoidal Hom-algebras and monoidal Hom-Lie algebras, to Sweedler [26] about Hopf algebras and Liu and Shen [18] about Hom-Yetter-Drinfeld categories.

If C is a coalgebra, we use the Sweedler-type notation for the comultiplication: $\Delta(c) = c_1 \otimes c_2$, for all $c \in C$, in which we often omit the summation symbols for convenience.

2.1 Hom-category

Let C be a category. We introduce a new category $\mathscr{H}(C)$ as follows: the objects are couples (X, α_X), with $X \in C$ and $\alpha_X \in Aut_C(X)$. A morphism $f : (X, \alpha_X) \to (Y, \alpha_Y)$ is a morphism $f : X \to Y$ in C such that $\alpha_Y \circ f = f \circ \alpha_X$.

Specially, let $\mathscr{M}_k$ denote the category of k-spaces. $\mathscr{H}(\mathscr{M}_k)$ will be called the Hom-category associated to $\mathscr{M}_k$. If $(X, \alpha_X) \in \mathscr{M}_k$, then $\alpha_X : X \to X$ is obviously an isomorphism in $\mathscr{H}(\mathscr{M}_k)$. It is easy to show that $\widetilde{\mathscr{H}}(\mathscr{M}_k) = (\mathscr{H}(\mathscr{M}_k), \otimes, (k, id), \widetilde{a}, \widetilde{l}, \widetilde{r})$ is a monoidal category by Proposition 1.1 in [4]:

• the tensor product of (X, α_X) and (Y, α_Y) in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is given by the formula $(X, \alpha_X) \otimes (Y, \alpha_Y) = (X \otimes Y, \alpha_X \otimes \alpha_Y)$;

• for any $x \in X$, $y \in Y$, $z \in Z$, the associator is given by the formulas

$$\widetilde{a}_{X,Y,Z}((x \otimes y) \otimes z) = \alpha_X(x) \otimes (y \otimes \alpha_Z^{-1}(z));$$

• for any $x \in X$, $\lambda \in k$, the unit constraints are given by the formulas

$$\widetilde{l}_X(\lambda \otimes x) = \widetilde{r}_X(x \otimes \lambda) = \lambda \alpha_X(x).$$

2.2 Monoidal Hom-Hopf algebras

Definition 2.1. A *monoidal Hom-algebra* is an object (A, α) in the Hom-category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with an element $1_A \in A$ and a linear map $m : A \otimes A \to A,\ a \otimes b \mapsto ab$ such that

$$\alpha(a)(bc) = (ab)\alpha(c),\ \alpha(ab) = \alpha(a)\alpha(b), \tag{1}$$

$$a1_A = 1_A a = \alpha(a),\ \alpha(1_A) = 1_A, \tag{2}$$

for all $a, b, c \in A$.

As noted in [4], the definition of monoidal Hom-algebras is different from the definition of Hom-associative algebras defined in [22]. Specifically, the unitality condition in [22] is the usual untwisted one: $a1_A = 1_A a = a$, for any $a \in A$, and the condition (2) is not desired there. These Hom-algebras are sometimes called multiplicative Hom-algebras.

Definition 2.2. A *monoidal Hom-coalgebra* is an object (C, γ) in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with linear maps $\Delta : C \to C \otimes C$, $\Delta(c) = c_1 \otimes c_2$ and $\epsilon : C \to k$ such that

$$\gamma^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \gamma^{-1}(c_2),\ \Delta(\gamma(c)) = \gamma(c_1) \otimes \gamma(c_2), \tag{3}$$

$$c_1\epsilon(c_2) = \epsilon(c_1)c_2 = \gamma^{-1}(c),\ \epsilon(\gamma(c)) = \epsilon(c), \tag{4}$$

for all $c \in C$.

The definition of monoidal Hom-coalgebras is different from the definition of Hom-coassociative coalgebras defined in [22]. The coassociativity condition is twisted by some endomorphism, not necessarily by the inverse of the automorphism γ. The counitality condition in [22] is the usual untwisted one: $c_1\epsilon(c_2) = \epsilon(c_1)c_2 = c$, for any $c \in C$, and the condition (4) is not needed there.

Definition 2.3. A *monoidal Hom-bialgebra* $H = (H, \alpha, m, 1_H, \Delta, \epsilon)$ is a bialgebra in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. This means that $(H, \alpha, m, 1_H)$ is a monoidal Hom-algebra and $(H, \alpha, \Delta, \epsilon)$ is a monoidal Hom-coalgebra such that Δ and ϵ are Hom-algebra maps, that is, for any $h, g \in H$,

$$\Delta(hg) = \Delta(h)\Delta(g), \quad \Delta(1_H) = 1_H \otimes 1_H,$$
$$\epsilon(hg) = \epsilon(h)\epsilon(g), \quad \epsilon(1_H) = 1_k.$$

A monoidal Hom-bialgebra (H, α) is called a *monoidal Hom-Hopf algebra* if there exists a morphism (called the antipode) $S : H \to H$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ (i.e. $S \circ \alpha = \alpha \circ S$), which is the convolution inverse of the identity morphism id_H (i.e. $S * id_H = \eta_H \circ \epsilon_H = id_H * S$), this means for any $h \in H$,

$$S(h_1)h_2 = \epsilon(h)1_H = h_1S(h_2).$$

2.3 Hom-Yetter-Drinfeld categories

Definition 2.4. Let (A, α) be a monoidal Hom-algebra. A *left* (A, α)*-Hom-module* consists of $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\psi : A \otimes M \to M$, $\psi(a \otimes m) = a \cdot m$ such that

$$\alpha(a) \cdot (b \cdot m) = (ab) \cdot \mu(m),\ 1_A \cdot m = \mu(m), \mu(a \cdot m) = \alpha(a) \cdot \mu(m),$$

for all $a, b \in A$ and $m \in M$.

Let $(M, \mu), (N, \nu)$ be (A, α)-modules and the corresponding structure maps. A morphism $f : M \to N$ of (A, α)-Hom-modules is called *left A-linear* if $f(a \cdot m) = a \cdot f(m)$, for any $a \in A, m \in M$ and $f \circ \mu = \nu \circ f$.

Definition 2.5. Let (C, γ) be a monoidal Hom-coalgebra. A *left* (C, γ)*-Hom-comodule* consists of $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\rho_M : M \to C \otimes M$, $\rho_M(m) = m_{(-1)} \otimes m_0$ (here we omit the summation for convenience) such that

$$\Delta_C(m_{(-1)}) \otimes \mu^{-1}(m_0) = \gamma^{-1}(m_{(-1)}) \otimes (m_{0(-1)} \otimes m_{00}),$$
$$\rho_M(\mu(m)) = \gamma(m_{(-1)}) \otimes \mu(m_0),\ \epsilon(m_{(-1)})m_0 = \mu^{-1}(m),$$

for all $m \in M$.

Let (M, μ) and (N, ν) be two left (C, γ)-Hom-comodules. A morphism $g : M \to N$ is called *left C-colinear* if $g \circ \mu = \nu \circ g$ and $m_{(-1)} \otimes g(m_0) = g(m)_{(-1)} \otimes g(m)_0$, for any $m \in M$.

Definition 2.6. Let (H, α) be a monoidal Hom-Hopf algebra. A monoidal Hom-algebra (A, β) is called a left (H, α) *Hom-module algebra*, if (A, β) is a left (H, α) Hom-module with action $\phi : H \otimes A \to A$, $\phi(h \otimes a) = h \cdot a$ such that the following conditions satisfy:

$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b),$$
$$h \cdot 1_A = \epsilon(h)1_A,$$

for all $a, b \in A$ and $h \in H$.

Definition 2.7. Let (H, α) be a monoidal Hom-Hopf algebra. A monoidal Hom-algebra (A, β) is called a left (H, α)*-Hom-comodule algebra* if (A, β) is a left (H, α) Hom-comodule with coaction $\rho : A \to H \otimes A$, $\rho(a) = a_{(-1)} \otimes a_0$ such that the following conditions satisfy,

$$\rho(ab) = a_{(-1)}b_{(-1)} \otimes a_0b_0,$$
$$\rho(1_A) = 1_H \otimes 1_A.$$

for all $a, b \in A$.

Definition 2.8. Let (H, α) be a monoidal Hom-Hopf algebra. A *left-left* (H, α)*-Hom-Yetter-Drinfeld module* is an object $(M, \beta) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$, such that (M, β) is both a left (H, α)-Hom-module and a left (H, α)-Hom-comodule with the following compatibility condition:

$$\rho(h \cdot m) = (h_{11}\alpha^{-1}(m_{(-1)}))S(h_2) \otimes \alpha(h_{12}) \cdot m_0, \tag{5}$$

for all $h \in H$ and $m \in M$.

By Proposition 4.2 in Ref. [16], Eq. (5) is equivalent to the following equation:

$$h_1m_{(-1)} \otimes h_2 \cdot m_0 = (h_1 \cdot \beta^{-1}(m))_{(-1)}h_2 \otimes \beta((h_1 \cdot \beta^{-1}(m))_0).$$

Definition 2.9. Let (H, α) be a monoidal Hom-Hopf algebra. A *Hom-Yetter-Drinfeld category* ${}^H_H\mathcal{HYD}$ is a braided monoidal category whose objects are left-left (H, α)-Hom-Yetter-Drinfeld modules, morphisms are both left (H, α)-linear and (H, α)-colinear maps, and its braiding $C_{-,-}$ is given by

$$C_{M,N}(m \otimes n) = m_{(-1)} \cdot \nu^{-1}(n) \otimes \mu(m_{(0)}),$$

for all $m \in (M, \mu) \in {}^H_H\mathcal{HYD}$ and $n \in (N, \nu) \in {}^H_H\mathcal{HYD}$.

Definition 2.10. Let (A, β) be an object in ${}^H_H\mathcal{HYD}$, the braiding C is called symmetric on A if the following condition holds:

$$a_{(-1)} \cdot \beta^{-1}(b) \otimes \beta(a_0) = \beta(b_0) \otimes S^{-1}(b_{(-1)}) \cdot \beta^{-1}(a);$$

A is called H-commutative if

$$(a_{(-1)} \cdot \beta^{-1}(b))\beta(a_0) = ab,$$

A is called H-cocommutative if

$$a_{1(-1)} \cdot \beta^{-1}(a_2) \otimes \beta(a_{10}) = a_1 \otimes a_2,$$

for all $a, b \in A$.

3. Braided Hom-Lie algebras

In this section, we first introduce the concept of braided Hom-Lie algebras and show that each monoidal Hom-algebra in ${}^H_H\mathcal{HYD}$ gives rise to a braided Hom-Lie algebras. Also we study the braided Lie structures of monoidal Hom-algebras in ${}^H_H\mathcal{HYD}$ as a generalization of results in [3], [28] and [30].

From now on, we always assume that (H,α) is a monoidal Hom-Hopf algebra and ${}^H_H\mathcal{HYD}$ the Hom-Yetter-Drinfeld category over (H,α).

Definition 3.1. A monoidal Hom-Lie algebra in ${}^H_H\mathcal{HYD}$, called a braided Hom-Lie algebra, is a pair (L,β), where L is an object in ${}^H_H\mathcal{HYD}$, $\beta : L \to L$ is a homomorphism in ${}^H_H\mathcal{HYD}$ and $[\cdot,\cdot] : L\otimes L \to L$ is a morphism in ${}^H_H\mathcal{HYD}$ satisfying

(i) Braided Hom-skew-symmetry:

$$[l,l'] = -[l_{(-1)}\cdot\beta^{-1}(l'),\beta(l_0)],\ l,l'\in L.$$

(ii) Braided Hom-Jacobi identity:

$$\{l\otimes l'\otimes l''\} + \{(C\otimes 1)(1\otimes C)(l\otimes l'\otimes l'')\} + \{(1\otimes C)(C\otimes 1)(l\otimes l'\otimes l'')\} = 0,$$

for all $l,l',l''\in L$, where $\{l\otimes l'\otimes l''\}$ denotes $[\beta(l),[l',l'']]$ and C the braiding for L.

Proposition 3.2. Let (A,β) be a monoidal Hom-algebra in ${}^H_H\mathcal{HYD}$. Assume that the braiding C is symmetric on A. Then the triple $(A,[\cdot,\cdot],\beta)$ is a braided Hom-Lie algebra, where the bracket product is defined

$$[\cdot,\cdot] : A\otimes A\to A \ by\ [a,b] = ab - (a_{(-1)}\cdot\beta^{-1}(b))\beta(a_0),$$

for all $a,b\in A$.

Proof. Denote $A^- = (A,[\cdot,\cdot],\beta)$. It is clear that the bracket product is a morphism in ${}^H_H\mathcal{HYD}$, so it remains to verify that the conditions (i) and (ii) of Definition 3.1 hold.

For the braided Hom-skew-symmetry, we have $[a_{(-1)}\cdot\beta^{-1}(b),\beta(a_0)] = (a_{(-1)}\cdot\beta^{-1}(b))\beta(a_0) - ((a_{(-1)}\cdot\beta^{-1}(b))_{(-1)}\cdot\beta(\beta^{-1}(a_0)))\beta((a_{(-1)}\cdot\beta^{-1}(b))_0) = (a_{(-1)}\cdot\beta^{-1}(b))\beta(a_0) - ab = -[a,b]$, as desired. The last equality holds since the braiding C is symmetric on A.

Similarly, one may check the braided Hom-Jacobi identity by the Hom-associativity of A routinely. And this finishes the proof. □

Example 3.3. Let (H,α) be a commutative involutive monoidal Hom-Hopf algebra. By Example 4.3 in [18], (H,α) is a Hom-Yetter-Drinfeld module with left (H,α)-action $h\cdot g = (h_1\alpha^{-1}(g))S(\alpha(h_2))$ and left (H,α)-coaction by the Hom-comultiplication Δ, note it by $H_1 = (H_1, \text{adjoint}, \Delta, \alpha)$. By Corollary 5.4 in [29], the braiding C is symmetric on H_1, then H_1^- is a braided Hom-Lie algebra.

Example 3.4. Let (H,α) be a cocommutative involutive monoidal Hom-Hopf algebra. By Example 2.7 in [29], (H,α) is a Hom-Yetter-Drinfeld module with left (H,α)-action by the Hom-multiplication m and left (H,α)-coaction $\rho(h) = h_{11}\alpha^{-1}(S(h_2))\otimes\alpha(h_{12})$, and note it by $H_2 = (H_2, m, \text{coadjoint}, \alpha)$. By Corollary 5.4 in [29], the braiding C is symmetric on H_2, then H_2^- is a braided Hom-Lie algebra.

Example 3.5. Let $H = k\{1_H, h\}$ be a monoidal Hom-Hopf algebra with an automorphism $\alpha : H\to H, \alpha(1_H) = 1_H, \alpha(h) = -h$, where the Hom-algebra structure is defined by

$$1_H 1_H = 1_H, 1_H h = h 1_H = -h, h^2 = 0,$$

the Hom-coalgebra structure is defined by

$$\Delta(1_H) = 1_H\otimes 1_H, \Delta(h) = (-h)\otimes 1_H + 1_H\otimes(-h), \epsilon(1_H) = 1, \epsilon(h) = 0,$$

and the antipode is defined by $S : H\to H, S(1_H) = 1_H, S(h) = -h$.

Recall from ([6]), $A = k\{1_A, x, g, gx\}$ is a Sweedler 4 dimensional monoidal Hopf algebra constructed from Sweedler 4-dimension Hopf algebra by Yau twist, where the twist map is defined by

$$\beta(1_A) = 1_A, \beta(g) = g, \beta(x) = -x, \beta(gx) = -gx,$$

the Hom-algebra structure m is defined by

$$m(1_A\otimes 1_A)=1_A, m(1_A\otimes g)=g, m(1_A\otimes x)=-x, m(1_A\otimes gx)=-gx,$$
$$m(g\otimes 1_A)=g, m(g\otimes g)=1, m(g\otimes x)=-gx, m(g\otimes gx)=-x,$$
$$m(x\otimes 1_A)=-x, m(x\otimes g)=gx, m(x\otimes x)=0, m(x\otimes gx)=0,$$
$$m(gx\otimes 1_A)=-gx, m(gx\otimes g)=x, m(gx\otimes x)=0, m(gx\otimes gx)=0,$$

the Hom-coalgebra structures ϵ and Δ are defined by

$$\epsilon(1_A)=1, \epsilon(g)=\epsilon(x)=\epsilon(gx)=0, \Delta(1_A)=1_A\otimes 1_A, \Delta(g)=g\otimes g,$$
$$\Delta(x)=(-x)\otimes 1_A+g\otimes(-x), \Delta(gx)=(-gx)\otimes g+1\otimes(-gx)$$

and the antipode is defined by $S:A\to A, S(1_A)=1_A, S(g)=g, S(x)=-gx, S(gx)=x$.

Now we define a left (H,α)-Hom-module structure on A:

$$h\cdot 1_A=h\cdot g=h\cdot x=h\cdot gx=0,$$
$$1_H\cdot 1_A=1_A, 1_H\cdot g=g, 1_H\cdot x=-x, 1_H\cdot gx=-gx.$$

One may check directly that A is a (H,α)-Hom-module algebra. Similarly, we can define a left (H,α)-Hom-comodule structure on A:

$$\rho(1_A)=1_H\otimes 1_A, \rho(g)=1_H\otimes g, \rho(x)=1_H\otimes(-x), \rho(gx)=1_H\otimes(-gx).$$

Then A is a (H,α)-Hom-comodule algebra and A is an object in ${}^H_H\mathcal{HYD}$.

Define the braiding C on A by the usual flip map. Clearly, C is symmetric on A. By Proposition 3.2, there is a braided Hom-Lie algebra A^- with the bracket product $[\cdot,\cdot]$ satisfying the following non-vanishing relation

$$[x,g]=-[g,x]=2gx, [gx,g]=-[g,gx]=2x.$$

Lemma 3.6. Let (A,β) be a monoidal Hom-algebra in ${}^H_H\mathcal{HYD}$ with monoidal Hom-subalgebras X and Y which are H-commutative such that $A=X+Y$. Then the following equality holds:

$$\begin{aligned}&\alpha^{-1}(u_{(-1)})\otimes\alpha^{-1}(y_{(-1)})\otimes(u_0y_0)^X_{(-1)}\otimes(u_0y_0)^X_0+\alpha^{-1}(u_{(-1)})\otimes\alpha^{-1}(y_{(-1)})\otimes(u_0y_0)^Y_{(-1)}\otimes(u_0y_0)^Y_0\\=\;&u_{(-1)1}\otimes y_{(-1)1}\otimes u_{(-1)2}y_{(-1)2}\otimes\beta^{-1}((u_0y_0)^X)+u_{(-1)1}\otimes y_{(-1)1}\otimes u_{(-1)2}y_{(-1)2}\otimes\beta^{-1}((u_0y_0)^Y),\end{aligned}\tag{6}$$

for all $u\in X$ and $y\in Y$, where $u_0y_0=(u_0y_0)^X+(u_0y_0)^Y\in X+Y$.

Proof. Since $\Delta(m_{(-1)})\otimes\beta^{-1}(m_0)=\alpha^{-1}(m_{(-1)})\otimes(m_{0(-1)}\otimes m_{00})$, by applying it to u and y respectively, we can get Eq. (6). □

Lemma 3.7. Let (A,β) be a monoidal Hom-algebra in ${}^H_H\mathcal{HYD}$ with monoidal Hom-subalgebras X and Y which are H-commutative such that $A=X+Y$. Assume that the braiding C is symmetric on A, then the following equality holds:

$$\begin{aligned}&\epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot\beta^{-1}(w))\beta((u_0y_0)^X)-\epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot\beta^{-1}(z))\beta((u_0y_0)^Y)\\=\;&\epsilon(u_{(-1)})\beta((u_0y_0)^X)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(w))-\epsilon(u_{(-1)})\beta((u_0y_0)^Y)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(z)),\end{aligned}\tag{7}$$

for all $u\in X$ and $y\in Y$, where $u_0y_0=(u_0y_0)^X+(u_0y_0)^Y\in X+Y$.

Proof. For Eq. (7), we show it by the following computation:

$$\begin{aligned}
& \epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot\beta^{-1}(w))\beta((u_0y_0)^X) - \epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot\beta^{-1}(z))\beta((u_0y_0)^Y) \\
= \;& \epsilon(y_{(-1)})((\alpha(u_{(-1)})\cdot\beta^{-1}(w))_{(-1)}\cdot(u_0y_0)^X)\beta((\alpha(u_{(-1)})\cdot\beta^{-1}(w))_0) - \\
& \epsilon(y_{(-1)})((\alpha(u_{(-1)})\cdot\beta^{-1}(z))_{(-1)}\cdot(u_0y_0)^Y)\beta((\alpha(u_{(-1)})\cdot\beta^{-1}(z))_0) \\
= \;& \epsilon(y_{(-1)})\beta(\beta((u_0y_0)^X)_0)(S^{-1}(\beta((u_0y_0)^X)_{(-1)})\cdot\beta^{-1}(\alpha(u_{(-1)})\cdot\beta^{-1}(w))) - \\
& \epsilon(y_{(-1)})\beta(\beta((u_0y_0)^Y)_0)(S^{-1}(\beta((u_0y_0)^Y)_{(-1)})\cdot\beta^{-1}(\alpha(u_{(-1)})\cdot\beta^{-1}(z))) \\
= \;& \epsilon(y_{(-1)})\beta^2((u_0y_0)^X_0)(S^{-1}(\alpha((u_0y_0)^X_{(-1)}))\cdot\beta^{-1}(\alpha(u_{(-1)})\cdot\beta^{-1}(w))) - \\
& \epsilon(y_{(-1)})\beta^2((u_0y_0)^Y_0)(S^{-1}(\alpha((u_0y_0)^Y_{(-1)}))\cdot\beta^{-1}(\alpha(u_{(-1)})\cdot\beta^{-1}(z)) \\
\overset{(6)}{=} \;& \epsilon(\alpha(y_{(-1)1}))\beta((u_0y_0)^X)(S^{-1}(\alpha(u_{(-1)2}y_{(-1)2}))\cdot\beta^{-1}(\alpha^2(u_{(-1)1})\cdot\beta^{-1}(w))) - \\
& \epsilon(\alpha(y_{(-1)1}))\beta((u_0y_0)^Y)(S^{-1}(\alpha(u_{(-1)2}y_{(-1)2}))\cdot\beta^{-1}(\alpha^2(u_{(-1)1})\cdot\beta^{-1}(z))) - \\
= \;& \epsilon(u_{(-1)})(\beta((u_0y_0)^X)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(w)) - \beta((u_0y_0)^Y)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(z))).
\end{aligned}$$

The last equality holds since

$$\begin{aligned}
& \epsilon(\alpha(y_{(-1)1}))S^{-1}(\alpha(u_{(-1)2}y_{(-1)2}))\cdot\beta^{-1}(\alpha^2(u_{(-1)1})\cdot\beta^{-1}(w)) \\
= \;& \epsilon(\alpha(y_{(-1)1}))S^{-1}(\alpha(u_{(-1)2}y_{(-1)2}))\cdot(\alpha(u_{(-1)1})\cdot\beta^{-2}(w)) \\
= \;& \epsilon(y_{(-1)1})((S^{-1}(y_{(-1)2})S^{-1}(u_{(-1)2}))\alpha(u_{(-1)1}))\cdot\beta^{-1}(w) \\
= \;& \epsilon(y_{(-1)1})(\alpha(S^{-1}(y_{(-1)2}))(S^{-1}(u_{(-1)2})u_{(-1)1}))\cdot\beta^{-1}(w) \\
= \;& (S^{-1}(y_{(-1)})(\epsilon(u_{(-1)})1_H))\cdot\beta^{-1}(w) \\
= \;& \epsilon(u_{(-1)})S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(w).
\end{aligned}$$

And this completes the proof. □

Theorem 3.8. Let (A,β) be a monoidal Hom-algebra in ${}^H_H\mathcal{HYD}$ with monoidal Hom-subalgebras X and Y which are H-commutative such that $A = X + Y$. Assume that the braiding C is symmetric on A, then $[A,A][A,A] = 0$.

Proof. It is sufficient to prove $[u,x][v,y] = 0$ holds for all $u,v\in X$ and $x,y\in Y$. For any $a,b,c,d\in A$, we first note that $(ab)(cd) = (a\beta^{-1}(bc))\beta(d)$ which can be verified easily from the Hom-associativity of A. By the definition of the bracket product, we have

$$\begin{aligned}
[u,x][v,y] \;=\;& (ux - (u_{(-1)}\cdot\beta^{-1}(x))\beta(u_0))(vy - (v_{(-1)}\cdot\beta^{-1}(y))\beta(v_0)) \\
=\;& (ux)(vy) + ((u_{(-1)}\cdot\beta^{-1}(x))\beta(u_0))((v_{(-1)}\cdot\beta^{-1}(y))\beta(v_0)) - \\
& (ux)((v_{(-1)}\cdot\beta^{-1}(y))\beta(v_0)) - ((u_{(-1)}\cdot\beta^{-1}(x))\beta(u_0))(vy).
\end{aligned}$$

Next we will compute the four expressions above respectively. For this purpose, let $xv = w + z$, where $w\in X, z\in Y$.

(1) $(ux)(vy) = ((u_{(-1)}\cdot\beta^{-2}(w))\beta(u_0))\beta(y) + (u\beta^{-1}(z_{(-1)}\cdot y))\beta(z_0)$. In fact,

$$\begin{aligned}
(ux)(vy) \;=\;& (u\beta^{-1}(xv))\beta(y) = (u\beta^{-1}(w))\beta(y) + \beta(u)(\beta^{-1}(z)y) \\
=\;& ((u_{(-1)}\cdot\beta^{-2}(w))\beta(u_0))\beta(y) + \beta(u)((\alpha^{-1}(z_{(-1)})\cdot\beta^{-1}(y))\beta(\beta^{-1}(z_0))) \\
=\;& ((u_{(-1)}\cdot\beta^{-2}(w))\beta(u_0))\beta(y) + \beta(u)((\alpha^{-1}(z_{(-1)})\cdot\beta^{-1}(y))z_0) \\
=\;& ((u_{(-1)}\cdot\beta^{-2}(w))\beta(u_0))\beta(y) + (u\beta^{-1}(z_{(-1)}\cdot y))\beta(z_0).
\end{aligned}$$

(2) $((u_{(-1)}\cdot\beta^{-1}(x))\beta(u_0))(vy) = ((u_{(-1)}\cdot\beta^{-2}(w))\beta(u_0))\beta(y) + \epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot\beta^{-1}(z))\beta((u_0y_0)^X) + \epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot$

$\beta^{-1}(z))\beta((u_0y_0)^Y)$. In fact,

$$\begin{aligned}
&((u_{(-1)}\cdot\beta^{-1}(x))\beta(u_0))(vy)\\
=\ &((u_{(-1)}\cdot\beta^{-1}(x))\beta^{-1}(\beta(u_0)v))\beta(y)\\
=\ &((u_{(-1)}\cdot\beta^{-1}(x))(u_0\beta^{-1}(v)))\beta(y)\\
=\ &((u_{(-1)}\cdot\beta^{-1}(x))((u_{0(-1)}\cdot\beta^{-2}(v))\beta(u_{00})))\beta(y)\\
=\ &((\alpha(u_{(-1)1})\cdot\beta^{-1}(x))((u_{(-1)2}\cdot\beta^{-2}(v))u_0))\beta(y)\\
=\ &(((u_{(-1)1}\cdot\beta^{-2}(x))(u_{(-1)2}\cdot\beta^{-2}(v)))\beta(u_0))\beta(y)\\
=\ &((u_{(-1)}\cdot\beta^{-2}(xv))\beta(u_0))\beta(y)\\
=\ &((u_{(-1)}\cdot\beta^{-2}(w))\beta(u_0))\beta(y)+((u_{(-1)}\cdot\beta^{-2}(z))\beta(u_0))\beta(y)\\
=\ &((u_{(-1)}\cdot\beta^{-2}(w))\beta(u_0))\beta(y)+(\alpha(u_{(-1)})\cdot\beta^{-1}(z))(\beta(u_0)\beta(y_0))\epsilon(y_{(-1)})\\
=\ &((u_{(-1)}\cdot\beta^{-2}(w))\beta(u_0))\beta(y)+(\alpha(u_{(-1)})\cdot\beta^{-1}(z))\beta(u_0y_0)\epsilon(y_{(-1)})\\
=\ &((u_{(-1)}\cdot\beta^{-2}(w))\beta(u_0))\beta(y)+\epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot\beta^{-1}(z))\beta((u_0y_0)^X)\\
&+\epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot\beta^{-1}(z))\beta((u_0y_0)^Y).
\end{aligned}$$

(3) $(ux)((v_{(-1)}\cdot\beta^{-1}(y))\beta(v_0))=(u\beta^{-1}(z_{(-1)}\cdot y))\beta(z_0)+\epsilon(u_{(-1)})\beta((u_0y_0)^X)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(w))$ $+\ \epsilon(u_{(-1)})\beta((u_0y_0)^Y)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(w))$. In fact,

$$\begin{aligned}
&(ux)((v_{(-1)}\cdot\beta^{-1}(y))\beta(v_0))\\
=\ &(u\beta^{-1}(x(v_{(-1)}\cdot\beta^{-1}(y))))\beta^2(v_0)\\
=\ &(u\beta^{-1}((x_{(-1)}\cdot\beta^{-1}(v_{(-1)}\cdot\beta^{-1}(y)))\beta(x_0)))\beta^2(v_0)\\
=\ &(u\beta^{-1}((x_{(-1)}\cdot(\alpha^{-1}(v_{(-1)})\cdot\beta^{-2}(y)))\beta(x_0)))\beta^2(v_0)\\
=\ &(u\beta^{-1}((\alpha^{-1}(x_{(-1)}v_{(-1)})\cdot\beta^{-1}(y))\beta(x_0))\beta^2(v_0)\\
=\ &\beta(u)(((\alpha^{-2}(x_{(-1)}v_{(-1)})\cdot\beta^{-2}(y))x_0)\beta(v_0))\\
=\ &\beta(u)((\alpha^{-1}(x_{(-1)}v_{(-1)})\cdot\beta^{-1}(y))(x_0v_0))\\
=\ &(u\beta^{-1}((x_{(-1)}v_{(-1)})\cdot y))\beta(x_0v_0)\\
=\ &(u\beta^{-1}((xv)_{(-1)}\cdot y))\beta((xv)_0)\\
=\ &(u\beta^{-1}(w_{(-1)}\cdot y))\beta(w_0)+(u\beta^{-1}(z_{(-1)}\cdot y))\beta(z_0)\\
=\ &(u\beta(y_0))(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(w))+(u\beta^{-1}(z_{(-1)}\cdot y))\beta(z_0)\\
=\ &\epsilon(u_{(-1)})\beta(u_0y_0)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(w))+(u\beta^{-1}(z_{(-1)}\cdot y))\beta(z_0)\\
=\ &\epsilon(u_{(-1)})\beta((u_0y_0)^X)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(w))+(u\beta^{-1}(z_{(-1)}\cdot y))\beta(z_0)+\\
&\epsilon(u_{(-1)})\beta((u_0y_0)^Y)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(w)).
\end{aligned}$$

(4) $((u_{(-1)}\cdot\beta^{-1}(x))\beta(u_0))((v_{(-1)}\cdot\beta^{-1}(y))\beta(v_0))=\epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot\beta^{-1}(w))\beta((u_0y_0)^X)$ $+\ \epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot\beta^{-1}(w))\beta((u_0y_0)^X)+\epsilon(u_{(-1)})\beta((u_0y_0)^Y)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(w))+$ $\epsilon(u_{(-1)})\beta((u_0y_0)^Y)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(z))$.

Here we first give two useful equalities:

$$(u_{(-1)2}y_{(-1)2})\cdot(S^{-1}(y_{(-1)1})\cdot\beta^{-2}(v)) = \epsilon(y_{(-1)})\alpha(u_{(-1)2})\cdot\beta^{-1}(v), \tag{8}$$

$$(S^{-1}(y_{(-1)2})S^{-1}(u_{(-1)2}))\cdot(u_{(-1)1}\cdot\beta^{-2}(v)) = \epsilon(u_{(-1)})S^{-1}(\alpha(y_{(-1)2}))\cdot\beta^{-1}(v). \tag{9}$$

In fact,

$$\begin{aligned}
&(u_{(-1)2}y_{(-1)2})\cdot(S^{-1}(y_{(-1)1})\cdot\beta^{-2}(v))\\
=\ &((\alpha^{-1}(u_{(-1)2})\alpha^{-1}(y_{(-1)2}))S^{-1}(y_{(-1)1}))\cdot\beta^{-1}(v)\\
=\ &(u_{(-1)2}(\alpha^{-1}(y_{(-1)2})\alpha^{-1}(S^{-1}(y_{(-1)1}))))\cdot\beta^{-1}(v)\\
=\ &(u_{(-1)2}\epsilon(y_{(-1)})1_H)\cdot\beta^{-1}(v)=\epsilon(y_{(-1)})\alpha(u_{(-1)2})\cdot\beta^{-1}(v).
\end{aligned}$$

So Eq. (8) holds and similarly for Eq. (9). Therefore,

$$
\begin{aligned}
& ((u_{(-1)}\cdot\beta^{-1}(x))\beta(u_0))((v_{(-1)}\cdot\beta^{-1}(y))\beta(v_0)) \\
= \; & ((u_{(-1)}\cdot\beta^{-1}(x))\beta(u_0))(\beta(y_0)(S^{-1}(y_{(-1)})\cdot\beta^{-1}(v))) \\
= \; & ((u_{(-1)}\cdot\beta^{-1}(x))(u_0y_0))\beta(S^{-1}(y_{(-1)})\cdot\beta^{-1}(v)) \\
= \; & ((u_{(-1)}\cdot\beta^{-1}(x))(u_0y_0))(S^{-1}(\alpha(y_{(-1)}))\cdot v) \\
= \; & \beta(u_{(-1)}\cdot\beta^{-1}(x))((u_0y_0)(S^{-1}(y_{(-1)})\cdot\beta^{-1}(v))) \\
= \; & \beta(u_{(-1)}\cdot\beta^{-1}(x))((u_0y_0)^X(S^{-1}(y_{(-1)})\cdot\beta^{-1}(v)))+ \\
& \beta(u_{(-1)}\cdot\beta^{-1}(x))((u_0y_0)^Y(S^{-1}(y_{(-1)})\cdot\beta^{-1}(v))) \\
= \; & \beta(u_{(-1)}\cdot\beta^{-1}(x))(((u_0y_0)^X_{(-1)}\cdot\beta^{-1}(S^{-1}(y_{(-1)})\cdot\beta^{-1}(v)))\beta((u_0y_0)^X_0))+ \\
& ((u_{(-1)}\cdot\beta^{-1}(x))(u_0y_0)^Y)\beta(S^{-1}(y_{(-1)})\cdot\beta^{-1}(v)) \\
= \; & ((u_{(-1)}\cdot\beta^{-1}(x))((u_0y_0)^X_{(-1)}\cdot\beta^{-1}(S^{-1}(y_{(-1)})\cdot\beta^{-1}(v))))\beta^2((u_0y_0)^X_0)+ \\
& (((u_{(-1)}\cdot\beta^{-1}(x))_{(-1)}\cdot\beta^{-1}((u_0y_0)^Y))\beta((u_{(-1)}\cdot\beta^{-1}(x))_0))\beta(S^{-1}(y_{(-1)})\cdot\beta^{-1}(v)) \\
= \; & ((u_{(-1)}\cdot\beta^{-1}(x))((u_0y_0)^X_{(-1)}\cdot\beta^{-1}(S^{-1}(y_{(-1)})\cdot\beta^{-1}(v))))\beta^2((u_0y_0)^X_0)+ \\
& (\beta((u_0y_0)^Y_0)(S^{-1}((u_0y_0)^Y_{(-1)})\cdot\beta^{-1}(u_{(-1)}\cdot\beta^{-1}(x))))\beta(S^{-1}(y_{(-1)})\cdot\beta^{-1}(v)) \\
= \; & ((\alpha(u_{(-1)1})\cdot\beta^{-1}(x))((u_{(-1)2}y_{(-1)2})\cdot\beta^{-1}(S^{-1}(\alpha(y_{(-1)1}))\cdot\beta^{-1}(v))))\beta((u_0y_0)^X)+ \\
& ((u_0y_0)^Y(S^{-1}(u_{(-1)2}y_{(-1)2})\cdot\beta^{-1}(\alpha(u_{(-1)1})\cdot\beta^{-1}(x))))\beta(S^{-1}(\alpha(y_{(-1)1}))\cdot\beta^{-1}(v)) \\
= \; & ((\alpha(u_{(-1)1})\cdot\beta^{-1}(x))((u_{(-1)2}y_{(-1)2})\cdot(S^{-1}(y_{(-1)1})\cdot\beta^{-2}(v))))\beta((u_0y_0)^X)+ \\
& ((u_0y_0)^Y((S^{-1}(u_{(-1)2})S^{-1}(y_{(-1)2}))\cdot(u_{(-1)1}\cdot\beta^{-2}(x))))\beta(S^{-1}(\alpha(y_{(-1)1}))\cdot\beta^{-1}(v)) \\
\overset{(8),(9)}{=} \; & \epsilon(y_{(-1)})((\alpha(u_{(-1)1})\cdot\beta^{-1}(x))(\alpha(u_{(-1)2})\beta^{-1}(v)))\beta((u_0y_0)^X)+ \\
& \epsilon(u_{(-1)})((u_0y_0)^Y(S^{-1}(\alpha(y_{(-1)2}))\cdot\beta^{-1}(x)))\beta(S^{-1}(\alpha(y_{(-1)1}))\cdot\beta^{-1}(v)) \\
= \; & \epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot\beta^{-1}(xv))\beta((u_0y_0)^X)+\epsilon(u_{(-1)})\beta((u_0y_0)^Y)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(xv)) \\
= \; & \epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot\beta^{-1}(w))\beta((u_0y_0)^X)+\epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot\beta^{-1}(w))\beta((u_0y_0)^X)+ \\
& \epsilon(u_{(-1)})\beta((u_0y_0)^Y)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(w))+\epsilon(u_{(-1)})\beta((u_0y_0)^Y)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(z)).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
[u,x][v,y] \;=\; & -\epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot\beta^{-1}(z))\beta((u_0y_0)^Y) \\
& -\epsilon(u_{(-1)})\beta((u_0y_0)^X)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(w)) \\
& +\epsilon(y_{(-1)})(\alpha(u_{(-1)})\cdot\beta^{-1}(w))\beta((u_0y_0)^X) \\
& +\epsilon(u_{(-1)})\beta((u_0y_0)^Y)(S^{-1}(\alpha(y_{(-1)}))\cdot\beta^{-1}(z)) \\
=\; & 0,
\end{aligned}
$$

as desired. And this completes the proof. □

Next we will give an interesting corollary, for this we first consider some H-analogous of classical concepts of ring theory and Lie theory as follows.

Let (A,β) be a monoidal Hom-algebra in ${}^H_H\mathcal{HYD}$. An *H-Hom-ideal* U of A is not only H-stable (i.e. $h\cdot a\in U$ for all $h\in H$ and $a\in U$) but also H-costable (i.e. $\rho(a)\in H\otimes U$ for all $a\in U$) such that $\beta(U)\subseteq U$ and $(AU)A=A(UA)\subseteq U$.

Let (L,β) be a braided Hom-Lie algebra. An *H-Hom-Lie ideal* U of L is not only H-stable but also H-costable such that $\beta(U)\subseteq U$ and $[U,L]\subseteq U$.

Define the *center* of L to be $Z(L)=\{l\in L|[l,L]=0\}$. It is easy to see that $Z(L)$ is not only H-stable but also H-costable.

L is called *H-prime* if the product of any two non-zero H-Hom-ideals of L is non-zero. It is called *H-semiprime* if it has no non-zero nilpotent H-Hom-ideals, and is called *H-simple* if it has no nontrivial H-Hom-ideals.

Corollary 3.9. Under the hypotheses of the theorem above, $[A, A]$ is nilpotent. If A is also H-semiprime, then A is H-commutative.

Proof. Straightforward from Theorem 3.8. □

4. Central invariants of braided Hom-Lie algebras

In this section, we study the central invariant of braided Hom-Lie algebras as a generalization of [27], we always assume that (H, α) is a monoidal Hom-Hopf algebra.

Definition 4.1. If (A, β) is a monoidal Hom-algebra in ${}^H_H\mathcal{HYD}$, the monoidal Hom-subalgebra of H-invariants is the set:

$$A_0 = \{a \in A | h \cdot a = \epsilon(h)a, \text{ for all } h \in H\}.$$

Recall from Proposition 3.2, a monoidal Hom-algebra (L, β) in ${}^H_H\mathcal{HYD}$ gives rise to a braided Hom-Lie algebra $(L, [\cdot, \cdot], \beta)$ in ${}^H_H\mathcal{HYD}$.

In what follows, we always assume that the bracket product in braided Hom-Lie algebra $(L, [\cdot, \cdot], \beta)$ is defined as Proposition 3.2, that is

$$[,] : A \otimes A \to A \ by \ [a, b] = ab - (a_{(-1)} \cdot \beta^{-1}(b))\beta(a_0), \ a, b \in A.$$

Lemma 4.2. Let (L, β) be a monoidal Hom-algebra in ${}^H_H\mathcal{HYD}$ and $(L, [\cdot, \cdot], \beta)$ the derived braided Hom-Lie algebra. Then

(1) $[\beta(a), bc] = [a, b]\beta(c) + (\alpha(a_{(-1)}) \cdot b)[\beta(a_0), c]$,

(2) $[ab, \beta(c)] = \beta(a)[b, c] + [a, b_{(-1)} \cdot \beta^{-1}(c)]\beta^2(b_0)$, for all $a, b, c \in L$.

Proof. (1) For all $a, b, c \in L$, it is clear that $[a, b]\beta(c) = (ab)\beta(c) - ((a_{(-1)} \cdot \beta^{-1}(b))\beta(a_0))\beta(c)$. Similarly,

$$\begin{aligned}
& (\alpha(a_{(-1)}) \cdot b)[\beta(a_0), c] \\
= \ & (\alpha(a_{(-1)}) \cdot b)(\beta(a_0)c) - (\alpha(a_{(-1)}) \cdot b)((\alpha(a_{0(-1)}) \cdot \beta^{-1}(c))\beta^2(a_{00})) \\
= \ & \beta(a_{(-1)} \cdot \beta^{-1}(b))(\beta(a_0)c) - \beta(a_{(-1)} \cdot \beta^{-1}(b))((\alpha(a_{0(-1)}) \cdot \beta^{-1}(c))\beta^2(a_{00})) \\
= \ & ((a_{(-1)} \cdot \beta^{-1}(b))\beta(a_0))\beta(c) - ((a_{(-1)} \cdot \beta^{-1}(b))(\alpha(a_{0(-1)}) \cdot \beta^{-1}(c)))\beta^3(a_{00})) \\
= \ & ((a_{(-1)} \cdot \beta^{-1}(b))\beta(a_0))\beta(c) - ((\alpha(a_{(-1)1}) \cdot \beta^{-1}(b))(\alpha(a_{(-1)2}) \cdot \beta^{-1}(c)))\beta^2(a_0)) \\
= \ & ((a_{(-1)} \cdot \beta^{-1}(b))\beta(a_0))\beta(c) - (\alpha(a_{(-1)}) \cdot \beta^{-1}(bc))\beta^2(a_0)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& [a, b]\beta(c) + (\alpha(a_{(-1)}) \cdot b)[\beta(a_0), c] \\
= \ & (ab)\beta(c) - (\alpha(a_{(-1)}) \cdot \beta^{-1}(bc))\beta^2(a_0)) \\
= \ & \beta(a)(bc) - ((\alpha(a_{(-1)}) \cdot \beta^{-1}(bc))\beta^2(a_0) \\
= \ & \beta(a)(bc) - ((\beta(a))_{(-1)} \cdot \beta^{-1}(bc))\beta((\beta(a))_0) \\
= \ & [\beta(a), bc].
\end{aligned}$$

(2) For all $a, b, c \in L$, on the one hand, we have

$$\begin{aligned}
\beta(a)[b, c] & = \beta(a)(bc) - \beta(a)((b_{(-1)} \cdot \beta^{-1}(c))\beta(b_0)) \\
& = (ab)\beta(c) - (a(b_{(-1)} \cdot \beta^{-1}(c)))\beta^2(b_0).
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
& [a, b_{(-1)} \cdot \beta^{-1}(c)]\beta^2(b_0) \\
= \ & (a(b_{(-1)} \cdot \beta^{-1}(c)))\beta^2(b_0) - ((a_{(-1)} \cdot \beta^{-1}(b_{(-1)} \cdot \beta^{-1}(c)))\beta(a_0))\beta^2(b_0) \\
= \ & (a(b_{(-1)} \cdot \beta^{-1}(c)))\beta^2(b_0) - ((a_{(-1)} \cdot (\alpha^{-1}(b_{(-1)}) \cdot \beta^{-2}(c)))\beta(a_0))\beta^2(b_0) \\
= \ & (a(b_{(-1)} \cdot \beta^{-1}(c)))\beta^2(b_0) - (((\alpha^{-1}(a_{(-1)})\alpha^{-1}(b_{(-1)})) \cdot \beta^{-1}(c))\beta(a_0))\beta^2(b_0) \\
= \ & (a(b_{(-1)} \cdot \beta^{-1}(c)))\beta^2(b_0) - (a_{(-1)}b_{(-1)} \cdot c)\beta(a_0 b_0).
\end{aligned}$$

It follows that

$$\begin{aligned}
& \beta(a)[b, c] + [a, b_{(-1)} \cdot \beta^{-1}(c)]\beta^2(b_0) \\
= \ & \beta(a)(bc) - (a_{(-1)}b_{(-1)} \cdot c)\beta(a_0 b_0) \\
= \ & (ab)\beta(c) - (a_{(-1)}b_{(-1)} \cdot c)\beta(a_0 b_0) \\
= \ & [ab, \beta(c)].
\end{aligned}$$

The proof is completed. □

Define $ad_x(l) = [x, l]$ for all $x, l \in L$, By Lemma 4.2(1) we have

$$ad_{\beta(x)}(lm) = ad_x(l)\alpha(m) + (\alpha^{-1}(x_{(-1)}) \cdot \beta(l))ad_{x_0}(m), \ x, l, m \in L.$$

Lemma 4.3. Let (L, β) be a monoidal Hom-algebra in ${}^H_H\mathcal{HYD}$ and x a β-invariant element in L_0. Then for any $y, z \in L$, the following equalities hold:
(1) $C_{L,L}(x \otimes y) = y \otimes x$, $C_{L,L}(y \otimes x) = x \otimes y$;
(2) $ad_x(y) = xy - yx$;
(3) $ad_x(yz) = ad_x(y)\beta(z) + \beta(y)ad_x(z)$;
(4) $ad_x^2(yz) = ad_x^2(y)\beta^2(z) + 2\beta(ad_x(y)ad_x(z)) + \beta^2(y)ad_x^2(z)$.

Proof. (1) Since $x \in L_0$, we have

$$\begin{aligned}
C_{L,L}(y \otimes x) &= y_{(-1)} \cdot \beta^{-1}(x) \otimes \beta(y_0) = y_{(-1)} \cdot x \otimes \beta(y_0) \\
&= \epsilon(y_{(-1)})x \otimes \beta(y_0) = x \otimes y, \\
C_{L,L}(x \otimes y) &= x_{(-1)} \cdot \beta^{-1}(y) \otimes \beta(x_0) = \beta(y_0) \otimes S^{-1}(y_{(-1)}) \cdot \beta^{-1}(x) \\
&= \beta(y_0) \otimes S^{-1}(y_{(-1)}) \cdot x = \beta(y_0) \otimes \epsilon(S^{-1}(y_{(-1)}))x = y \otimes x.
\end{aligned}$$

(2) Straightforward from (1).
(3) Straightforward from Lemma 4.2 (1).
(4) By (2) and (3), we have

$$\begin{aligned}
ad_x^2(yz) &= ad_x(ad_x(y)\beta(z) + \beta(y)ad_x(z)) \\
&= ad_x(ad_x(y)\beta(z)) + ad_x(\beta(y)ad_x(z)) \\
&= ad_x^2(y)\beta^2(z) + \beta(ad_x(y))ad_x\beta(z) + \\
&\quad\ ad_x\beta(y)\beta(ad_x(z)) + \beta^2(y)ad_x^2(z) \\
&= ad_x^2(y)\beta^2(z) + \beta(ad_x(y))ad_{\beta(x)}\beta(z) + \\
&\quad\ ad_{\beta(x)}\beta(y)\beta(ad_x(z)) + \beta^2(y)ad_x^2(z) \\
&= ad_x^2(y)\beta^2(z) + 2\beta(ad_x(y)ad_x(z)) + \beta^2(y)ad_x^2(z).
\end{aligned}$$

The proof is finished. □

Lemma 4.4. Let $(L, [\cdot,\cdot], \beta)$ be the derived braided Hom-Lie algebra. Assume that L is H-simple, then $Z(L)_0$ is a field.

Proof. Note that $Z(L)_0 = Z(L) \cap L_0 = Z(L)_0$, where $Z(L)$ is the usual center of L. Taking $0 \neq x \in Z(L)_0$, we have that $Lx = I \neq 0$ is an H-Hom-ideal, thus $I = L$ since L is H-simple. That is to say that for some $y \in L$, we obtain $xy = yx = 1$. Since

$$\begin{aligned}
\beta^2(h \cdot y) &= \beta(h \cdot y)1 = \beta(h \cdot y)(xy) \\
&= \beta(\alpha(h_1) \cdot y)(\epsilon(\alpha(h_2))xy) \\
&= \beta(\alpha(h_1) \cdot y)((\alpha(h_2) \cdot x)y) \\
&= ((\alpha(h_1) \cdot y)(\alpha(h_2) \cdot x))\beta(y) \\
&= (\alpha(h) \cdot (xy))\beta(y) = (\alpha(h) \cdot 1)\beta(y) \\
&= (\epsilon(\alpha(h))1)\beta(y) = \epsilon(h)\beta^2(y) \\
&= \beta^2(\epsilon(h)y)
\end{aligned}$$

We can get $h \cdot y = \epsilon(h)y$ since β is bijective, that is, $y \in L_0$.

We need to show $y \in Z(L)$. For any $z \in L$, by Lemma 4.3(1), $[z, x] = zx - xz = 0$. Then we have

$$\begin{aligned}
&\beta^2(yz - zy) = \beta^2(yz) - \beta^2(zy) \\
= \;& \beta(yz)\beta(1) - \beta(yx)\beta(zy) \\
= \;& \beta^2(y)(\beta(z)1) - \beta^2(y)(\beta(x)(zy)) \\
= \;& \beta^2(y)(\beta(z)(xy)) - \beta^2(y)(\beta(x)(zy)) \\
= \;& \beta^2(y)((zx)\beta(y)) - \beta^2(y)((xz)\beta(y)) \\
= \;& \beta^2(y)((zx - xz)\beta(y)) \\
= \;& 0.
\end{aligned}$$

Since β is bijective, it follows that $yz = zy$, i.e. $[y, z] = yz - zy = 0$ by Lemma 4.3 (2). This shows that $y \in Z(L)$, as desired. □

Lemma 4.5. Let $(L, [\cdot, \cdot], \beta)$ be the derived braided Hom-Lie algebra and x a β-invariant element in L_0, $l, m \in L$. Then

(1) $ad_x^2(xl) = xad_x^2(l)$;

(2) If $ad_x^2(L) = 0$ and $\mathrm{char}(k) \neq 2$, then $ad_x(l)(Lad_x(m)) = 0$.

Proof. (1) It is straightforward from Lemma 4.3 (4).

(2) For all $l, m \in L$, we have

$$\begin{aligned}
0 &= ad_x^2(lm) = ad_x^2(l)\beta^2(m) + 2\beta(ad_x(l)ad_x(m)) + \beta^2(l)ad_x^2(m) \\
&= 2ad_x(\beta(l))ad_x(\beta(m)).
\end{aligned}$$

So $ad_x(l)ad_x(m) = 0$ since $\mathrm{char}(k) \neq 2$. For any $z \in L$, by Lemma 4.3 (3), $zad_x(m) = ad_x(\beta^{-1}(z)m) - ad_x(\beta^{-1}(z))\beta(m)$. Therefore,

$$\begin{aligned}
ad_x(l)(zad_x(m)) &= ad_x(l)ad_x(\beta^{-1}(z)m) - ad_x(l)(ad_x(\beta^{-1}(z))\beta(m)) \\
&= 0 - \beta(ad_x(\beta^{-1}(l)))(ad_x(\beta^{-1}(l))\beta(m)) \\
&= -(ad_x(\beta^{-1}(l))ad_x(\beta^{-1}(l)))m \\
&= 0.
\end{aligned}$$

By the arbitrary of z, $ad_x(l)(Lad_x(m)) = 0$. And this finishes the proof. □

Lemma 4.6. Let $(L, [,], \beta)$ be the derived braided Hom-Lie algebra and I an H-Hom-Lie ideal of $[L, L]$. Assume that L is H-simple and $\mathrm{char}(k) \neq 2$. If x is a β-invariant element in I_0 satisfying (i) $ad_x(I) = 0$, (ii) $ad_x^2([L, L]) = 0$. Then $x \in Z(L)$.

Proof. For any $m \in L$, $l \in [L, L]$ and $y \in I$. By Lemma 4.2 (1),

$$0 = ad_x^2([\beta(l), my]) = ad_x^2([l, m]\beta(y)) + ad_x^2((\alpha(l_{(-1)}) \cdot m)[\beta(l_0), y]).$$

First, we have

$$\begin{aligned}
& ad_x^2([l,m]\beta(y)) \\
= \;& ad_x^2([l,m])\beta^3(y) + 2\beta(ad_x([l,m])ad_x(\beta(y))) + \beta^2([l,m])ad_x^2(\beta(y)) \\
\overset{(i)}{=} \;& ad_x^2([l,m])\beta^3(y) \overset{(ii)}{=} 0.
\end{aligned}$$

So $ad_x^2((\alpha(l_{(-1)})\cdot m)[\beta(l_0),y])$. On the other hand, since $l \in [L,L]$ and $[,]$ is H-colinear, it follows that $\beta(l_0) \in [L,L]$, $ad_x([l_0,y]) \overset{(i)}{=} 0$ and $ad_x^2([l_0,y]) \overset{(ii)}{=} 0$. Therefore,

$$\begin{aligned}
& ad_x^2(\alpha(l_{(-1)})\cdot m)[\beta(l_0),y]) \\
= \;& ad_x^2(\alpha(l_{(-1)})\cdot m)\beta^2([\beta(l_0),y]) + 2\beta(ad_x(\alpha(l_{(-1)})\cdot m)ad_x([\beta(l_0),y])) \\
& +\beta^2(\alpha(l_{(-1)})\cdot m)ad_x^2([\beta(l_0),y]) \\
= \;& ad_x^2(\alpha(l_{(-1)})\cdot m)\beta^2([\beta(l_0),y]).
\end{aligned}$$

Thus we obtain $ad_x^2(\alpha(l_{(-1)})\cdot m)\beta^2([\beta(l_0),y]) = 0$. We completes the proof by the following two cases:

Case (1): If $[I,[L,L]] = 0$, then we have $ad_x^2(L) = 0$. By Lemma 4.5 (2), $ad_x(l)(Lad_x(m)) = 0$. Since L is H-simple, we get $ad_x(l) = 0$. So $x \in Z(L)$ since l is an arbitrary element in L.

Case (2): If $[I,[L,L]] \neq 0$, let $U = [I,[L,L]]$. It is easy to see that U is an H-Hom-Lie ideal of $[L,L]$. Since $ad_x^2(\alpha(l_{(-1)})\cdot m)\beta^2([\beta(l_0),y]) = 0$, we have $ad_x^2(L)U = 0$. Let $Q = \{y \in L | yU = 0\}$, then Q is an H-stable H-costable left Hom-ideal of L, we claim $Q = 0$. If not, then $L = QL$ since L is H-simple. By Proposition 3.2, we have

$$QL \subseteq [Q,L] + LQ \subseteq [Q,L] + Q \subseteq L.$$

Thus $L = Q + [Q,L]$. Let $y \in Q$, $l \in [L,L]$ and $u \in U$. Since Q is an H-Hom-ideal, $\beta^2(y_0) \in Q$. Then

$$\begin{aligned}
[y,l]u &= (yl)u - ((y_{(-1)}\cdot\beta^{-1}(l))\beta(y_0))u \\
&= (yl)u - \beta^{-1}(y_{(-1)}\cdot\beta^{-1}(l))(\beta(y_0)\beta^{-1}(u)) \\
&= (yl)u - \beta^{-1}(y_{(-1)}\cdot\beta^{-1}(l))\beta^{-1}(\beta^2(y_0)u) \\
&= (yl)u = \beta(y)(l\beta^{-1}(u)) \\
&= \beta(y)[l,\beta^{-1}(u)] + \beta(y)((l_{(-1)}\cdot\beta^{-2}(u))\beta(l_0)) \\
&= \beta(y)[l,\beta^{-1}(u)] + (y(l_{(-1)}\cdot\beta^{-2}(u)))l_0 \\
&= \beta(y)[l,\beta^{-1}(u)].
\end{aligned}$$

Since $\beta^{-1}(u) \in U, \beta(y) \in Q$, we obtain $[l,\beta^{-1}(u)] \in U$, $\beta(y)[l,\beta^{-1}(u)] = 0$, and thus $[y,l]u$. Which means $[Q,[L,L]] \subseteq Q$ and $Q[L,L] \subseteq Q$. Hence

$$L = QL = Q(Q + [Q,L]) \subseteq Q.$$

This implies $LU = 0$, which contradicts the assumption $U \neq 0$. Hence, $Q = 0$, and so $ad_x^2(L) = 0$. Similarly to case (1), one get $x \in Z(L)$. □

Theorem 4.7. Let $(L,[\cdot,\cdot],\beta)$ be the derived braided Hom-Lie algebra. Assume that char$(k) \neq 2$ and L is H-simple. If V is an H-Hom-Lie ideal of $[L,L]$ such that any element in V_0 is β-invariant and $[V_0,V] \subseteq Z(L)_0$. Then $V_0 \subseteq Z(L)_0$.

Proof. For any $x \in V_0$. We consider the following two cases:

(1) If $ad_x(V) = 0$, then $x \in Z(L)_0$ by Lemma 4.6.

(2) If $ad_x(V) \neq 0$, then for any $v \in V$ and $l \in L$, we have

$$\begin{aligned}
[[x,[x,l]],v] &= -[[x,[x,l]]_{(-1)}\cdot\beta^{-1}(v),\beta([x,[x,l]]_0)] \\
&= -[\beta(v_0),S^{-1}(v_{(-1)})\cdot\beta^{-1}([x,[x,l]])] \\
&= -[\beta(v_0),\beta^{-1}(S^{-1}(\alpha(v_{(-1)}))\cdot[x,[x,l]])] \\
&= -[\beta(v_0),\beta^{-1}([x,[x,S^{-1}(v_{(-1)})\cdot l]])] \\
&= -[\beta(v_0),[x,[x,S^{-1}(\alpha^{-1}(v_{(-1)}))\cdot\beta^{-1}(l)]]].
\end{aligned}$$

The fourth equality and the fifth equality hold since $x \in V_0$ is β-invariant. By Lemma 4.3 (1), we get

$$\begin{aligned} & (1\otimes C)(C\otimes 1)(v_0\otimes x\otimes [x, S^{-1}(\alpha^{-1}(v_{(-1)}))\cdot \beta^{-1}(l)]) \\ = \ & (1\otimes C)(x\otimes v_0\otimes [x, S^{-1}(\alpha^{-1}(v_{(-1)}))\cdot \beta^{-1}(l)]) \\ = \ & x\otimes v_{0(-1)}\cdot \beta^{-1}([x, S^{-1}(\alpha^{-1}(v_{(-1)}))\cdot \beta^{-1}(l)])\otimes \beta(v_{00}) \\ = \ & x\otimes v_{0(-1)}\cdot [x, S^{-1}(\alpha^{-2}(v_{(-1)}))\cdot \beta^{-2}(l)]\otimes \beta(v_{00}) \\ = \ & x\otimes v_{(-1)2}\cdot [x, S^{-1}(\alpha^{-1}(v_{(-1)1}))\cdot \beta^{-2}(l)]\otimes v_0 \\ = \ & x\otimes [v_{(-1)21}\cdot x, v_{(-1)22}\cdot (S^{-1}(\alpha^{-1}(v_{(-1)1})\cdot \beta^{-2}(l))]]\otimes v_0 \\ = \ & x\otimes [x, (\alpha^{-1}(v_{(-1)2})S^{-1}(\alpha^{-1}(v_{(-1)1})))\cdot \beta^{-2}(l)]\otimes v_0 \\ = \ & x\otimes [x, \epsilon(v_{(-1)})1\cdot \beta^{-2}(l)]\otimes v_0 \\ = \ & x\otimes [x, \beta^{-1}(l)]\otimes \beta^{-1}(v). \end{aligned}$$

Similarly, $(1\otimes C)(C\otimes 1)(v_0\otimes x\otimes [x, S^{-1}(\alpha^{-1}(v_{(-1)}))\cdot \beta^{-1}(l)]) = [x, \beta^{-1}(l)]\otimes \beta^{-1}(v)\otimes x$. By braided Hom-Jacobi identity, we have

$$\begin{aligned} [[x,[x,l]],v] &= -[\beta(v_0), [x, [x, S^{-1}(\alpha^{-1}(v_{(-1)}))\cdot \beta^{-1}(l)]]] \\ &= [[\beta(x), l], [v, x]] + [\beta(x), [[x, \beta^{-1}(l)], \beta^{-1}(v)]] \\ &= [[x, l], [v, x]] + [x, [[x, \beta^{-1}(l)], \beta^{-1}(v)]] \\ &\subseteq [[x, L], [V, x]] + [x, [[x, L], \beta^{-1}(v)]] \\ &\subseteq 0 + [x, [[L, L], V]] \subseteq [x, V] \subseteq Z_H(L)_0. \end{aligned}$$

We obtain $[ad_x^2(L), V] \subseteq Z(L)_0$. By Lemma 4.5 (1), we have $ad_x^2(xl) = \beta^2(x)ad_x^2(l)$.

(2.1) If $ad_x^2(l) \neq 0$ for some $l \in L$, then $(ad_x^2(l))^{-1} \in Z(L)_0$ by Lemma 4.4. In this case, it is easy to see that $x \in Z(L)_0$.

(2.2) Now we assume $ad_x^2(L) \subsetneq Z(L)_0$. Let $y \in L$ with $ad_x^2(y) \notin Z(L)_0$. Then we choose $z \in V$ such that $0 \neq ad_z(x) = u \in Z(L)_0$. Thus there exist $v_1, v_2, v_3 \in Z(L)_0$ such that $[z, ad_x^2(y)] = v_1$, $[\beta(z), ad_x^2(xy)] = v_2$ and $[\beta^2(z), ad_x^2(x^2y)] = v_3$. Now we have

$$\begin{aligned} v_2 &= [\beta(z), ad_x^2(xy)] = [\beta(z), xad_x^2(y)] \\ &= [z, x]\beta(ad_x^2(y)) + (\alpha(z_{(-1)})\cdot x)[\beta(z_0), ad_x^2(y)] \\ &= [z, x]\beta(ad_x^2(y)) + x[z, ad_x^2(y)] \\ &= u\beta(ad_x^2(y)) + xv_1. \end{aligned}$$

By Lemma 4.4, u is invertible. Thus $ad_x^2(y) = \beta^{-1}(u^{-1}v_2 - u^{-1}(xv_1))$. However, $v_1 \in Z(L)$, $x \in V_0$, by Lemma 4.3 (1), we have $xv_1 = v_1x$, and so $ad_x^2(y) = \beta^{-1}(u^{-1}v_2 - u^{-1}(v_1x))$. Similarly, we have

$$\begin{aligned} v_3 &= [\beta^2(z), ad_x^2(x^2y)] = [\beta(\beta(z)), xad_x^2(xy)] \\ &= [\beta(z), x]\beta(ad_x^2(xy)) + (\alpha((\beta(z))_{(-1)})\cdot x)[\beta((\beta(z))_0), ad_x^2(xy)] \\ &= [\beta(z), x]\beta(ad_x^2(xy)) + (\alpha^2(z_{(-1)})\cdot x)[\beta^2(z_0), ad_x^2(xy)] \\ &= [\beta(z), \beta(x)]\beta(ad_x^2(xy)) + x[\beta(z), ad_x^2(xy)] \\ &= \beta(u)\beta(ad_x^2(xy)) + xv_2 \\ &= u\beta(ad_x^2(xy)) + xv_2. \end{aligned}$$

The last equality holds since $u = ad_z(x) \in V_0$. Thus $ad_x^2(xy) = \beta^{-1}(u^{-1}v_3 - u^{-1}(v_2x))$. Using Lemma 4.5 (1), we have

$$\begin{aligned}
ad_x^2(xy) &= xad_x^2(y) = x\beta^{-1}(u^{-1}v_2 - u^{-1}(v_1x)) \\
&= \beta^{-1}(\beta(x)(u^{-1}v_2) - \beta(x)(u^{-1}(v_1x))) \\
&= \beta^{-1}((xu^{-1})\beta(v_2) - (xu^{-1})\beta(v_1x)) \\
&= \beta^{-1}((u^{-1}x)\beta(v_2) - (u^{-1}x)\beta(v_1x)) \\
&= \beta^{-1}(\beta(u^{-1})(xv_2) - \beta(u^{-1}x)(\beta(v_1)\beta(x))) \\
&= \beta^{-1}(\beta(u^{-1})(v_2x) - ((u^{-1}x)\beta(v_1))\beta^2(x)) \\
&= \beta^{-1}((u^{-1}v_2)\beta(x) - (\beta(u^{-1})(xv_1))\beta^2(x)) \\
&= \beta^{-1}((u^{-1}v_2)\beta(x) - u^{-1}((xv_1)\beta(x))) \\
&= \beta^{-1}(\beta(u^{-1})(v_2x) - u^{-1}((v_1x)\beta(x))) \\
&= \beta^{-1}(u^{-1}(v_2x) - u^{-1}(\beta(v_1)x^2)).
\end{aligned}$$

Hence, $\beta(v_1)x^2 - 2v_2x + v_3 = 0$, that is, $x^2 + \theta^1x + \theta^0 = 0$, where $\theta^1 = -2v_2/\beta(v_1)$, $\theta^0 = v_3/\beta(v_1)$, and $\theta^1, \theta^0 \in Z(L)$. It is easy to see that $\theta^0 = v_3/\beta(v_1) = (-\beta(v_1)x^2 + 2v_2x)/\beta(v_1) = -x^2 - \theta^1x$. By Lemma 4.2 (2) and Lemma 4.3 (1) we have

$$\begin{aligned}
0 &= [-\theta^0, \beta(z)] = [x^2, \beta(z)] + [\theta^1x, \beta(z)] \\
&= \beta([x^2, z]) + \beta(\theta^1)[x, z] + [\theta^1, x_{(-1)} \cdot \beta^{-1}(z)]\beta^2(x_0) \\
&= \beta([x^2, z]) + \beta(\theta^1)[x, z].
\end{aligned}$$

By Lemma 4.3(1), one has $\beta([x^2, z]) = -\beta(\theta^1)[x, z] = \beta(\theta^1)u$. Similarly,

$$\beta([x^2, z]) = \beta(x[x, z] + [x, z]x) = 2\beta([x, z]x) = -2\beta(ux) = -2ux.$$

Since $u \in Z_H(L)_0$, $\beta(\theta^1) = -2x$, it follows that $\theta^1 = -2\beta^{-1}(x) = -2x$. As $\mathrm{char}(k) \neq 2$, we have $x = -(1/2)\theta^1 \in Z(L)$, as desired. □

5. Universal enveloping algebras of braided Hom-Lie algebras

In this section, we will first present the structure of the universal enveloping algebra $U(L)$ of a braided Hom-Lie algebra L, then we show that $U(L)$ is a cocommutative Hom-Hopf algebra.

Definition 5.1. Let $(L, [\cdot,\cdot], \beta)$ be a braided Hom-Lie algebra. A universal enveloping algebra of L is a monoidal Hom-algebra

$$U(L) = (U(L), m_U, \beta_U)$$

together with a morphism $\psi : L \to U(L)^-$ of Hom-Lie algebras in ${}_H^H\mathcal{HYD}$ such that the following universal property holds: for any monoidal Hom-algebra $A = (A, m_A, \beta_A)$ and any Hom-Lie algebra morphism $f : L \to A^-$ in ${}_H^H\mathcal{HYD}$, there exists a unique morphism $g : U(L) \to A$ of monoidal Hom-algebra in ${}_H^H\mathcal{HYD}$ such that $g \circ \psi = f$.

Definition 5.2. Let (M, β_M) be an involutive (i.e., $\beta_M^2 = id$) Hom-Yetter-Drinfeld module. A free involutive monoidal Hom-algebra on M is an involutive monoidal Hom-algebra $(F_M, *, \beta_M)$ together with a morphism $j : M \to F_M$ in ${}_H^H\mathcal{HYD}$, satisfying the following property: for any involutive monoidal Hom-algebra (A, β_A) together with a morphism $f : M \to A$ in ${}_H^H\mathcal{HYD}$, there is a unique morphism $\bar{f} : M \to F_M$ in ${}_H^H\mathcal{HYD}$ such that $\bar{f} \circ j = f$.

The well-known construction of the (non-unitary) free associative algebra on a module is the tensor algebra equipped with the concatenation tensor product. Recently, Guo, Zhang and Zheng generalized this method to Hom-associative algebras in [13], Armakan, Silvestrov and Farhangdoost generalized the work to color Hom-associative algebras in [2]. Next we hope to extend the above work to monoidal Hom-algebras in ${}^H_H\mathcal{HYD}$.

Let (M,β) be an involutive Hom-Yetter-Drinfeld module and $T(M)=\bigoplus_{i\geq 0}M^{\otimes i}$, where $M^{\otimes 0}=k$. Obviously, $T(M)$ is an object in ${}^H_H\mathcal{HYD}$. Define the linear map β_T and the binary operation $\odot$ on $T(M)$ as follows:

$$\beta_T(x)=\beta_T(x_1\otimes x_2\otimes\cdots\otimes x_i)=\beta(x_1)\otimes\beta(x_2)\otimes\cdots\otimes\beta(x_i),$$
$$x\odot y=(x_1\otimes x_2\otimes\cdots\otimes x_i)\odot(y_1\otimes y_2\otimes\cdots\otimes y_j)=\beta_T^{j-1}(x)\otimes y_1\otimes\beta_T(y_2\otimes\cdots\otimes y_j).$$

One may check directly that β_T and $\odot$ are morphisms in ${}^H_H\mathcal{HYD}$. Similar to the proof in [13], $(T(M),\odot,\beta_T)$ is an involutive monoidal Hom-algebra in ${}^H_H\mathcal{HYD}$.

Theorem 5.3. Let (H,α) be an involutive monoidal Hom-Hopf algebra and $(L,[\cdot,\cdot],\beta)$ an involutive braided Hom-Lie algebra. Let $U(L)=T(L)/I$, where I is the H-Hom-ideal of $T(L)$ generated by

$$\{x\otimes y-(x_{-1}\cdot\beta(y))\otimes\beta(x_0)-[x,y]\|\ x,y\in L\}.$$

Let ψ be the composition of the natural inclusion $i:L\to T(L)$ with the canonical map $\pi:T(L)\to T(L)/I$. Then $(U(L),\psi,\beta_T)$ is an universal enveloping algebra of L.

Proof. We first show that I is an object in ${}^H_H\mathcal{HYD}$. For any $x,y\in L$ and $h\in H$, it is clear that $\rho(h_1\cdot x)=(h_{111}\alpha^{-1}(x_{(-1)}))S(h_{12})\otimes\alpha(h_{112})\cdot x_0=(\alpha^{-1}(h_{11})\alpha^{-1}(x_{(-1)}))S\alpha(h_{122})\otimes\alpha(h_{121})\cdot x_0$. Then we have

$$\begin{aligned}
&h\cdot(x\otimes y-(x_{-1}\cdot\beta(y))\otimes\beta(x_0)-[x,y])\\
=\ &h_1\cdot x\otimes h_2\cdot y-h_1\cdot(x_{-1}\cdot\beta(y))\otimes h_2\cdot\beta(x_0)-[h_1\cdot x,h_2\cdot y]\\
=\ &h_1\cdot x\otimes h_2\cdot y-(\alpha^{-1}(h_1)x_{-1})\cdot y\otimes h_2\cdot\beta(x_0)-[h_1\cdot x,h_2\cdot y]\\
=\ &h_1\cdot x\otimes h_2\cdot y-(h_1\cdot x)_{-1}\cdot\beta(h_2\cdot y)\otimes\beta((h_1\cdot x)_0)-[h_1\cdot x,h_2\cdot y]\in I.
\end{aligned}$$

The last equality holds since

$$\begin{aligned}
&(h_1\cdot x)_{-1}\cdot\beta(h_2\cdot y)\otimes\beta((h_1\cdot x)_0)\\
=\ &((\alpha^{-1}(h_{11})\alpha^{-1}(x_{(-1)}))S\alpha(h_{122}))\cdot(\alpha(h_2)\cdot\beta(y))\otimes\alpha^2(h_{121})\cdot\beta(x_0)\\
=\ &(((\alpha^{-2}(h_{11})\alpha^{-2}(x_{(-1)}))S(h_{122}))\alpha(h_2))\cdot y\otimes\alpha^2(h_{121})\cdot\beta(x_0)\\
=\ &((\alpha^{-1}(h_{11})\alpha^{-1}(x_{(-1)}))(S(h_{122})h_2))\cdot y\otimes\alpha^2(h_{121})\cdot\beta(x_0)\\
=\ &((\alpha^{-2}(h_1)\alpha^{-1}(x_{(-1)}))(S(h_{212})\alpha(h_{22})))\cdot y\otimes\alpha^2(h_{211})\cdot\beta(x_0)\\
=\ &((\alpha^{-2}(h_1)\alpha^{-1}(x_{(-1)}))(S(h_{221})\alpha^2(h_{222})))\cdot y\otimes\alpha(h_{21})\cdot\beta(x_0)\\
=\ &((\alpha^{-2}(h_1)\alpha^{-1}(x_{(-1)}))(\epsilon(h_{22})1_H))\cdot y\otimes\alpha(h_{21})\cdot\beta(x_0)\\
=\ &(\alpha^{-1}(h_1)x_{(-1)})\cdot y\otimes h_2\cdot\beta(x_0).
\end{aligned}$$

So I is H-stable. Now we prove that I is also H-costable, that is, $\rho(x\otimes y-(x_{(-1)}\cdot\beta(y))\otimes\beta(x_0)-[x,y])\in H\otimes I$, we note that $\rho(x_{(-1)}\cdot\beta(y))=(x_{(-1)11}y_{(-1)})S(x_{(-1)2})\otimes\alpha(x_{(-1)12})\cdot\beta(y_0)$ and compute

$$\begin{aligned}
&\rho(x_{-1}\cdot\beta(y)\otimes\beta(x_0))\\
=\ &(x_{-1}\cdot\beta(y))_{(-1)}\alpha(x_{0(-1)})\otimes(x_{-1}\cdot\beta(y))_0\otimes\beta(x_{00})\\
=\ &((x_{(-1)11}y_{(-1)})S(x_{(-1)2}))\alpha(x_{0(-1)})\otimes\alpha(x_{(-1)12})\cdot\beta(y_0)\otimes\beta(x_{00})\\
=\ &((\alpha(x_{(-1)111})y_{(-1)})S(x_{(-1)12}))\alpha(x_{(-1)2})\otimes\alpha^2(x_{(-1)112})\cdot\beta(y_0)\otimes x_0\\
=\ &((x_{(-1)11}y_{(-1)})S(x_{(-1)21}))\alpha^2(x_{(-1)22})\otimes\alpha(x_{(-1)12})\cdot\beta(y_0)\otimes\beta^2(x_0)\\
=\ &(\alpha(x_{(-1)11})\alpha(y_{(-1)}))(S(x_{(-1)21})\alpha(x_{(-1)22}))\otimes\alpha(x_{(-1)12})\cdot\beta(y_0)\otimes x_0\\
=\ &(\alpha(x_{(-1)11})\alpha(y_{(-1)}))(\epsilon(x_{(-1)2})1_H)\otimes\alpha(x_{(-1)12})\cdot\beta(y_0)\otimes x_0\\
=\ &(\alpha^2(x_{(-1)1})\alpha(y_{(-1)}))1_H\otimes\alpha^2(x_{(-1)2})\cdot\beta(y_0)\otimes x_0\\
=\ &\alpha(x_{(-1)1})y_{(-1)}\otimes x_{(-1)2}\cdot\beta(y_0)\otimes x_0\\
=\ &x_{(-1)}y_{(-1)}\otimes x_{0(-1)}\cdot\beta(y_0)\otimes\beta(x_{00}).
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
& \rho(x\otimes y-(x_{(-1)}\cdot\beta(y))\otimes\beta(x_0)-[x,y]) \\
= \ & x_{(-1)}y_{(-1)}\otimes x_0\otimes y_0-x_{(-1)}y_{(-1)}\otimes x_{0(-1)}\cdot\beta(y_0)\otimes\beta(x_{00})-x_{(-1)}y_{(-1)}\otimes[x_0,y_0] \\
= \ & x_{(-1)}y_{(-1)}\otimes(x_0\otimes y_0-x_{0(-1)}\cdot\beta(y_0)\otimes\beta(x_{00})-[x_0,y_0])\in H\otimes I,
\end{aligned}
$$

as desired, where $\rho[x,y]=x_{(-1)}y_{(-1)}\otimes[x_0,y_0]$ since $[\cdot,\cdot]$ is a morphism in ${}^H_H\mathcal{HYD}$.

Next, we show that ψ is a morphism of braided Hom-Lie algebras. It is easy to see that ψ is a morphism in ${}^H_H\mathcal{HYD}$. Now we prove that ψ is compatible with the bracket product, we denote the multiplication in $U(L)$ by $*$ and calculate

$$
\begin{aligned}
\psi([x,y]) &= \pi([x,y])=\pi(x\otimes y-(x_{(-1)}\cdot\beta(y))\otimes\beta(x_0)) \\
&= \pi(x\odot y-(x_{(-1)}\cdot\beta(y))\odot\beta(x_0)) \\
&= \pi(x)*\pi(y)-\pi(x_{(-1)}\cdot\beta(y))*\pi(\beta(x_0)) \\
&= \psi(x)*\psi(y)-\psi(x_{(-1)}\cdot\beta(y))*\psi(\beta(x_0)) \\
&= \psi(x)*\psi(y)-(x_{(-1)}\cdot\psi(\beta(y)))*\psi(\beta(x_0)) \\
&= \psi(x)*\psi(y)-((\psi(x))_{(-1)}\cdot\beta(\psi(y)))*\beta((\psi(x))_0) \\
&= [\psi(x),\psi(y)].
\end{aligned}
$$

Finally, we show that the following statement holds: for any involutive monoidal Hom-algebra of (A,m_A,β_A) and any homomorphism $f:L\longrightarrow A^-$ of Hom-Lie algebras in ${}^H_H\mathcal{HYD}$, there exists a unique morphism $g:U(L)\longrightarrow A$ in ${}^H_H\mathcal{HYD}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
L & \stackrel{\psi}{\longrightarrow} & U(L) \\
f\downarrow & \swarrow g & \\
A & &
\end{array}
$$

To prove this statement, we first consider a unique homomorphism f^* of $T(L)$ which maps $T(L)$ into A by extending the homomorphism f of L into A. For any $x,y\in L$, we have

$$
\begin{aligned}
& f^*(x\otimes y-(x_{(-1)}\cdot\beta(y))\otimes\beta(x_0)) \\
= \ & f^*(x\odot y-(x_{(-1)}\cdot\beta(y))\odot\beta(x_0)) \\
= \ & f^*(x)f^*(y)-f^*(x_{(-1)}\cdot\beta(y))f^*(\beta(x_0)) \\
= \ & f(x)f(y)-f(x_{(-1)}\cdot\beta(y))f(\beta(x_0)) \\
= \ & f(x)f(y)-x_{(-1)}\cdot\beta(f(y))\beta(f(x_0)) \\
= \ & [f(x),f(y)]=f([x,y])=f^*([x,y]).
\end{aligned}
$$

This shows that $I\subset kerf^*$, and we have a unique homomorphism g of $U(L)=T(L)/I$ into A such that $g(x+I)=f(x)$ or $g\psi(x)=f(x)$. Hence $f=g\psi$, since L generates $T(L)$.

Furthermore, it is easy to see that $\alpha_A\circ g=g\circ\beta_T$. We still need to check that g is a morphism in ${}^H_H\mathcal{HYD}$. Since $\rho_Af=(1\otimes f)\rho_L$ by our assumption, where ρ_A and ρ_L are the (H,α)-Hom-comodule structure of A and L respectively, for any $\overline{x},\overline{y}\in U(L)$, we have

$$
\begin{aligned}
\rho_Ag(\overline{x}*\overline{y}) &= \rho_A(g(\overline{x})g(\overline{y}))=\rho_A(f(x)f(y)) \\
&= (f(x))_{(-1)}(f(y))_{(-1)}\otimes(f(x))_0(f(x))_0 \\
&= x_{(-1)}y_{(-1)}\otimes f(x_0)f(y_0)=x_{(-1)}y_{(-1)}\otimes g(\overline{x_0})f(\overline{y_0}) \\
&= (1\otimes g)(x_{(-1)}y_{(-1)}\otimes(\overline{x_0}*\overline{y_0}))=(1\otimes g)\rho_U(\overline{x}*\overline{y}),
\end{aligned}
$$

It follows that g is indeed (H,α)-linear. Similarly, one may check that g is also (H,α)-colinear. And the proof is completed. □

Now we will define a Hom-Hopf algebra structure on the universal enveloping algebra $U(L)$, we first present a useful Lemma.

Lemma 5.4. Let (H,α) be an involutive monoidal Hom-Hopf algebra and $(L,[\cdot,\cdot],\beta)$ an involutive braided Hom-Lie algebra. Assume $U(L)$ is the universal enveloping algebra of L. Then there exists a homomorphism $g: U(L\oplus L)\longrightarrow U(L)\otimes U(L)$ of monoidal Hom-algebras in ${}^H_H\mathcal{HYD}$.

Proof. Define $f: L\oplus L\longrightarrow U(L)\otimes U(L)$ by

$$(x,y)\mapsto \beta_T(\overline{x})\otimes 1+1\otimes\beta_T(\overline{y}).$$

We first show that f is a morphism in ${}^H_H\mathcal{HYD}$. In fact, for any $h\in H$ and $x,y\in L$, we have

$$\begin{aligned}
h\cdot f(x,y)) &= h_1\cdot\beta_T(\overline{x})\otimes h_2\cdot 1+h_1\cdot 1\otimes h_2\cdot\beta_T(\overline{y})\\
&= h_1\cdot\beta_T(\overline{x})\otimes\epsilon(h_2)1+\epsilon(h_1)1\otimes h_2\cdot\beta_T(\overline{y})\\
&= \alpha(h)\cdot\beta_T(\overline{x})\otimes 1+1\otimes\alpha(h)\cdot\beta_T(\overline{y})\\
&= \beta_T(h\cdot\overline{x})\otimes 1+1\otimes\beta_T(h\cdot\overline{y})\\
&= \beta_T(\overline{h\cdot x})\otimes 1+1\otimes\beta_T(\overline{h\cdot y})\\
&= f(h\cdot x,h\cdot y)=f(h\cdot(x,y)).
\end{aligned}$$

It follows that f is H-linear. Similarly, one may check that f is H-colinear.

Second, we prove that f is a Hom-Lie homomorphism. For any $x,y,x',y'\in L$, we have

$$\begin{aligned}
[f(x,y),f(x',y')]] &= [\beta_T(\overline{x})\otimes 1+1\otimes\beta_T(\overline{y}),\beta_T(\overline{x'})\otimes 1+1\otimes\beta_T(\overline{y'})]\\
&= [\beta_T(\overline{x})\otimes 1,\beta_T(\overline{x'})\otimes 1]+[\beta_T(\overline{x})\otimes 1,1\otimes\beta_T(\overline{y'})]+\\
&\quad [1\otimes\beta_T(\overline{y}),\beta_T(\overline{x'})\otimes 1]+[1\otimes\beta_T(\overline{y}),1\otimes\beta_T(\overline{y'})].
\end{aligned}$$

Recall that multiplication in $U(L)\otimes U(L)$ is

$$(\overline{x}\otimes\overline{y})(\overline{x'}\otimes\overline{y'})=\overline{x}(y_{(-1)}\cdot\beta_T^{-1}(\overline{x'}))\otimes(\beta_T(y_0)y').$$

Obviously, we have $(\overline{x}\otimes 1)(1\otimes\overline{y})=\beta_T(\overline{x})\otimes\beta_T(\overline{y})$ and $(1\otimes\overline{x})(\overline{y}\otimes 1)=\alpha(x_{(-1)})\cdot\overline{y}\otimes x_0$. Therefore,

$$\begin{aligned}
[\beta_T(\overline{x})\otimes 1,1\otimes\beta_T(\overline{y'})] &= (\beta_T(\overline{x})\otimes 1)(1\otimes\beta_T(\overline{y'}))-((\alpha(x_{(-1)})1)\cdot(1\otimes\overline{y'}))(\overline{x}_0\otimes 1)\\
&= \overline{x}\otimes\overline{y'}-(x_{(-1)}\cdot(1\otimes\overline{y'}))(\overline{x}_0\otimes 1)\\
&= \overline{x}\otimes\overline{y'}-(1\otimes\alpha(x_{(-1)})\cdot\overline{y'}))(\overline{x}_0\otimes 1)\\
&= \overline{x}\otimes\overline{y'}-((\alpha^2(x_{(-1)11})y_{(-1)})S\alpha(x_{(-1)2}))\cdot\overline{x_0}\otimes x_{(-1)12}\cdot\overline{y_0}\\
&= \overline{x}\otimes\overline{y'}-\overline{x}\otimes\overline{y'}=0,
\end{aligned}$$

where $((\alpha^2(x_{(-1)11})y_{(-1)})S\alpha(x_{(-1)2}))\cdot\overline{x_0}\otimes x_{(-1)12}\cdot\overline{y_0}=\overline{x}\otimes\overline{y'}$ since the braiding is symmetric on L. Similarly, we have $[1\otimes\beta_T(\overline{y}),1\otimes\beta_T(\overline{y'})]=0$. Also,

$$\begin{aligned}
[\beta_T(\overline{x})\otimes 1,\beta_T(\overline{x'})\otimes 1] &= (\beta_T(\overline{x})(1\cdot\overline{x'}))\otimes\beta_T(1)1-((\alpha(x_{(-1)})1)\cdot(\overline{x'}\otimes 1))(\overline{x_0}\otimes 1)\\
&= \beta_T(\overline{x})\beta_T(\overline{x'})\otimes 1-(\alpha(x_{(-1)})\cdot\overline{x'}\otimes 1)(\overline{x_0}\otimes 1)\\
&= \beta_T(\overline{x})\beta_T(\overline{x'})\otimes 1-(\alpha(x_{(-1)})\cdot\overline{x'})\overline{x_0}\otimes 1\\
&= \beta_T(\overline{x})\beta_T(\overline{x'})\otimes 1-((\beta_T(\overline{x}))_{(-1)}\cdot\beta_T^{-1}(\beta_T(\overline{x'})))\beta_T((\beta_T(\overline{x}))_0)\otimes 1\\
&= [\beta_T(\overline{x}),\beta_T(\overline{x'})]\otimes 1.
\end{aligned}$$

Similarly, we have $[1\otimes\beta_T(\overline{y}),1\otimes\beta_T(\overline{y'})]=1\otimes[\beta_T(\overline{y}),\beta_T(\overline{y'})]$. Then we have

$$\begin{aligned}
[f(x,y),f(x',y')]] &= [\beta_T(\overline{x}),\beta_T(\overline{x'})]\otimes 1+1\otimes[\beta_T(\overline{y}),\beta_T(\overline{y'})]\\
&= \beta_T([\overline{x},\overline{x'}])\otimes 1+1\otimes\beta_T([\overline{y},\overline{y'}])\\
&= f([(x,y),(x',y')]).
\end{aligned}$$

So f is a Hom-Lie homomorphism. Now by the universal property of $U(L\oplus L)$, there exists a homomorphism $g : U(L\oplus L) \longrightarrow U(L)\otimes U(L)$ of monoidal Hom-algebras in ${}^H_H\mathcal{HYD}$.

Theorem 5.5. Let (H,α) be an involutive monoidal Hom-Hopf algebra and $(L,[\cdot,\cdot],\beta)$ an involutive braided Hom-Lie algebra. Then $U(L)$ in Theorem 5.3 is a monoidal Hom-Hopf algebra in ${}^H_H\mathcal{HYD}$ with

$$\Delta(\bar{l}) = \beta_T(\bar{l})\otimes 1 + 1\otimes\beta_T(\bar{l});$$
$$\Delta(1) = 1\otimes 1,\ \ \epsilon(\bar{l}) = 0,\ \ \epsilon(1) = 1;$$
$$S(\bar{l}) = -\bar{l},\ \ S(\bar{x}\bar{y}) = (x_{(-1)}\cdot S(\beta_T^{-1}(\overline{y})))S(\beta_T(\overline{x_0})).$$

for all $l\in L$ and $\overline{x},\overline{y}\in U(L)$.

Proof. We first consider the diagonal mapping $d : L \longrightarrow L\oplus L$ defined by $l\mapsto (l,l)$. It is easy to check that d is a Hom-Lie homomorphism in ${}^H_H\mathcal{HYD}$. Let f be the map described in Lemma 5.4. Then $f\circ d$ is a Hom-Lie homomorphism from L to $U(L)\otimes U(L)$, therefore there exists a homomorphism $\Delta : U(L)\to U(L)\otimes U(L)$, which is a homomorphism of monoidal Hom-algebras in ${}^H_H\mathcal{HYD}$ satisfying the following condition

$$\Delta(\bar{l}) = ((f\circ d)(l)) = \beta_T(\bar{l})\otimes 1 + 1\otimes\beta_T(\bar{l}),$$

for all $\bar{l}\in\bar{L}$. It is now straightforward to check that $(\beta_T^{-1}\otimes\Delta)\Delta = (\Delta\otimes\beta_T^{-1})\Delta$ and $(\eta\otimes\beta_T)\Delta = (\beta_T\otimes\epsilon)\Delta = \beta_T^{-1}$.

It is easy to see that S is a well-defined morphism in ${}^H_H\mathcal{HYD}$, since if we define $\widetilde{S}$ on the free generators of $T(L)$ by $\widetilde{S}(\bar{l}) = -\bar{l}, \widetilde{S}(1) = 1$, and set $\widetilde{S}(\bar{x}\bar{y}) = (x_{(-1)}\cdot\widetilde{S}(\beta_T^{-1}(\overline{y})))\widetilde{S}(\beta_T(\overline{x_0}))$, then $\widetilde{S}$ is a morphism in ${}^H_H\mathcal{HYD}$ which vanishes on I. Thus S is well defined.

To show that S is an antipode, we first note that

$$\begin{aligned}(m(id\otimes S)\circ\Delta)(\bar{l}) &= m(id\otimes S)(\beta_T(\bar{l})\otimes 1 + 1\otimes\beta_T(\bar{l}))\\ &= m(\beta_T(\bar{l})\otimes 1 - 1\otimes\beta_T(\bar{l})) = 0 = \epsilon(\bar{l}),\\ (m(S\otimes id)\circ\Delta)(\bar{l}) &= m(S\otimes id)(\beta_T(\bar{l})\otimes 1 + 1\otimes\beta_T(\bar{l}))\\ &= m(-\beta_T(\bar{l})\otimes 1 + 1\otimes\beta_T(\bar{l})) = 0 = \epsilon(\bar{l}),\end{aligned}$$

for any generator $l\in L$. Similarly, one may check that $(m(id\otimes S)\circ\Delta)(1) = (m(S\otimes id)\circ\Delta)(1) = \epsilon(1)$. Therefore, we can derive that

$$\begin{aligned}(m(id\otimes S)\circ\Delta)(\overline{x}\,\overline{y}) &= m(id\otimes S)(\overline{x_1}(x_{2(-1)}\cdot\beta_T^{-1}(\overline{y_1}))\otimes\beta_T(\overline{x_{20}})\overline{y_2})\\ &= m(\overline{x_1}(x_{2(-1)}\cdot\beta_T^{-1}(\overline{y_1}))\otimes S(\beta_T(\overline{x_{20}})\overline{y_2}))\\ &= \{\overline{x_1}(x_{2(-1)}\cdot\beta_T^{-1}(\overline{y_1}))\}\{(\alpha(x_{20(-1)})\cdot S\beta_T(\overline{y_2}))S(\overline{x_{200}})\}\\ &= \{(\overline{x_1}(\alpha(x_{2(-1)1})\cdot\beta_T(\overline{y_1}))\}\{(\alpha(x_{2(-1)2})\cdot S\beta_T(\overline{y_2}))S\beta_T(\overline{x_{20}})\}\\ &= \{\overline{x_1}\beta_T(x_{2(-1)1}\cdot\overline{y_1})\}\beta_T((x_{2(-1)2}\cdot S(\overline{y_2}))S(\overline{x_{20}}))\\ &= \beta_T(\overline{x_1})(\beta_T(x_{2(-1)1}\cdot\overline{y_1})\{x_{2(-1)2}\cdot S(\overline{y_2}))S(\overline{x_{20}})\})\\ &= \beta_T(\overline{x_1})(\{(x_{2(-1)1}\cdot\overline{y_1})(x_{2(-1)2}\cdot S(\overline{y_2}))\}S\beta_T(\overline{x_{20}}))\\ &= \beta_T(\overline{x_1})\{(x_{2(-1)}\cdot\epsilon(\overline{y})1)S\beta_T(\overline{x_{20}})\}\\ &= \epsilon(\overline{y})\beta_T(\overline{x_1})S\beta_T(\overline{x_2}) = \epsilon(\overline{y})\epsilon(\overline{x}).\end{aligned}$$

Similarly, we can show that $(m(S\otimes id)\circ\Delta)(\overline{x}\,\overline{y}) = \epsilon(\overline{y})\epsilon(\overline{x})$. So S is an antipode on $U(L)$, and this finishes the proof. □

Corollary 5.6. Under the hypotheses of the Theorem 5.5, the universal enveloping algebra $U(L)$ is H-cocommutative.

Proof. For any $\overline{x}\in U(L)$, we have $C_{U,U}\Delta(\overline{x}) = C_{U,U}(\beta_T(\overline{x})\otimes 1 + 1\otimes\beta_T(\overline{x})) = \alpha(x_{(-1)})\cdot\beta_T^{-1}(1)\otimes\beta_T^2(\overline{x}_0) + 1\cdot\beta_T^{-1}\beta_T(\overline{x})\otimes\beta_T(1) = 1\otimes\beta_T(\overline{x})) + \beta_T(\overline{x})\otimes 1 = \Delta(\overline{x})$. It follows that $C_{U,U}\Delta = \Delta$, as desired. □

As an application of Theorem 5.5, we will define a Hom-Yetter-Drinfeld module structure on the $End(V)$ and construct a Radford's Hom-biproduct. In order to define a good (H, α)-Hom-module operation on $End(V)$, it is necessary to assume that $\alpha = id_H$.

Lemma 5.7. Let H be a Hopf algebra with a bijective antipode and (V, ν) a finite-dimensional Hom-Yetter-Drinfeld module in ${}^H_H\mathcal{HYD}$. Then $(End(V), \delta)$ is a Hom-Yetter-Drinfeld module under the following structures

$$(h \cdot f)(v) = h_1 \cdot f(S(h_2) \cdot v), \ \delta(f)(v) = f(\nu^2(v)),$$
$$\rho(f)(v) = (f(v_0))_{(-1)} S^{-1}(v_{(-1)}) \otimes (f(v_0))_0,$$

for any $v \in V$.

Proof. We first show that $(End(V), \delta)$ is a Hom-module. In fact, for any $h, g \in H$, $f \in End(V)$ and $v \in V$, we have

$$\begin{aligned}
(h \cdot (g \cdot f))(v) &= h_1 \cdot (g \cdot f)(S(h_2) \cdot v) = h_1 \cdot (g_1 \cdot f(S(g_2) \cdot (S(h_2) \cdot v))) \\
&= h_1 \cdot (g_1 \cdot f(S(g_2)S(h_2) \cdot \nu(v))) = (h_1 g_1) \cdot f(S(g_2)S(h_2) \cdot \nu^2(v)), \\
((hg) \cdot \delta(f))(v) &= (hg)_1 \cdot \delta(f)(S((hg)_2)) \cdot v) = (h_1 g_1) \cdot f(S(h_2 g_2) \cdot \nu^2(v)).
\end{aligned}$$

It follows that $h \cdot (g \cdot f) = (hg) \cdot \delta(f)$. Now we verify $1_H \cdot f = \delta(f)$ and $\delta(h \cdot f) = h \cdot \delta(f)$ as follows

$$\begin{aligned}
(1_H \cdot f)(v) &= 1 \cdot f(1 \cdot v) = 1 \cdot f(\nu(v)) = f(\nu^2(v)) \\
\delta(h \cdot f)(v) &= (h \cdot f)(\nu^2(v)) = h_1 \cdot f(S(h_2) \cdot \nu^2(v)) \\
&= h_1 \cdot \delta(f)(S(h_2) \cdot v) = (h \cdot \delta(f))(v).
\end{aligned}$$

So $(End(V), \delta)$ is a Hom-module, as desired. Similarly, one may check that $(End(V), \delta)$ is a Hom-comodule.

Now we show that for any $f \in End(V)$ and $h \in H$, the following compatibility condition

$$h_1 f_{(-1)} \otimes h_2 \cdot f_0 = (h_1 \cdot \delta^{-1}(f))_{(-1)} h_2 \otimes \delta((h_1 \cdot \delta^{-1}(f))_0),$$

holds. For this, we take $h \in H, f \in End(V), v \in V$. On the one hand, we have

$$\begin{aligned}
& (h_1 \cdot \delta^{-1}(f))_{(-1)} h_2 \otimes \delta((h_1 \cdot \delta^{-1}(f))_0)(v) \\
= \ & (h_1 \cdot \delta^{-1}(f))_{(-1)} h_2 \otimes (h_1 \cdot \delta^{-1}(f))_0(\nu^2(v)) \\
= \ & ((h_1 \cdot \delta^{-1}(f))(\nu^2(v_{00})))_{(-1)} S^{-1}(v_{(-1)}) h_2 \otimes ((h_1 \cdot \delta^{-1}(f))(\nu^2(v_{00})))_0 \\
= \ & (h_1 \cdot f(S(h_3) \cdot v_0))_{(-1)} S^{-1}(v_{(-1)}) h_3 \otimes (h_1 \cdot f(S(h_3) \cdot v_0))_0 \\
= \ & h_1 (f(S(h_4) \cdot v_0))_{(-1)} S(h_3) S^{-1}(v_{(-1)}) h_5 \otimes h_3 \cdot (f(S(h_4) \cdot v_0))_0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& h_1 f_{(-1)} \otimes (h_2 \cdot f_0)(v) \\
= \ & h_1 f_{(-1)} \otimes h_2 \cdot (f_0((S(h_3)) \cdot v)) \\
= \ & h_1 (f(S(h_3)) \cdot v)_0)_{(-1)} S^{-1}(S(h_3) \cdot v)_{(-1)} \otimes h_2 \cdot (f(((Sh_3)) \cdot v)_0)_0 \\
= \ & h_1 (f(S(h_4) \cdot v_0))_{(-1)} S^{-1}(S(h_5) v_{(-1)} S^2 h_3) \otimes h_2 \cdot (f(S(h_4)) \cdot v_0))_0 \\
= \ & h_1 (f(S(h_4) \cdot v_0))_{(-1)} S(h_3) S^{-1}(v_{(-1)}) h_5 \otimes h_2 \cdot (f((Sh_4)) \cdot v_0)_0.
\end{aligned}$$

So $(End(V), \delta) \in^H_H \mathcal{HYD}$. The proof is finished. □

Lemma 5.8. Let H be a Hopf algebra with a bijective antipode and (V, ν) a finite-dimensional involutive Hom-Yetter-Drinfeld module in ${}^H_H\mathcal{HYD}$. Then $(End(V), \delta)$ is an algebra in ${}^H_H\mathcal{HYD}$.

Proof. We first show that $End(V)$ is a H-module algebra. Indeed, for any $h \in H, f, g \in End(V)$ and $v \in V$, we have

$$\begin{aligned}((h_1 \cdot f)(h_2 \cdot g))(v) &= (h_1 \cdot f)(h_2 \cdot g(S(h_3) \cdot v))\\ &= h_1 \cdot f(S(h_2) \cdot (h_3 \cdot g(S(h_4) \cdot v)))\\ &= h_1 \cdot f((S(h_2)h_3) \cdot g(S(h_4) \cdot \nu(v)))\\ &= h_1 \cdot f((\epsilon(h_2)1_H) \cdot g(S(h_3) \cdot \nu(v)))\\ &= h_1 \cdot f(g(S(h_2) \cdot \nu^2(v)))\\ &= (h_1 \cdot (fg))(S(h_2) \cdot v).\end{aligned}$$

It follows that $h \cdot (fg) = (h_1 \cdot f)(h_2 \cdot g)$. Also, we have

$$\begin{aligned}(h \cdot id)(v) &= h_1 \cdot id(S(h_2) \cdot v) = h_1 \cdot (S(h_2) \cdot v)\\ &= (h_1 S(h_2)) \cdot \nu(v) = \epsilon(h)1_H \cdot \nu(v) = \epsilon(h)v.\end{aligned}$$

So $h \cdot id = \epsilon(h)id$. Therefore, $End(V)$ is a H-module algebra.

Next, we will show that $End(V)$ is a H-comodule algebra. In fact, for any $f, g \in End(V)$ and $v \in V$, we have

$$\begin{aligned}(fg)_{(-1)} \otimes (fg)_0(v) &= ((fg)(v_0))_{(-1)}S^{-1}(v_{(-1)}) \otimes ((fg)(v_0))_0\\ &= (fg(v_0))_{(-1)}S^{-1}(v_{(-1)}) \otimes (fg(v_0))_0,\\ f_{(-1)}g_{(-1)} \otimes f_0 g_0(v) &= f_{(-1)}(g(v_0))_{(-1)}S^{-1}(v_{(-1)}) \otimes f_0((g(v_0))_0)\\ &= (f((g(v_0))_{00}))_{(-1)}S^{-1}((g(v_0))_{0(-1)})(g(v_0))_{(-1)}S^{-1}(v_{(-1)}) \otimes (f((g(v_0))_{00}))_0\\ &= (f(\nu^{-1}(g(v_0))_0))_{(-1)}S^{-1}((g(v_0))_{(-1)2})(g(v_0))_{(-1)1}S^{-1}(v_{(-1)}) \otimes (f(\nu^{-1}(g(v_0))_0))_0\\ &= (f(\nu^{-1}(g(v_0))_0))_{(-1)}\epsilon(g(v_0)_{(-1)})S^{-1}(v_{(-1)}) \otimes (f(\nu^{-1}(g(v_0))_0))_0\\ &= (f((g(v_0))_0))_{(-1)}S^{-1}(v_{(-1)}) \otimes (f((g(v_0))_0))_0\\ &= (fg(v_0))_{(-1)}S^{-1}(v_{(-1)}) \otimes (fg(v_0))_0.\end{aligned}$$

It follows that $(fg)_{(-1)} \otimes (fg)_0 = f_{(-1)}g_{(-1)} \otimes f_0 g_0$. Also, we have

$$\begin{aligned}\rho(id)(v) &= v_{0(-1)}S^{-1}(v_{(-1)}) \otimes v_{00} = v_{(-1)2}S^{-1}(v_{(-1)1}) \otimes \nu^{-1}(v_0)\\ &= \epsilon(v_{(-1)})1_H \otimes \nu^{-1}(v_0) = 1_H \otimes v = 1_H \otimes id(v).\end{aligned}$$

So $\rho(id) = 1_H \otimes id$, as desired. And this completes the proof. □

Lemma 5.9. Let H be a Hopf algebra with a bijective antipode and (V, ν) a finite-dimensional involutive Hom-Yetter-Drinfeld module in ${}^H_H\mathcal{HYD}$. Assume that the braiding C is symmetric on V. Then $(End(V), \delta)$ is a braided Hom-Lie algebra, where the bracket product is defined by

$$[f, g] = fg - (f_{(-1)} \cdot \delta^{-1}(g))\delta(f_0),$$

for any $f, g \in End(V)$.

Proof. Since the braiding C is symmetric on V, one may check that C is symmetric on $End(V)$, too. By Proposition 3.2, $(End(V), \delta)$ is a braided Hom-Lie algebra. □

Proposition 5.10. Let H be a Hopf algebra with a bijective antipode and (V, ν) a finite-dimensional involutive Hom-Yetter-Drinfeld module. Assume that the braiding C is symmetric on V. Then the Radford's Hom-biproduct $(U(End(V))^{\times}_{\sharp}H, \delta \otimes id)$ is a monoidal Hom-Hopf algebra, where the multiplication is defined by

$$(f \times h)(f' \times h') = f(h_1 \cdot \delta^{-1}(f)) \times h_2 h',$$

the coproduct is defined by

$$\Delta(f \times h) = (f_1 \times f_{2(-1)}h_1) \otimes (\delta(f_{20}) \times h_2),$$

the antipode is defined by

$$S(f \times h) = (1 \times S(f_{(-1)}h))(S(f_0) \times 1),$$

for all $f \times h, f' \times h' \in U(End(V))^{\times}_{\sharp}H$.

Proof. By Lemma 5.9 and Theorem 5.5, $(U(End(V)), \delta)$ is a monoidal Hom-Hopf algebra in ${}^H_H\mathcal{HYD}$. By Proposition 4.6 in [18], $(U(End(V))^{\times}_{\sharp}H, \delta \otimes id)$ is a monoidal Hom-Hopf algebra.

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STATEMENT

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