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Sequentially Cohen-Macaulay Matroidal Ideals

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Abstract. Let $R = K[x_1, ..., x_n]$ be the polynomial ring in n variables over a field K and let I be a matroidal ideal of degree d in R. Our main focus is determining when matroidal ideals are sequentially Cohen-Macaulay. In particular, all sequentially Cohen-Macaulay matroidal ideals of degree 2 are classified. Furthermore, we give a classification of sequentially Cohen-Macaulay matroidal ideals of degree $d \geq 3$ in some special cases.

Introduction

Our goal is to classify the sequentially Cohen-Macaulay matroidal ideals. While for the Cohen-Macaulay property of matroidal ideals, a complete classification was given by Herzog and Hibi [10], the classification of the sequentially Cohen-Macaulay matroidal ideals seems to be much harder. In the present paper partial answers to this problem are given. Herzog and Hibi [9] were the first to give a systematic treatment of polymatroidal ideals and they studied some combinatoric and algebraic properties related to it. They defined the polymatroidal ideal, a monomial ideal having the exchange property. A square-free polymatroidal ideal is called a matroidal ideal. Herzog and Takayama [13] proved that all polymatroidal ideals have linear quotients which implies that they have linear resolutions. Herzog and Hibi [10] proved that a polymatroidal ideal I is Cohen-Macaulay (i.e. CM) if and only if I is a principal ideal, a Veronese ideal, or a square-free Veronese ideal.

Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminate over a field K and $I \subset R$ be a homogeneous ideal. For a positive integer i, let (I_i) be the ideal generated by all forms in I of degree i. We say that I is componentwise linear if for each positive integer i, (I_i) has a linear resolution. Componentwise linear ideals were first introduced by Herzog and Hibi [8] to generalize Eagon and Reiner's result that the Stanley-Reisner ideal I_Δ of simplicial complex Δ has a linear resolution if and only if the Alexander dual $\Delta^\vee$ is C.M [5]. In particular, Herzog and Hibi [8] and Herzog, Reiner, and Welker [12] showed that the Stanley-Reisner ideal I_Δ is componentwise linear if and only if $\Delta^\vee$ is sequentially Cohen-Macaulay(i.e. SCM).

It is of interest to understand the SCM matroidal ideals, and this paper may be considered as a first attempt to characterize such ideals for matroidal ideals in low degree or in a small number of variables. The remainder of this paper is organized as follows. Section 1 and 2 recall some definitions and results of componentwise linear ideals, simplicial complexes, and polymatroidal ideals. Section 3 classifies all SCM

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matroidal ideals of degree 2. Section 4 studies SCM matroidal ideals of degree $d \geq 3$ over polynomial rings of small dimensional.

For any unexplained notion or terminology, we refer the reader to [11] and [21]. Several explicit examples were performed with help of the computer algebra systems Macaulay2 [7].

1. Preliminaries

In this section, we recall some definitions and results used throughout the paper. As in the introduction, let K be a field and $R = K[x_1, ..., x_n]$ be the polynomial ring in n variables over K with each $\deg x_i = 1$. Let $I \subset R$ be a monomial ideal and $G(I)$ be its unique minimal set of monomial generators of I.

We say that a monomial ideal I with $G(I) = \{u_1, ..., u_r\}$ has *linear quotients* if there is an ordering $\deg(u_1) \leq \deg(u_2) \leq ... \leq \deg(u_r)$ such that for each $2 \leq i \leq r$ the colon ideal $(u_1, ..., u_{i-1}) : u_i$ is generated by a subset $\{x_1, ..., x_n\}$ It is known that if a monomial ideal I generated in single degree has linear quotients, then I has a linear resolution (see [3, Lemma 4.1]). In particular, a monomial ideal I generated in degree d has a linear resolution if and only if the Castelnuovo-Mumford regularity of I is $\mathrm{reg}(I) = d$ (see [20, Lemma 49]).

Lemma 1.1. *[4, Corollary 20.19] If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence of graded finitely generated R-modules, then*

(a) $\mathrm{reg}\, A \leq \max(\mathrm{reg}\, B, \mathrm{reg}\, C + 1)$.

(b) $\mathrm{reg}\, B \leq \max(\mathrm{reg}\, A, \mathrm{reg}\, C)$,

(c) $\mathrm{reg}\, C \leq \max(\mathrm{reg}\, A - 1, \mathrm{reg}\, B)$.

(d) If A has a finite length, set $s(A) = \max\{s : A_s \neq 0\}$, then $\mathrm{reg}(A) = s(A)$ and the equality holds in (b).

One of the important classes of monomial ideals with linear quotients is the class of polymatroid ideals.

Let $I \subset R$ be a monomial ideal generated in one degree. We say that I is *polymatroidal* if the following "exchange condition" is satisfied: For any two monomials $u = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ and $v = x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n}$ belong to $G(I)$ such that $\deg_{x_i}(v) < \deg_{x_i}(u)$, there exists an index j with $\deg_{x_j}(u) < \deg_{x_j}(v)$ such that $x_j(u/x_i) \in G(I)$. The polymatroidal ideal I is called *matroidal* if I is generated by square-free monomials. Note that if I is a matroidal ideal of degree d, then $\mathrm{depth}(R/I) = d - 1$ (see [2]).

Theorem 1.2. *[10, theorem 4.2] A polymatroidal ideal I is CM if and only if I is a principal ideal, a Veronese ideal, or a square-free Veronese ideal.*

2. review on componentwise linear ideals

For a homogeneous ideal I, we write (I_i) to denote the ideal generated by the degree i elements of I. Note that (I_i) is different from I_i, the vector space of all degree i elements of I. Herzog and Hibi introduced the following definition in [8].

Definition 2.1. *A monomial ideal I is componentwise linear if (I_i) has a linear resolution for all i.*

A number of familiar classes of ideals are componentwise linear. For example, all ideals with linear resolutions, all stable ideals, all square-free strongly stable ideals are componentwise linear (see [11]).

Proposition 2.2. *[6, Proposition 2.6] If Iis a homogeneous ideal with linear quotients, then I is componentwise linear.*

If I is generated by square-free monomials, then we denote by $I_{[i]}$ the ideal generated by the square-free monomials of degree i of I.

Theorem 2.3. *[8, Proposition 1.5] Let I be a monomial ideal generated by square-free monomials. Then I is componentwise linear if and only if $I_{[i]}$ has a linear resolution for all i.*

The notion of componentwise linearity is intimately related to the concept of sequential Cohen-Macaulayness.

Definition 2.4. *[18] A graded R-module M is called sequentially Cohen-Macaulay (SCM) if there exists a finite filtration of graded R-modules* $0 = M_0 \subset M_1 \subset ... \subset M_r = M$ *such that each* M_i/M_{i-1} *is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:*

$$\dim(M_1/M_0) < \dim(M_2/M_1) < ... < \dim(M_r/M_{r-1}).$$

The theorem connecting sequentially Cohen-Macaulayness to componentwise linearity is based on the idea of Alexander duality. We recall the definition of Alexander duality for square-free monomial ideals and then state the fundamental result of Herzog and Hibi [8] and Herzog, Reiner, and Welker [12].

Let Δ be a simplicial complex on the vertex set $V = \{x_1, x_2, ..., x_n\}$, i.e., Δ is a collection of subsets V such that (1) $\{x_i\} \in \Delta$ for each $i = 1, 2, ..., n$ and (2) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. Let $\Delta^\vee$ denote the dual simplicial complex of Δ, that is to say, $\Delta^\vee = \{V \setminus F \mid F \notin \Delta\}$.

If I is a square-free monomial ideal, then the square-free *Alexander dual* of $I = (x_{1,1}...x_{1,n_1}, ..., x_{t,1}...x_{t,n_t})$ is the ideal $I^\vee = (x_{1,1}, ..., x_{1,n_1}) \cap ... \cap (x_{t,1}, ..., x_{t,n_t})$.

We quote the following results which are proved in [5], [8], [19] and [15].

Theorem 2.5. *Let I be a square-free monomial ideal of R. Then the following conditions hold:*

(a) R/I is CM if and only if the Alexander dual $I^\vee$ *has a linear resolution.*

(b) R/I is SCM if and only if the Alexander dual $I^\vee$ *is componentwise linear.*

(c) $\operatorname{proj}\dim(R/I) = \operatorname{reg}(I^\vee)$.

(d) If $y_1, ..., y_r$ *is an R-sequence with* $\deg(y_i) = d_i$ *and* $I = (y_1, ..., y_r)$*, then* $\operatorname{reg}(I) = d_1 + ... + d_r - r + 1$.

In the following if $G(I) = \{u_1, ..., u_t\}$, then we set $\operatorname{supp}(I) = \cup_{i=1}^t \operatorname{supp}(u_i)$, where $\operatorname{supp}(u) = \{x_i : u = x_1^{a_1}...x_n^{a_n}, a_i \neq 0\}$. Also we set $\gcd(I) = \gcd(u_1, ..., u_m)$ and $\deg(I) = \max\{\deg(u_1), ..., \deg(u_m)\}$.

Throughout this paper we assume that all matroidal ideals are full supported, that is, $\operatorname{supp}(I) = \{x_1, ..., x_n\}$.

Corollary 2.6. *[6, Corollary 6.6] Let* Δ *be a simplicial complex on n vertices, and let* I_Δ *be it's Stanley-Reisner ideal, minimally generated by square-free monomials* $m_1, ..., m_s$*. If* $s \leq 3$*, so that* Δ *has at most three minimal nonfaces, or if* $\operatorname{Supp}(m_i) \cup \operatorname{Supp}(m_j) = \{x_1, ..., x_n\}$ *for all* $i \neq j$*, then* Δ *is SCM.*

Definition 2.7. *Let I be a monomial ideal of R. Then the big height of I, denoted by bight(I), is* $\max\{\operatorname{height}(\mathfrak{p}) | \mathfrak{p} \in \operatorname{Ass}(R/I)\}$.

Note that, if I is a matroidal ideal of degree d, then by Auslander-Buchasbum formula $bight(I) \leq n - d + 1$.

Proposition 2.8. *[21, Corollary 6.4.20]. Let I be a monomial ideal of R such that R/I is SCM. Then* $\operatorname{proj}\dim(R/I) = bight(I)$.

The following examples say that the converse of Proposition 2.8 is not true even if I is matroidal with $\gcd(I) = 1$.

Example 2.9. *Let* $n = 5$ *and* $I = (x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_3, x_2x_4, x_3x_5, x_4x_5)$ *be an ideal of R. Then I is a matroidal ideal of R with* $\operatorname{proj}\dim(R/I) = bight(I)$ *but I is not SCM.*

Proof. It is clear that I is a matroidal ideal and

$$\operatorname{Ass}(R/I) = \{(x_1, x_3, x_4), (x_1, x_2, x_5), (x_2, x_3, x_4, x_5)\}.$$

Thus $I^\vee_{[3]} = (x_1x_3x_4, x_1x_2x_5)$ and so $\operatorname{reg}(I^\vee_{[3]}) = 4$. Hence $I^\vee$ is not componentwise linear resolution. Therefore I is not SCM but $\operatorname{proj}\dim(R/I) = 4 = bight(I)$. □

3. SCM matroidal ideals of degree 2

In this section, we classify all SCM matroid ideals of degree 2.

Lemma 3.1. *Let $n = 3$ and I be a matroidal ideal in R generated in degree d. Then I is a SCM ideal.*

Proof. Let $n = 3$, then every matroidal ideal in R generated by at most three square-free monomials and so by Corollary 2.6 we have the result. □

Lemma 3.2. *Let I be a monomial ideal of R such that $I = (u_1, ..., u_d)$ and $\deg(u_i) \leq \deg(u_d) = d$ for all i. If $\operatorname{reg}(I) = d$, then $\operatorname{reg}(I_i) = i$ for all $i > d$.*

Proof. Consider the following exact sequence for $i > d$,

$$0 \longrightarrow \frac{I}{(I_i)} \longrightarrow \frac{R}{(I_i)} \longrightarrow \frac{R}{I} \longrightarrow 0.$$

$l(\frac{I}{(I_i)}) < \infty$, so by Lemma 1.1 (d) $\operatorname{reg}(\frac{I}{(I_i)}) = i - 1$ and

$$\operatorname{reg}(\frac{R}{(I_i)}) = \max\{\operatorname{reg}(\frac{R}{I}), \operatorname{reg}(\frac{I}{(I_i)})\} = \max\{d-1, i-1\} = i-1.$$

On the other hand $\operatorname{reg}(I_i) = \operatorname{reg}(\frac{R}{(I_i)}) + 1$, that is, $\operatorname{reg}(I_i) = i$ for all $i > d$. □

Proposition 3.3. *Let I be monomial ideal which is componentwise linear in R. Then $J = (x_{n+1}, I)$ is componentwise linear in $R' = K[x_1, ..., x_n, x_{n+1}]$.*

Proof. Suppose that $I = (u_1, ..., u_m)$, where $\deg(u_i) = d_i$ and $d_{i-1} \leq d_i$ for $i = 2, ..., m$. We induct on m, the number of minimal generators of I. If $m = 1$, then $I = (x_{n+1}, u_1)$. Set $J' = x_{n+1}R'$. Note that $(J_j) = (J'_j)$ for all $j < d_1$ and so (J_j) has a linear resolution for all $j < d_1$. By theorem 2.5, $\operatorname{reg}(J) = d_1$. Thus (J_{d_1}) has a linear resolution and also (J_j) has a linear resolution for all $j > d_1$, by using Lemma 3.2.

Now, let $m > 1$ and assume that the ideal $L = (x_{n+1}, u_1, ..., u_{m-1})$ is componentwise linear. Set $J = (L, u_m) = (I, x_{n+1})$. Note that $(J_j) = (L_j)$ for all $j < d_m$ and so (J_j) has a linear resolution for all $j < d_m$. Hence by using [14, Lemma 3.2] we have $\operatorname{reg}(J) = \operatorname{reg}(I) = d_m$. Therefore (J_{d_m}) has a linear resolution. Again, by using Lemma 3.2, we have (J_j) has a linear resolution for all $j > d_m$. This completes the proof. □

Corollary 3.4. *Let I be a SCM matroidal ideal in R and let $J = x_{n+1}I$ be a monomial ideal in $R' = K[x_1, ..., x_n, x_{n+1}]$. Then J is a SCM matroidal ideal in $R' = k[x_1, ..., x_n, x_{n+1}]$.*

Proof. The Alexander dual of J is $J^{\vee} = (x_{n+1}, I^{\vee})$ and by our hypothesis on I, $I^{\vee}$ is componentwise linear resolution. Thus by Proposition 3.3, $J^{\vee}$ is componentwise linear resolution. Thus J is a SCM matroidal ideal of R'. □

One of the most distinguished polymatroidal ideals is the ideal of Veronese type. Consider the fixed positive integers d and $1 \leq a_1 \leq ... \leq a_n \leq d$. The ideal of *Veronese type* of R indexed by d and $(a_1, ..., a_n)$ is the ideal $I_{(d;a_1,...,a_n)}$ which is generated by those monomials $u = x_1^{i_1}...x_n^{i_n}$ of R of degree d with $i_j \leq a_j$ for each $1 \leq j \leq n$.

Remark 3.5. Let I be a SCM matroidal ideal in R and let $J = x_{n+1}...x_m I$ be a monomial ideal in $R' = K[x_1, ..., x_n, x_{n+1}, ..., x_m]$. Then, by induction on m, J is a SCM matroidal ideal in $R' = K[x_1, ..., x_n, x_{n+1}, ..., x_m]$. Hence for a SCM matroidal ideal J, we can assume that $\gcd(J) = 1$. By using [16, Lemma 2.16] all fully supported matroid ideals of degree $n - 1 (n \geq 2)$ are Veronese type ideals and then by theorem 1.2, all matroidal ideals generated in degrees $d = 1, n-1, n$ are SCM.

Definition 3.6. *Let I be a square-free Veronese ideal of degree d. We say that J is an almost square-free Veronese ideal of degree d when $J \neq 0$, $G(J) \subseteq G(I)$ and*
$\mid G(J) \mid \geq \mid G(I) \mid -1$. *Note that every square-free Veronese ideal is an almost quare-free Veronese ideal. Also, if J is an almost square-free Veronese ideal of degree n, then J is a square-free Veronese ideal.*

Lemma 3.7. *Let J be an almost square-free Veronese ideal of degree $d < n$. Then J is a SCM matroidal ideal of R.*

Proof. Suppose that $y_1, ..., y_n$ is an arbitrary permutation of the variables of R such that $\{y_1, ..., y_n\} = \{x_1, ..., x_n\}$ and let I be a square-free Veronese ideal of degree d. We may assume that $I = J + (y_{n-d+1}y_{n-d+2}...y_n)$. Then we have $J = (y_1, ..., y_{n-d}) \cap I$ and so J is a matroidal ideal. Therefore $J^\vee = (y_1...y_{n-d}, I^\vee)$. Set $J' = (y_1...y_{n-d})$. Then, for all $i \leq n-d$, $J^\vee{}_{[i]} = J'_{[i]}$ and so it is componentwise linear. For all $i \geq n-d+1$, $J^\vee{}_{[i]}$ is a square-free Veronese ideal and so $J^\vee$ is a componentwise linear ideal. Hence J is a SCM matroidal ideal, as required. □

From now on, we will let $y_1, ..., y_n$ be an arbitrary permutation variables of R such that $\{x_1, ..., x_n\} = \{y_1, ..., y_n\}$.

Theorem 3.8. *Let J be a matroidal ideal of R with $\deg(J) = 2$ and $\gcd(J) = 1$. Then J is SCM if and only if there exists a permutation of variables such that the following hold:*

(a) $J = y_1\mathfrak{p} + J'$, *where $\mathfrak{p}$ is a monomial prime ideal with $y_1 \notin \mathfrak{p}$,* $\mathrm{height}(\mathfrak{p}) = n-1$ *and J' is a SCM matroidal ideal with* $\mathrm{Supp}(J') = \{y_2, ..., y_n\}$ *and* $\gcd(J') = 1$, *or*

(b) $J = y_1\mathfrak{p} + y_2\mathfrak{q}$, *where $\mathfrak{p}$ and $\mathfrak{q}$ are monomial prime ideals with $y_1 \notin \mathfrak{p}$ and $y_1, y_2 \notin \mathfrak{q}$ such that* $\mathrm{height}(\mathfrak{p}) = n-1$, $\mathrm{height}(\mathfrak{q}) = n-2$.

Proof. ($\Longleftarrow$). Consider the case (a). We have $J = \mathfrak{p} \cap (y_1, J')$, then $J^\vee = (\mathfrak{p}^\vee, y_1J'^\vee)$ and $\mathfrak{p}^\vee \in (u)$ for all $u \in J'^\vee$. Since $J^\vee{}_{[i]} = y_1J'^\vee{}_{[i-1]}$ for all $i \leq n-2$, and $J'^\vee{}_{[i-1]}$ is componentwise linear, it follows that $J^\vee{}_{[i]}$ is componentwise linear for all $i \leq n-2$. now consider the exact sequence

$$0 \longrightarrow R/(y_1J'^\vee : \mathfrak{p}^\vee)(-n+1) \xrightarrow{y_2y_3\cdots y_n} R/y_1J'^\vee \longrightarrow R/(\mathfrak{p}^\vee, y_1J'^\vee) \longrightarrow 0.$$

From $(y_1J'^\vee : \mathfrak{p}^\vee) = (y_1)$, we have $\mathrm{reg}(R/(y_1J'^\vee : \mathfrak{p}^\vee)) = 0$. Since $\deg(\mathfrak{p}^\vee) = n-1$, we have $\mathrm{reg}(R/(\mathfrak{p}^\vee, y_1J'^\vee)) \geq n-2$. Since $y_1J'^\vee$ is componentwise linear and $\deg(u) \leq n-2$ for all $u \in J'^\vee$, by [11, Corollary 8.2.14] we have $\mathrm{reg}(R/y_1J'^\vee) \leq n-2$. By using Lemma 1.1,

$$\begin{aligned}\mathrm{reg}(R/(\mathfrak{p}^\vee, y_1J'^\vee)) &\leq \max\{\mathrm{reg}(R/(y_1J'^\vee : \mathfrak{p}^\vee)(-n+1)) - 1, \mathrm{reg}(R/y_1J'^\vee)\}\\ &= \max\{n-2, \mathrm{reg}(R/y_1J'^\vee)\}.\end{aligned}$$

It therefore follows $\mathrm{reg}(R/(\mathfrak{p}^\vee, y_1J'^\vee)) = n-2$. Thus $J^\vee{}_{[n-1]}$ has a linear resolution and so J is a SCM ideal.

Let us consider the case (b). $J = (y_1, y_2) \cap (y_1, \mathfrak{q}) \cap \mathfrak{p}$ and so $J^\vee = (y_1y_2, y_1\mathfrak{q}^\vee, \mathfrak{p}^\vee)$. It is clear that $J^\vee$ is a monomial ideal with linear quotients. Thus, by Proposition 2.2, $J^\vee$ is componentwise linear and so J is a SCM ideal.

($\Longrightarrow$). Let J be a SCM ideal. Then there exists $\mathfrak{p} \in \mathrm{Ass}(R/J)$ such that $\mathrm{height}(\mathfrak{p}) = \mathrm{proj\,dim}(R/J) = n-1$. Since $J = \cap_{i=1}^n (J : y_i)$ and $\deg(J) = 2$, we can consider $\mathfrak{p} = (J : y_1)$ and $\mathfrak{p} = (y_2, ..., y_n)$. Hence $J = y_1\mathfrak{p} + J'$, where J' is a matroidal ideal of degree 2 in $K[y_2, ..., y_n]$. We claim that $\mathrm{Supp}(J') = \{y_2, ..., y_n\}$. Let $y_l \notin \mathrm{Supp}(J')$, where $l \geq 2$. Thus $y_1y_l, y_jy_k \in J$, where $j, k \geq 2$. Since J is a matroidal ideal, it follows y_ly_k or $y_ly_j \in J$. Hence y_ly_k or $y_ly_j \in J'$ and this is a contradiction. Therefore $\mathrm{Supp}(J') = \{y_2, ..., y_n\}$. $J = \mathfrak{p} \cap (J', y_1)$, it follows that $J^\vee = (\mathfrak{p}^\vee, y_1J'^\vee)$. For all $i \leq n-2$, we have $J^\vee{}_{[i]} = y_1J'^\vee{}_{[i-1]}$ and so $J'^\vee{}_{[i-1]}$ has a linear resolution for all $i \leq n-2$. Since $J^\vee{}_{[n-1]} = y_1J'^\vee{}_{[n-2]} + (\mathfrak{p}^\vee)$ and $\mathrm{reg}(J^\vee{}_{[n-1]}) = n-1$, it follows that $\mathrm{reg}(J'^\vee{}_{[n-2]}) \leq n-2$. Therefore $J'^\vee{}_{[n-2]}$ has a linear resolution and so $J'^\vee$ is componentwise linear. That is J' is a SCM matroidal ideal of degree 2. If $\gcd(J') = 1$, then J satisfy in the case (a). If $\gcd(J') \neq 1$, then we have the case (b). This completes the proof. □

4. SCM matroidal ideals over polynomial rings of small dimensional

We start this section by the following fundamental lemma.

Lemma 4.1. *Let $n \geq 5$ and J be a matroidal ideal of degree d in R and* $\gcd(J) = 1$. *If J is SCM, then*

$$\begin{aligned} J =& y_1y_2...y_{d-1}\mathfrak{p} + y_1y_2...y_{d-2}J_1 + y_1y_2...y_{d-3}y_{d-1}J_2 + ...+ \\ & y_1y_3...y_{d-1}J_{d-2} + y_2y_3...y_{d-1}J_{d-1} + J_d, \end{aligned}$$

where $\mathfrak{p} = (y_d, ..., y_n)$ is a monomial prime ideal, J_i is a SCM matroidal ideal of degree 2 with Supp$(J_i) = \{y_d, y_{d+1}, ..., y_n\}$ for $i = 1, ..., d-1$ and $J_d \subseteq \cap_{i=1}^{d-1} J_i$.

Proof. J is a SCM matroidal ideal, then there is a prime ideal $\mathfrak{p} \in \mathrm{Ass}(R/J)$ such that $\mathrm{height}(\mathfrak{p}) = \mathrm{proj\,dim}(R/J)$. Since $depth(R/J) = d-1$, it follows that $\mathrm{height}(\mathfrak{p}) = n-d+1$. For every square-free monomial ideal in R, we have $J = \cap_{i=1}^{n}(J : y_i)$. It follows that $\mathfrak{p} = (J : y_1y_2...y_{d-1})$ and we can write $J = y_1...y_{d-1}\mathfrak{p} + J'$, where J' is a square-free monomial ideal of degree d. It is clear that J' has a presentation

$$J' = y_1y_2...y_{d-2}J_1 + y_1y_2...y_{d-3}y_{d-1}J_2 + ... + y_1y_3...y_{d-1}J_{d-2} + y_2y_3...y_{d-1}J_{d-1} + J_d$$

and $J_d \subseteq \cap_{i=1}^{d-1} J_i$. Note that $\gcd(J) = 1$ and

$$(J : y_1y_2...y_{d-i-1}y_{d-i+1}...y_{d-1}) = y_{d-i}\mathfrak{p} + J_i,$$

we have $\mathrm{height}(J) \geq 2$ and so $J_i \neq 0$ for $i = 1, ..., d-1$. It is known that the localization of every SCM ideal is SCM and so

$$(J : y_1y_2...y_{d-i-1}y_{d-i+1}...y_{d-1}) = y_{d-i}\mathfrak{p} + J_i$$

is a SCM matroidal ideal of degree 2 for $i = 1, ..., d-1$. By using the proof of theorem 3.8, J_i is a SCM matroidal ideal with $\mathrm{Supp}(J_i) = \{y_d, y_{d+1}, ..., y_n\}$ for $i = 1, ..., d-1$. □

It is known that the localization of each SCM matroidal ideal is a SCM matroidal ideal. The following example shows that the converse is not true.

Example 4.2. *Let $n = 4$ and $J = (x_1x_3, x_1x_4, x_2x_3, x_2x_4)$. Then J is a matroidal ideal and $(J : x_i)$ is SCM matroidal for $i = 1, 2, 3, 4$; but J is not SCM.*

Proof. It is clear that J is matroidal and $(J : x_i)$ is SCM matroidal for $i = 1, 2, 3, 4$. Since $J^\vee = (x_1x_2, x_3x_4)$, it follows that $\mathrm{reg}(J^\vee) = 3$. Therefore J is not SCM. □

From now on, as Lemma 4.1, for a SCM matroidal ideal J of degree d and $\gcd(J) = 1$ in R with $n \geq 5$, we can write

$$J = y_1...y_{d-1}\mathfrak{p} + y_1y_2...y_{d-2}J_1 + y_1y_2...y_{d-3}y_{d-1}J_2 + ... + y_2y_3...y_{d-1}J_{d-1} + J_d,$$

where $\mathfrak{p} = (y_d, ..., y_n)$ is a monomial prime ideal, J_i is a SCM matroidal ideal of degree 2 with $\mathrm{Supp}(J_i) = \{y_d, y_{d+1}, ..., y_n\}$ for $i = 1, ..., d-1$ and $J_d \subseteq \cap_{i=1}^{d-1} J_i$.

Note that if for instance $\gcd(J_1) = y_d$, then we have

$$J = y_1...y_{d-1}\mathfrak{p} + y_1y_2...y_{d-2}y_d\mathfrak{q} + y_1y_2...y_{d-3}y_{d-1}J_2 + ... + y_1y_3...y_{d-2}J_{d-2} + y_2y_3...y_{d-1}J_{d-1} + J_d,$$

where $\mathfrak{q} = (y_{d+1}, ..., y_n)$.

Bandari and Herzog in [1, Proposition 2.7] proved that if $n = 3$ and J is a matroidal ideal with $\gcd(J) = 1$, then J is a square-free Veronese ideal and so by theorem 1.2, it is CM (see also [17, Proposition 1.5]). In the following proposition we prove this result in the case $n = 4$ for SCM ideals.

Proposition 4.3. *Let $n = 4$ and J be a matroidal ideal of R of degree d and* $\gcd(J) = 1$. *Then J is a SCM ideal if and only if J is*

(a) a square-free Veronese ideal, or

(b) an almost square-free Veronese ideal.

Proof. ($\Longleftarrow$) is clear by theorem 1.2 and Lemma 3.7.

($\Longrightarrow$). If $d = 1, 3, 4$, then by theorem 1.2 and [16, Lemma 2.16] J is a square-free Veronese ideal. If $d = 2$, then by theorem 3.8, $J = y_1\mathfrak{p} + J'$, where $\mathfrak{p}$ is a monomial prime ideal with $y_1 \notin \mathfrak{p}$, height($\mathfrak{p}$) = 3 and J' is a SCM matroidal ideal with Supp(J') = $\{y_2, y_3, y_4\}$. If gcd(J') = 1, then J' is a square-free Veronese ideal and so is J. If gcd(J') $\neq 1$, then J' is an almost square-free Veronese ideal. □

Proposition 4.4. *Let $n = 4$ and J be a matroidal ideal of R of degree d. Then J is a SCM ideal if and only if* $\operatorname{proj\,dim}(R/J) = bight(J)$.

Proof. ($\Longrightarrow$). It follows by Proposition 2.8.

($\Longleftarrow$). If $d = 1, 3, 4$, then by Remark 3.5 J is SCM. Let $d = 2$. By our hypothesis, there exists $\mathfrak{p} \in \operatorname{Ass}(R/J)$ such that $\mathfrak{p} = (J : y_1)$. Thus $J = y_1\mathfrak{p} + J'$, where J' is matroidal ideal of degree 2 in $K[y_2, y_3, y_4]$. Hence J' is a square-free Veronese ideal or an almost square-free Veronese ideal. Therefore by Proposition 4.3, J is SCM. □

Lemma 4.5. *Let $n \geq 5$ and J be a matroidal ideal of degree 3 in R such that $J = y_1y_2\mathfrak{p} + y_1y_3\mathfrak{q} + y_2y_3\mathfrak{q}$, where $\mathfrak{p}$ and $\mathfrak{q}$ are monomial prime ideals with $y_1, y_2 \notin \mathfrak{p}$ and $y_1, y_2, y_3 \notin \mathfrak{q}$ such that* height($\mathfrak{p}$) = $n - 2$, height($\mathfrak{q}$) = $n - 3$. *Then J is SCM.*

Proof. Since $J = \mathfrak{p} \cap (y_1y_2, y_1y_3\mathfrak{q}, y_2y_3\mathfrak{q})$, it follows that $J = \mathfrak{p} \cap (y_1, y_2) \cap (y_1, y_3) \cap (y_2, y_3) \cap (y_1, \mathfrak{q}) \cap (y_2, \mathfrak{q})$. Therefore $J^\vee = (y_1y_2, y_1y_3, y_2y_3, y_1\mathfrak{q}^\vee, y_2\mathfrak{q}^\vee, \mathfrak{p}^\vee)$. It is clear that $J^\vee$ is a monomial ideal with linear quotients and so by Proposition 2.2, $J^\vee$ is componentwise linear. Thus J is SCM. □

Lemma 4.6. *Let $n \geq 5$ and J be a matroidal ideal of degree 3 such that*

$$J = y_1y_2\mathfrak{p} + y_1y_3\mathfrak{q}_1 + y_2y_4\mathfrak{q}_2 + J_1,$$

where $\mathfrak{p}$, $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are monomial prime ideals with $y_1, y_2 \notin \mathfrak{p}$, $y_1, y_2, y_3 \notin \mathfrak{q}_1$ and $y_1, y_2, y_4 \notin \mathfrak{q}_2$ such that height($\mathfrak{p}$) = $n - 2$, height($\mathfrak{q}_1$) = $n - 3$ = height($\mathfrak{q}_2$) *and J_1 is a matroidal ideal in $R' = K[y_3, ..., y_n]$. Then $G(J_1) = \{y_3y_4y_i \mid i = 5, 6, ..., n\}$. In particular, J is not SCM.*

Proof. We consider two cases:

Case (a) $J_1 = 0$, then we have $y_1y_3y_5, y_2y_3y_4 \in J$ but $y_2y_3y_5$ or $y_3y_4y_5$ are not elements of J. Thus J is not a matroidal ideal and this is a contradiction.

Case (b) $J_1 \neq 0$.

1) For $n = 5$, $J_1 = (y_3y_4y_5)$ and

$$J = (y_1, y_4) \cap (y_2, y_3) \cap (y_1, y_2, J_1) \cap (y_2, y_3, J_1) \cap (y_1, \mathfrak{q}_2) \cap (y_2, \mathfrak{q}_1) \cap \mathfrak{p}.$$

Therefore $\operatorname{reg}(J^\vee_{[2]}) = 3$ and so J is not SCM.

2) Suppose that $n \geq 6$. Then $(J : y_3) = (y_1y_2, y_2y_4, y_1\mathfrak{q}_1, (J_1 : y_3))$. If $y_iy_j \in (J : y_3)$ for $5 \leq i \neq j \leq n$, then $y_2y_i \in (J : y_3)$ for $i \geq 5$, since $y_2y_4 \in (J : y_3)$. But this is a contradiction. Therefore $y_3y_iy_j \notin J$ for all $5 \leq i \neq j \leq n$. Consider $(J : y_4)$, we have $y_4y_iy_j \notin J$ for all $5 \leq i \neq j \leq n$. Also, if $y_iy_jy_t \in J$ for different numbers i, j, t with $5 \leq i, j, t \leq n$, then since $y_1y_3y_i \in J$, we have $y_3y_iy_j \in J$ or $y_3y_iy_t \in J$ and this is a contradiction. Thus $G(J_1) \subseteq \{y_3y_4y_i \mid i = 5, 6, ..., n\}$. On the other hand, since $y_2y_4y_i$ and $y_1y_3y_i$ are elements in J for $i \geq 5$ we have $y_3y_4y_i \in J$ for $i \geq 5$. Hence $G(J_1) = \{y_3y_4y_i \mid i = 5, 6, ..., n\}$. Therefore

$$J = (y_1, y_4) \cap (y_2, y_3) \cap (y_1, y_2, J_1) \cap (y_1, \mathfrak{q}_2) \cap (y_2, \mathfrak{q}_1) \cap \mathfrak{p}$$

and so $J^\vee = (y_1y_4, y_2y_3, y_1y_2J_1^\vee, y_1\mathfrak{q}_2^\vee, y_2\mathfrak{q}_1^\vee, \mathfrak{p}^\vee)$. Thus $\operatorname{reg}(J^\vee_{[2]}) = 3$ and so J is not SCM.

□

Lemma 4.7. *Let $n \geq 6$ and J be a matroidal ideal of degree 3 such that $J = y_1y_2\mathfrak{p} + y_1y_3\mathfrak{q} + y_2J_1$, where $\mathfrak{p}$ and $\mathfrak{q}$ are monomial prime ideals with $y_1, y_2 \notin \mathfrak{p}$, $y_1, y_2, y_3 \notin \mathfrak{q}$ such that $\text{height}(\mathfrak{p}) = n-2$, $\text{height}(\mathfrak{q}) = n-3$ and J_1 is a matroidal ideal in $R' = K[y_3, ..., y_n]$ with $\gcd(J_1) = 1$. Then J is not SCM matroidal.*

Proof. By contrary, we assume that J is SCM matroidal. Then $(J : y_2) = y_1\mathfrak{p} + J_1$ is SCM matroidal and so by theorem 3.8 J_1 is SCM matroidal of degree 2. From $\gcd(J_1) = 1$, we have $J_1 = y_i\mathfrak{q}_1 + J_2$, where $\mathfrak{q}_1$ and J_2 are a monomial prime ideal of height $n-3$ and a matroidal ideal respectively in $R' = K[y_3, ..., y_{i-1}, y_{i+1}, ..., y_n]$. There are two main cases to consider.

a) $i = 3$, then $(J : y_j) = (y_1y_2, y_1y_3, y_2y_3, y_2(J_2 : y_j))$ when $j \neq 1, 2, 3$. Since $y_t \in (J_2 : y_j)$ for $t \neq 1, 2, 3, j$, we have y_2y_t and y_1y_3 are elements of $(J : y_j)$ but y_1y_t or y_3y_t are not elements of $(J : y_j)$. This is a contradiction.

b) $i \neq 3$, then $(J : y_i) = (y_1y_2, y_1y_3, y_2\mathfrak{q}_1)$. Thus y_2y_t and y_1y_3 for $t \neq 3$ are elements of $(J : y_i)$ but y_1y_t or y_3y_t are not elements of $(J : y_i)$ and this is a contradiction. Thus J is not SCM matroidal.

□

Lemma 4.8. *Let $n \geq 6$ and J be a matroidal ideal of degree 3 such that $J = y_1y_2\mathfrak{p} + y_1y_3\mathfrak{q} + y_2J_1 + J_2$ or $J = y_1y_2\mathfrak{p} + y_1y_3\mathfrak{q} + y_2y_3\mathfrak{q} + J_2$, where $\mathfrak{p}$ and $\mathfrak{q}$ are monomial prime ideals with $y_1, y_2 \notin \mathfrak{p}$, $y_1, y_2, y_3 \notin \mathfrak{q}$ such that $\text{height}(\mathfrak{p}) = n-2$, $\text{height}(\mathfrak{q}) = n-3$ and J_1 is a nonzero matroidal ideal in $R' = K[y_3, ..., y_n]$ with $\gcd(J_1) = 1$. Then $G(J_2) \subseteq \{y_3y_iy_j \mid 4 \leq i \neq j \leq n\}$ and if $J_2 \neq 0$, then $\text{Supp}(J_2) = \{y_3, y_4, ..., y_n\}$. In particular, if $J = y_1y_2\mathfrak{p} + y_1y_3\mathfrak{q} + y_2y_3\mathfrak{q} + J_2$, then $J_2 = 0$.*

Proof. Let us consider $J = y_1y_2\mathfrak{p} + y_1y_3\mathfrak{q} + y_2J_1 + J_2$. Then we have $(J : y_t) = (y_1y_2, y_1y_3, y_2(J_1 : y_t), (J_2 : y_t))$ for some $t \geq 4$. If $y_iy_jy_t \in J$ for some different numbers $4 \leq i, j, t \leq n$, then $y_iy_j \in (J : y_t)$. Since $y_1y_3 \in (J : y_t)$, it follows that $y_1y_i \in (J : y_t)$ for some $i \geq 4$ and this is a contradiction. It therefore follows that $G(J_2) \subseteq \{y_3y_iy_j \mid 4 \leq i \neq j \leq n\}$. Also, $(J : y_3) = (y_1y_2, y_1\mathfrak{q}, y_2(J_1 : y_3), (J_2 : y_3))$. If $y_iy_j \in (J : y_3)$ for some $4 \leq i \neq j \leq n$, then $y_iy_t \in (J : y_3)$ for all t with $4 \leq i \neq t \leq n$ since $y_1y_t \in (J : y_3)$. Hence $\text{Supp}(J_2) = \{y_3, y_4, ..., y_n\}$. The proof for the case $J = y_1y_2\mathfrak{p} + y_1y_3\mathfrak{q} + y_2y_3\mathfrak{q} + J_2$ is similar to the above argument. In particular, if $y_3y_iy_j \in J_2$ for some $4 \leq i \neq j \leq n$ then from $y_1y_2y_t \in J$ for some $4 \leq i \neq t \neq j \leq n$ we have $y_iy_jy_t \in J$. This is a contradiction. Thus $J_2 = 0$. □

Proposition 4.9. *Let $n = 5$ and J be a matroidal ideal of degree 3 such that $\gcd(J) = 1$. Then J is a SCM ideal if and only if $J = y_1y_2\mathfrak{p} + y_1J_1 + y_2J_2 + J_3$, where J_1 and J_2 are SCM ideals with $\text{Supp}(J_1) = \text{Supp}(J_2) = \{y_3, y_4, y_5\}$, $J_3 \subseteq J_1 \cap J_2$ and satisfying in the one of the following cases:*

(a) $\gcd(J_1) = 1$, $\gcd(J_2) = 1$, *or*

(b) $\gcd(J_1) = y_3 = \gcd(J_2)$ *and* $J_3 = 0$.

Proof. ($\Longleftarrow$). Consider (a). Then J_1 and J_2 are square-free Veronese ideal and $G(J_3) \subseteq \{y_3y_4y_5\}$. If $J_3 = 0$, then J is an almost square-free Veronese ideal and so by using Lemma 3.7, J is a SCM matroidal ideal. If $J_3 \neq 0$, then J is a square-free Veronese ideal and so J is a SCM matroidal ideal.

If we have the case (b), then by Lemma 4.5 the result follows.

($\Longrightarrow$). Let J be a SCM, then by Lemma 4.1, J has the presentation $J = y_1y_2\mathfrak{p} + y_1J_1 + y_2J_2 + J_3$, where J_1 and J_2 are SCM matroidal ideals with $\text{Supp}(J_1) = \text{Supp}(J_2) = \{y_3, y_4, y_5\}$ and $J_3 \subseteq J_1 \cap J_2$.

1) If $\gcd(J_1) = y_3$ and $\gcd(J_2) = y_4$, then by Lemma 4.6 J is not a SCM matroidal ideal and we don't have this case.

2) If $\gcd(J_1) = \gcd(J_2) = y_3$, then $J_3 = 0$. Let contrary, then $G(J_3) = \{y_3y_4y_5\}$ and $y_1y_2y_5, y_3y_4y_5 \in J$ but $y_1y_4y_5$ or $y_2y_4y_5$ are not elements of J. This is a contradiction.

3) If $\gcd(J_1) = y_3$, $\gcd(J_2) = 1$ and $J_3 = 0$, then $y_1y_3y_5, y_2y_4y_5 \in J$ but $y_1y_4y_5$ or $y_3y_4y_5$ are not elements of J. Therefore J is not matroidal and we don't have this case.

4) If $\gcd(J_1) = y_3$, $\gcd(J_2) = 1$ and $G(J_3) = \{y_3y_4y_5\}$, then by change of variables (a) follows with $J_3 = 0$. □

Proposition 4.10. *Let $n = 6$ and let J be a matroidal ideal of degree 4 such that $\gcd(J) = 1$. Then J is a SCM ideal if and only if $J = y_1y_2y_3\mathfrak{p} + y_1y_2J_1 + y_1y_3J_2 + y_2y_3J_3 + J_4$ such that J_1, J_2, J_3 are SCM matroidal ideals and satisfying in one of the following conditions:*

(a) for $i = 1, 2, 3$, $\gcd(J_i) = 1$ and $| G(J_4) |= 3$,

(b) for $i = 1, 2, 3$, $\gcd(J_i) = 1$ and $| G(J_4) |= 2$,

(c) for $i = 1, 2, 3$, $\gcd(J_i) = 1$ and $J_4 = 0$, or

(d) for $i = 1, 2, 3$, $\gcd(J_i) = y_4$ and $J_4 = 0$.

Proof. ($\Longleftarrow$). If we have (a), then J is a square-free Veronese ideal and so by theorem 1.2, J is SCM. Consider case (b), then J is an almost square-free Veronese ideal and so by Lemma 3.7, J is SCM. If we consider (d), then by using the same proof of Lemma 4.5 $J^\vee$ has linear quotients and so J is SCM. Let (c), then we have $J = \mathfrak{p} \cap (y_1, y_2) \cap (y_1, y_3) \cap (y_2, y_3) \cap (y_1, J_3) \cap (y_2, J_2) \cap (y_3, J_1)$ and so $J^\vee = (y_1y_2, y_1y_3, y_2y_3, y_1J_3^\vee, y_2J_2^\vee, y_3J_1^\vee, \mathfrak{p}^\vee)$. That is, $J^\vee$ has linear quotients. Thus J is SCM.

($\Longrightarrow$). Let J be a SCM ideal. Then by Lemma 4.1, $J = y_1y_2y_3\mathfrak{p} + y_1y_2J_1 + y_1y_3J_2 + y_2y_3J_3 + J_4$ and J_1, J_2, J_3 are SCM matroidal ideals. Let $\gcd(J_1) = y_4$. Since $(J : y_1) = y_2y_3\mathfrak{p} + y_2J_1 + y_3J_2 + (J_4 : y_1)$, $\gcd(J : y_1) = 1$ and $(J : y_1)$ is a SCM matroidal ideal, by Proposition 4.9 it follows $\gcd(J_2) = y_4$ and $(J_4 : y_1) = 0$. Again by using $(J : y_2)$ and $(J : y_3)$, we obtain $\gcd(J_1) = \gcd(J_3) = \gcd(J_2) = y_4$ and $J_4 = 0$. Also, if for some i, $\gcd(J_i) = 1$, then by Proposition 4.9 and by using $(J : y_1), (J : y_2)$ and $(J : y_3)$ we have $\gcd(J_i) = 1$ for $i = 1, 2, 3$. If $G(J_4) = \{y_1y_4y_5y_6\}$, then J is not a matroidal ideal since $y_1y_4y_5y_6, y_2y_3y_5y_6 \in J$, but $y_2y_4y_5y_6$ or $y_3y_4y_5y_6$ are not elements of J. Thus $J_4 = 0$ or $| G(J_4) |= 2$ or $| G(J_4) |= 3$ and this completes the proof. □

Proposition 4.11. *Let $n \geq 6$ and let J be a matroidal ideal of degree $n - 2$ such that $\gcd(J) = 1$. Then J is a SCM ideal if and only if*

$$\begin{aligned} J =& y_1y_2...y_{n-3}\mathfrak{p} + y_1y_2...y_{n-4}J_1 + y_1y_2...y_{n-5}y_{n-3}J_2 + ...+ \\ & y_1y_3...y_{n-3}J_{n-4} + y_2y_3...y_{n-3}J_{n-3} + J_{n-2} \end{aligned}$$

such that J_i is SCM matroidal ideal for all $i = 1, .., n - 3$ and satisfying in one of the following conditions:

(a) for $i = 1, ..., n - 3$, $\gcd(J_i) = 1$ and $| G(J_{n-2}) |= \binom{n-3}{2}$,

(b) for $i = 1, ..., n - 3$, $\gcd(J_i) = 1$ and $| G(J_{n-2}) |= \binom{n-3}{2} - 1$,

(c) for $i = 1, ..., n - 3$, $\gcd(J_i) = 1$ and $J_{n-2} = 0$, or

(d) for $i = 1, ...n - 3$, $\gcd(J_i) = y_{n-2}$ and $J_{n-2} = 0$.

Proof. ($\Longleftarrow$).

If case (a) holds, then J is a square-free Veronese ideal and so by theorem 1.2, J is SCM. Let (b), then J is an almost square-free Veronese ideal and so by Lemma 3.7, J is SCM. If (d), then by using the same proof of Lemma 4.5, $J^\vee$ has linear quotients and so J is SCM. Let (c), then we have

$$\begin{aligned} J =& \mathfrak{p} \cap (y_1, y_2) \cap ... \cap (y_1, y_{n-3}) \cap (y_2, y_3) \cap ... \cap (y_2, y_{n-3}) \\ & \cap ... \cap (y_{n-4}, y_{n-3}) \cap (y_1, J_{n-3}) \cap ... \cap (y_{n-3}, J_1) \end{aligned}$$

and so

$$J^\vee = (y_1y_2, ..., y_1y_{n-3}, y_2y_3, ..., y_2y_{n-3}, ..., y_{n-4}y_{n-3}, y_1J_{n-3}^\vee, ..., y_{n-3}J_1^\vee, \mathfrak{p}^\vee).$$

Since J_i are square-free Veronese ideals, it follows that $J^\vee$ has linear quotients. That is, J is SCM.

($\Longrightarrow$). Let J be a SCM ideal. Then by Lemma 4.1,

$$\begin{aligned} J =& y_1y_2...y_{n-3}\mathfrak{p} + y_1y_2...y_{n-4}J_1 + y_1y_2...y_{n-5}y_{n-3}J_2 + ...+ \\ & y_1y_3...y_{n-3}J_{n-4} + y_2y_3...y_{n-3}J_{n-3} + J_{n-2} \end{aligned}$$

and J_i are SCM matroidal ideals for all $i = 1, ..., n-3$. We use induction on $n \geq 6$. If $n = 6$, then the result follows by Proposition 4.10. Let $n > 6$ and $\gcd(J_1) = y_{n-2}$.

$$(J : y_1) = y_2y_3...y_{n-3}\mathfrak{p} + y_2...y_{n-4}J_1 + y_2...y_{n-5}y_{n-3}J_2 + ... + y_3...y_{n-3}J_{n-4} + (J_{n-2} : y_1),$$

$\gcd(J : y_1) = 1$ and $(J : y_1)$ is a SCM matroidal ideal, by induction hypothesis it follows $\gcd(J_i) = y_{n-2}$ for $i = 1, ..., n-4$ and $(J_{n-2} : y_1) = 0$. Again by using $(J : y_i)$ for $i = 2, ..., n-3$ and by using induction hypothesis, $\gcd(J_i) = y_{n-2}$ for $i = 1, ...n-3$ and $J_{n-2} = 0$. Also, if for some i, $\gcd(J_i) = 1$, then again by using $(J : y_i)$ for $i = 1, ..., n-3$ and by using induction hypothesis we have $\gcd(J_i) = 1$ for $i = 1, ..., n-3$. If $| G(J_{n-2}) |< \binom{n-3}{2} - 1$, then there exists $1 \leq i \leq n-3$ such that $| G(I : y_i) |< \binom{n-4}{2} - 1$ and this is a contradiction. Thus $J_{n-2} = 0$ or $| G(J_{n-2}) |= \binom{n-3}{2}$ or $| G(J_{n-2}) |= \binom{n-3}{2} - 1$ and this completes the proof. □

Theorem 4.12. *Let $n = 6$ and let J be a matroidal ideal of degree 3 such that $\gcd(J) = 1$. Then J is a SCM ideal if and only if $J = y_1y_2\mathfrak{p} + y_1J_1 + y_2J_2 + J_3$ such that J_1 and J_2 are SCM matroidal ideals and satisfying in one of the following conditions:*

(a) $| G(J_3) |= 4$ *and one of J_1 or J_2 is an almost square-free Veronese ideal and the other is a square-free Veronese ideal,*

(b) $| G(J_3) |= 3$, *J_1, J_2 are square-free Veronese ideals,*

(c) *$J_3 = 0$ and $J_1 = J_2$ are square-free Veronese ideals or almost square-free Veronese ideals either $J_3 = 0$ and* $\gcd(J_1) = y_3 = \gcd(J_2)$.

Proof. ($\Longleftarrow$). If we consider the (a) or (b), then J is a square-free Veronese ideal or an almost square-free Veronese ideal and so J is SCM. Consider (c) and suppose that $\gcd(J_1) = \gcd(J_2) = y_3$. Then by using Lemma 4.5, J is SCM. Also, for (c) if $J_1 = J_2$ are square-free Veronese ideals or almost square-free Veronese ideals, we have $J^\vee = (y_1y_2, y_1J_2^\vee, y_2J_1^\vee, \mathfrak{p}^\vee)$ and so $J^\vee$ has linear quotients. Thus J is SCM.

($\Longrightarrow$). Let J be a SCM ideal. Then by Lemma 4.1, $J = y_1y_2\mathfrak{p} + y_1J_1 + y_2J_2 + J_3$ and J_1 and J_2 are SCM matroidal ideals and $J_3 \subseteq J_1 \cap J_2$ with $(J_3) = \{y_3, y_4, y_5, y_6\}$. Therefore $| G(J_3) |\leq 4$. We have four cases:

Case (i) Suppose that $| G(J_3) |= 4$,then by Lemmas 4.6 and 4.8 we have $\gcd(J_1) = 1 = \gcd(J_2)$. By Proposition 4.3, we have the case (a) if we prove J_1 and J_2 aren't almost square-free Veronese ideals in the same time. Let contrary, if $y_1y_3y_5, y_2y_3y_5$ are not elements of J, then $y_1y_2y_3, y_3y_4y_5 \in J$. But $y_1y_3y_5$ or $y_2y_3y_5$ are not elements of J and this is a contradiction.

If $y_1y_3y_5, y_2y_3y_6$ are not elements of J, then

$$\begin{aligned} (J : y_3) =& (y_1y_2, y_1(y_4, y_6), y_2(y_4, y_5), y_4y_5, y_4y_6, y_5y_6) \\ =& y_4(y_1, y_2, y_5, y_6) + (y_1y_2, y_1y_6, y_2y_5, y_5y_6). \end{aligned}$$

By theorem 3.8, $(y_1y_2, y_1y_6, y_2y_5, y_5y_6)$ is not SCM and this is a contradiction.

If $y_1y_3y_5, y_2y_4y_6$ are not elements of J, then $(J^\vee_{[3]}) = (y_1y_3y_5, y_2y_4y_6)$ and so $\mathrm{reg}(J^\vee_{[3]}) = 5$. Thus J is not SCM and this is a contradiction.

Case (ii) Let $| G(J_3) |= 3$. We consider the following cases.

1) If $\gcd(J_1) = y_3$ and $\gcd(J_2) = 1$, then $G(J_3) = \{y_3y_4y_5, y_3y_4y_6, y_3y_5y_6\}$, by Lemma 4.8. $\gcd(J_2) = 1$, so by Proposition 4.3, J_2 is a square-free Veronese ideal or an almost square-free Veronese ideal. If $J_2 = (y_2y_3y_4, y_2y_3y_5, y_2y_3y_6, y_2y_4y_5, y_2y_4y_6)$ is an almost square-free Veronese ideal, then $y_3y_5y_6, y_1y_2y_4 \in J$ but $y_1y_5y_6$ or $y_2y_5y_6$ either $y_4y_5y_6$ are not elements of J and this is a contradiction. So J_2 is a square-free Veronese ideal and by using a new presentation for J and change of variables we get J_1 and J_2 are square-free Veronese ideals and $J_3 = 0$ and this is the case (c).

2) If $\gcd(J_1) = y_3$ and $\gcd(J_2) = y_4$, then by Lemma 4.6 we have $|\ G(J_3)\ |= 2$ and this is a contradiction.

3) If $\gcd(J_1) = y_3 = \gcd(J_2)$, then $y_1y_2y_4, y_3y_4y_5 \in J$ but $y_1y_4y_5$ or $y_2y_4y_5$ are not elements of J and this is a contradiction.

4) Let $\gcd(J_1) = 1 = \gcd(J_2)$. Suppose that J_1 is a square-free Veronese ideal and J_2 is an almost square-free Veronese ideal. We assume that $J_2 = (y_2y_3y_4, y_2y_3y_5, y_2y_3y_6, y_2y_4y_5, y_2y_4y_6)$. Since $|\ G(J_3)\ |= 3$, we can assume that one of the element $y_3y_5y_6$ or $y_3y_4y_6$ are not in J. If $y_3y_5y_6 \notin J$, then $y_2y_3y_5, y_1y_5y_6 \in J$ but $y_2y_5y_6$ or $y_3y_5y_6$ are not elements of J and this is a contradiction. If $y_3y_4y_6 \notin J$, then $(J : y_6) = (y_1y_2, y_1(y_3, y_4, y_5), y_2(y_3, y_4), y_3y_5, y_4y_5)$. Therefore by using theorem 3.8 this is not SCM. Thus we do not have this case. Also, by the same argument of the **Case (i)**, J_1 and J_2 are not almost square-free Veronese ideals in the same time. Therefore J_1, J_2 are square-free Veronese ideals and we have the case (b).

Case (iii) Let $|\ G(J_3)\ |= 2$. Then by Lemmas 4.6, 4.8, we have $\gcd(J_1) = y_3$, $\gcd(J_2) = 1$ or $\gcd(J_1) = 1 = \gcd(J_2)$. If $\gcd(J_1) = y_3$, $\gcd(J_2) = 1$, then we can assume that $G(J_3) = \{y_3y_4y_5, y_3y_4y_6\}$. Since $\gcd(J_2) = 1$, by Proposition 4.3 J_2 is square-free Veronese ideal or almost Veronese ideal. If J_2 is square-free Veronese ideal, then $y_2y_5y_6, y_3y_4y_5 \in J$ but $y_3y_5y_6$ or $y_4y_5y_6$ are not elements of J and this is a contradiction. Let J_2 be an almost square-free Veronese ideal and we assume that y_5y_6 is the only element which is not in J_2. In this case by change of variables we have $J_3 = 0$ and $J_1 = J_2$ are almost square-free Veronese ideals and and this is the case (c). If y_4y_5 is the only element which is not in J_2, then $y_3y_4y_5, y_2y_4y_6$ are elements of J but $y_2y_4y_5$ or $y_4y_5y_6$ are not elements of J and this is a contradiction. Also, if y_4y_6 is the only element which is not in J_2, then again J is not matroidal and this is a contradiction. Now we can assume that $J_3 = 0$. If $\gcd(J_1) = y_3$, then by Lemmas 4.6, 4.8 we have $\gcd(J_2) = 1$ or $\gcd(J_2) = y_3$. If $\gcd(J_2) = 1$, then $y_1y_3y_5$ and $y_2y_iy_j$ are elements of J for some $i, j = 4, 5, 6$, but $y_1y_iy_j$ or $y_3y_iy_j$ are not elements of J and this is a contradiction. Therefore $\gcd(J_2) = y_3$ and this is the case (c). Also, if $\gcd(J_1) = 1$ then $\gcd(J_2) = 1$. If $J_1 \neq J_2$ are almost square-free Veronese ideals, then again by using the above argument J is not matroidal and this is a contradiction. Therefore $J_1 = J_2$ are square-free Veronese ideals or almost square-free Veronese ideals.

Case (iv) Let $|\ G(J_3)\ |= 1$. Then by Lemmas 4.6, 4.8, we have $\gcd(J_1) = 1 = \gcd(J_2)$. Therefore by Proposition 4.3 J_1 and J_2 are square-free Veronese ideals or almost Veronese ideals. By choosing one element from J_1 and the only element from J_3, we have $|\ G(J_3)\ |\geq 2$. This is a contradiction.

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