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A Fixed Point Theorem and an Application for the Cauchy Problem in the Scale of Banach Spaces

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Abstract. The main aim of this paper is to prove the existence of the fixed point of the sum of two operators in setting of the cone-normed spaces with the values of cone-norm belonging to an ordered locally convex space. We apply this result to prove the existence of global solution of the Cauchy problem with perturbation of the form

$$\begin{cases} x'(t) = f[t, x(t)] + g[t, x(t)], \ t \in [0, \infty), \\ x(0) = x_0 \in F_1, \end{cases}$$

in a scale of Banach spaces $\{(F_s, \|.\|_s) : s \in (0, 1]\}$.

1. Introduction.

One of the trend in fixed point theory is to extend the structure on which the functions are defined on. Among several of them, we mention one of the natural extension of metric space:Cone metric space. Indeed, both cone metric and cone normed spaces (also called a K-metric spaces and K-normed spaces) are expected extension of the standard metric spaces and standard normed spaces that are obtained by replacing an ordered Banach space instead of the set of real numbers. The history of the discussion on cone metric (normed spaces) are back to about 1950. In fact, these spaces have been used in Differential Equations and Theory Fixed Point in the researches of Kantorovich [15, 16], Collatz [5], P.Zabreiko and other mathematicians [28]. In 2007, the notion of cone metric is re-introduced by L.G.Huang and X.Zhang [10], the investigation of fixed point theory in cone-metric spaces (in most cases for contractive mappings) has again attracted much attention from mathematicians. We refer to the papers [2, 4, 6, 9, 14, 25, 27, 17, 18, 19, 20, 11, 12, 13] for some historical notes, discussion on obtained results and further references. Recently, it was understood that the cone metric space defined over a normal-cone are equivalent to the standard metric spaces, see e.g. [1, 7, 21]. On the other hand, for our purpose, we prefer to use the structures of cone metric (normed) spaces

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The purpose of this paper is to present a type of the Krasnoselkii fixed point theorem for operator $T+S$ in cone normed spaces with the values of cone norm belonging to an ordered locally convex space. We use obtained result to prove the existence of global solution of the Cauchy problem with perturbation of the form

$$x'(t) = f[t, x(t)] + g[t, x(t)] \tag{1}$$

in a scale of Banach spaces $\{(F_s, \|.\|_s) : s \in (0,1]\}$.

The existence of solutions of (1) with f satisfying Lipschitz-Ovcjannikov condition of the form

$$\|f(t,u) - f(t,v)\|_s \leq \frac{c}{r-s}\|u-v\|_r,\ 0 < s < r \leq 1$$

and $g(t,u) = 0$ were studied by F.Treves, L.Ovcjannikov, L.Nirenber, T.Nishida, et al [22, 23, 24, 26]. In case g is compact, the problem was studied by H.Begehr [3], M.Ghisi [8], these author was proved the existence of locally solution of (1). However, we study the problem with a condition that seems stronger than Lipschitz-Ovcjannikov condition, that is,

$$\left\|f(t,u) - f(t,v)\right\|_s \leq l(s)\|u-v\|_s$$

if $\|u-v\|_s$ is sufficiently small. Our paper may be the first study on such condition as above.
This paper is organized as follows. In section 2, we give some premilinaries on some definitions of ordered locally convex space and their properties. In section 3, we state the main results which shows the existence of the fixed point of the sum of two operators in cone-normed spaces. Section 4 illustrate main results in Section 3 in order to give applications for concrete equations.

2. Preliminaries.

Let E be a real locally convex space. A subset K of E is called a *cone* if K is a closed convex subset satisfying $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{\theta\}$. Assume that the topology on E defined by a separating family of seminorms Γ satisfying the following condition

$$\theta \leq x \leq y \Rightarrow \varphi(x) \leq \varphi(y) \quad \forall \varphi \in \Gamma. \tag{2}$$

If in E we define a partial order by $x \leq y$ if and only if $y - x \in K$, then the triple (E, K, Γ) is called an *ordered locally convex* space. An operator $g : E \longrightarrow E$ is said to be *positive* if $g(x) \geq \theta$ for all $x \in K$.

A sequence (x_n), $x_n \in E$, is called *fundamental in the Cauchy sense,* if for any $(\varepsilon, \varphi) \in (0, \infty) \times \Gamma$, there exist $n_0 \in \mathbb{N}$ such that

$$\varphi(x_{n+m} - x_n) < \varepsilon \text{ for all } n \geq n_0,\ m \in \mathbb{N}.$$

The space E is called *sequentially complete* if each fundamental sequence (in the Cauchy sense) is convergent.

Definition 1 ([28]). Let (E, K, Γ) be an ordered locally convex space and X be a real linear space. A mapping $p : X \longrightarrow E$ is called a cone norm (or K-norm) if

(i) $p(x) \in K$ or equivalently $p(x) \geq \theta_E$ $\forall x \in X$ and $p(x) = \theta_E$ if and only if $x = \theta_X$, where θ_E, θ_X are the zero elements of E and X respectively,

(ii) $p(\lambda x) = |\lambda| p(x)$ $\forall \lambda \in \mathbb{R}, \forall x \in X$,

(iii) $p(x+y) \leq p(x) + p(y)$ $\forall x, y \in X$.

If p is a cone norm in X, then the pair (X, p) is called a cone normed space (or K-normed space). The cone normed space (X, p) endowed with a topology τ will be denoted by (X, p, τ), and it is said to be *sequentially* *complete* in the Weierstrass sense if each sequence $\{x_n\}$, $x_n \in X$, such that $\sum_{n=1}^{\infty} p(x_{n+1} - x_n)$ converges in E is convergent in (X, p, τ).

In what follows, we always suppose that (E, K, Γ) is an ordered locally convex space with the family of seminorms Γ satisfies the condition (2) and (X, p, τ) is a cone normed space with the topology τ defined by the family of seminorms $\{\varphi \circ p : \varphi \in \Gamma\}$. It is easily seen that a sequence $\{x_n\}$, $x_n \in X$, converges to x in (X, p, τ) if and only if $\{\varphi p(x_n - x)\}$ converges to 0 in $\mathbb{R}$ for every $\varphi \in \Gamma$, or, equivalently,

$$x_n \xrightarrow{\tau} x \Leftrightarrow (\lim \varphi p(x_n - x) = 0 \; \forall \varphi \in \Gamma).$$

Definition 2. A operator $T : C \subset X \to X$ is said to be uniformly continuous on C if for every $(\varphi, \varepsilon) \in \Gamma \times (0, \infty)$ there exists $\left(\phi, \delta\right) \in \Gamma \times (0, \infty)$ such that

$$\varphi p(Tx - Ty) < \varepsilon \text{ if } \phi p(x - y) < \delta \; (x, y \in C).$$

3. Main results.

Theorem 3. *Let (E, K, Γ) be sequentially complete in the Cauchy sense and (X, p, τ) be sequentially complete in the Weierstrass sense. Assume that C is a closed subset in (X, p, τ) and $T : C \to X$ is an operator satisfying the following conditions:*

(1) T is uniformly continuous on C and $T_z(x) = T(x) + z \in C$ for all $x, z \in C$,

(2) there is a sequence of positive continuous operators $\{Q_n : E \to E\}_{n=1,2,...}$ such that the following conditions hold:

(2a) $\forall (\varphi, \xi) \in \Gamma \times K$, the series $\sum_{n=1}^{\infty} \varphi[Q_n(\xi)]$ is convergent in $\mathbb{R}$,

(2b) $\forall (\varphi, \varepsilon) \in \Gamma \times (0, \infty)$ and $z \in C$, there exists $(\delta, r) \in (0, \varepsilon) \times \mathbb{N}^*$ such that if $\varphi p(x - y) < \delta + \varepsilon$, then $\varphi p(T_z^r(x) - T_z^r(y)) < \varepsilon$ for all $x, y \in C$,

(2c) $\forall \varphi \in \Gamma$, $\exists \varphi' \in \Gamma$ $(\varphi' \geq \varphi)$ such that

$$\varphi[p(T_z^n(x) - T_z^n(y))] \leq \varphi'(Q_n[p(x - y)]) \; \forall n \in \mathbb{N}^*, x, y, z \in C.$$

Then the operator $(I - T)^{-1}$ is well defined and continuous on C.

Proof. We first prove that for any $z \in C$, the operator T_z has a unique fixed point in C. In addition, for any $x \in C$ the iterated sequence $\{T_z^n(x)\}_n$ converges to this fixed point.

Starting with fixed element $x_0 \in C$, we construct the iterated sequence $x_n = T_z(x_{n-1})$, $n = 1, 2, ...$ By induction we have $x_n = T_z^n(x_0)$, $x_{n+1} = T_z^n(x_1)$. For every $(\varphi, \varepsilon) \in \Gamma \times (0, \infty)$, from the hypothesis (2c), $\exists \varphi' \in \Gamma$ such that

$$\varphi p(x_n - x_{n+1}) = \varphi p(T_z^n(x_0) - T_z^n(x_1)) \leq \varphi' Q_n p(x_0 - x_1).$$

By the hypothesis (2a) if follows that $\sum_{n=0}^{\infty} \varphi' Q_n p(x_0 - x_1) < \infty$. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{n+m} \varphi' Q_k p(x_0 - x_1) < \varepsilon \text{ for all } n \geq n_0, m \in \mathbb{N}.$$

Let $s_n = \sum_{k=0}^{n} p(x_k - x_{k+1})$. We have

$$\begin{aligned}\varphi(s_{n+m} - s_n) &= \varphi\left[\sum_{k=n+1}^{n+m} p(x_k - x_{k+1})\right] \\ &\leq \sum_{k=n+1}^{n+m} \varphi p(x_k - x_{k+1}) \leq \sum_{k=n+1}^{n+m} \varphi' Q_k p(x_0 - x_1) < \varepsilon\end{aligned}$$

for $n \geq n_0$, $m \in \mathbb{N}$. Hence, $\{s_n\}$ is fundament in E. Since E is sequentially complete, we see that the series $\sum_{n=0}^{\infty} p(x_n - x_{n+1})$ is convergent. By the sequentially complete property (in the Weierstrass sense) of (X, p, τ) shows that $\exists x_* \in X$ such that $x_n \xrightarrow{\tau} x_*$. We have

$$\begin{aligned}\varphi p(x_* - T_z(x_*)) &\leq \varphi p(x_* - x_{n+1}) + \varphi p(x_{n+1} - T_z(x_*)) \\ &= \varphi p(x_* - x_{n+1}) + \varphi p(T_z(x_n) - T_z(x_*)) \\ &\leq \varphi p(x_* - x_{n+1}) + \varphi' Q_1(p(x_n - x_*)). \end{aligned} \tag{3}$$

By letting $n \to \infty$ in (3) we conclude that $T_z(x_*) = x_*$. We shall prove that x_* is unique. Indeed, if we also have $T_z(a) = a$, then for every $\varphi \in \Gamma$, by condition (2c), $\exists \varphi' \in \Gamma$ such that

$$\varphi[p(x_* - a)] = \varphi p(T_z^n(x_*) - T_z^n(a)) \leq \varphi' Q_n p(x_* - a) \ \forall n \in \mathbb{N}^*. \tag{4}$$

Since $\sum_{n=0}^{\infty} \varphi' Q_n p(x_0 - x_1) < \infty$ (by (2a)) shows that $\lim \varphi' Q_n p(x_* - a) = 0$. It follows that $\lim \varphi Q_n p(x_* - a) = 0$ from (4). Hence that $p(x_* - a) = \theta_E$, so $x_* = a$. We have thus proved that the operator T_z has a unique fixed point for all $z \in C$. From this it follows that $(I - T)^{-1} = \phi$ is well defined on C, where $\phi(z)$ is a fixed point of T_z.

We next prove that $(I - T)^{-1}$ is continuous on C. Fix $y \in C$, set $x = \phi(y)$, for every $(\varphi, \varepsilon') \in \Gamma \times (0, \infty)$, we shall construct a neighborhood V of θ_E such that

$$\text{if } y' \in C,\ x' = \phi(y') \text{ and } p(y' - y) \in V \text{ imply that } \varphi(p(x - x')) < \varepsilon'. \tag{5}$$

Indeed, by the condition (2b) for $\varepsilon = \frac{1}{3}\varepsilon'$, $\exists (\delta, r) \in (0, \varepsilon) \times \mathbb{N}^*$ such that

$$\text{if } \varphi p(a - b) < \delta + \varepsilon \text{ implies that } \varphi p(T_z^r(a) - T_z^r(b)) < \varepsilon \ \forall z \in C. \tag{6}$$

Let $\phi_0 = \varphi$, $\delta_0 = \delta$ and $\delta_0' = \dfrac{1}{2}\delta_0$. By uniformly continuous of T, for the pair $\left(\phi_0, \delta_0'\right)$, we find a pair $\left(\phi_1, \delta_1\right) \in \Gamma \times \left(0, \delta_0'\right)$ such that

$$\phi_1[p(a - b)] < \delta_1 \Rightarrow \phi_0 p[T(a) - T(b)] < \delta_0' \ (a, b \in C). \tag{7}$$

By induction, we can construct a family $\left\{\left(\phi_j, \delta_j, \delta_j'\right)\right\}_{i=0,1,\ldots,r-1}$ satisfying the following conditions

$$\delta_j < \delta_{j-1}',\ \delta_j' + \delta_j' \leq \delta_j \ \forall j = 1, 2, \ldots, r - 1 \tag{8}$$

and

$$\phi_j p(a - b) < \delta_j \Rightarrow \phi_{j-1} p[T(a) - T(b)] < \delta_{j-1}' \ \forall j = 1, 2, \ldots, r - 1. \tag{9}$$

Now, set $\delta_* = \delta_{r-1}$, and

$$V = \left\{\xi \in E : \phi_{r-j}(\xi) < \delta_* \ \forall j = \overline{1,r}\right\},$$

Clearly, V is a neighborhood of θ_E. We can prove this by induction that

$$\phi_{r-k}\left[p\left(T_y^k(z) - T_{y'}^k(z)\right)\right] < \delta_{r-k} \ \forall z \in C, \forall k \in \{1, 2, ..., r\} \tag{10}$$

where $y' \in C$ and $p(y - y') \in V$. Indeed, we have

$$\begin{aligned}\phi_{r-1}\left[p\left(T_y(z) - T_{y'}(z)\right)\right] &= \phi_{r-1}(y - y') \\ &= \phi_{r-1}p(y - y') < \delta_* = \delta_{r-1}.\end{aligned}$$

Hence, (10) holds for $k = 1$. Assume that (10) holds for $k \in \{1, 2, ..., j\}$, i.e.

$$\phi_{r-j}\left[p\left(T_y^j(z) - T_{y'}^j(z)\right)\right] < \delta_{r-j} \ \forall z \in C. \tag{11}$$

We have

$$\begin{aligned}T_y^{j+1}(z) - T_{y'}^{j+1}(z) &= T_y\left(T_y^j(z)\right) - T_{y'}\left(T_{y'}^j(z)\right) \\ &= T(a) - T(b) + y - y', \end{aligned} \tag{12}$$

here $a = T_y^j(z)$, $b = T_{y'}^j(z)$. Combining the hypothesis (11) with (9) we have

$$\phi_{r-j-1}p\left[T(a) - T(b)\right] < \delta'_{r-j-1} \tag{13}$$

Since $p(y' - y) \in V$, it follows that $\phi_{r-j-1}p(y - y') < \delta_* \leq \delta'_{r-j-1}$. From (12) and (13) we deduce that

$$\phi_{r-j-1}p\left(T_y^{j+1}(z) - T_{y'}^{j+1}(z)\right) \leq \delta'_{r-j-1} + \delta'_{r-j-1} < \delta_{r-j-1}. \tag{14}$$

Hence (10) holds. In particular, for $k = r$ we have

$$\varphi\left[p\left(T_y^r(z) - T_{y'}^r(z)\right)\right] < \delta \ \forall z \in C. \tag{15}$$

We now prove by induction that

$$\varphi\left[p\left(T_y^{rn}(z) - T_{y'}^{rn}(z)\right)\right] < \delta + \varepsilon \ \forall z \in C, \forall n \in \mathbb{N}^*. \tag{16}$$

Indeed, since (15) shows that (16) holds for $n = 1$. Assume that (16) holds for $k = n$, i.e.

$$\varphi\left[p\left(T_y^{rk}(z) - T_{y'}^{rk}(z)\right)\right] < \delta + \varepsilon \ \forall z \in C. \tag{17}$$

We have

$$p\left(T_y^{r(k+1)}(z) - T_{y'}^{r(k+1)}(z)\right) = p\left(T_y^r\left(T_y^{rk}(z\right) - T_{y'}^r\left(T_{y'}^{rk}(z)\right)\right)$$

$$\leq p\left(T_y^r(a) - T_y^r(b)\right) + p\left(T_y^r(b) - T_{y'}^r(b)\right),$$

here $a := T_y^{rk}(z)$, $b := T_{y'}^{rk}(z)$. Combining (17), (6) and (15) gives

$$\varphi p\left(T_y^{r(k+1)}(z) - T_{y'}^{r(k+1)}(z)\right) < \varepsilon + \delta \text{ whenever } p(y - y') \in V. \tag{18}$$

This completes the induction process. Now, $y' \in C$, $x' = \phi(y)$, $p(y - y') \in V$. From $T^{nr}_{y'}(x) \xrightarrow{\tau} x'$ $(n \to \infty)$ there exists a number $n_{y'} \in \mathbb{N}^*$ such that

$$\varphi\left[p\left(T^{n_{y'}r}_{y'}(x) - x'\right)\right] < \varepsilon. \tag{19}$$

We have

$$p(x - x') \leq p\left(T^{n_{y'}r}_{y}(x) - T^{n_{y'}r}_{y'}(x)\right) + p\left(T^{n_{y'}r}_{y'}(x) - x'\right). \tag{20}$$

Combining (16), (19) and (20) yields

$$\varphi\left[p(x - x')\right] < \varepsilon + \delta + \varepsilon < \varepsilon'.$$

Therefore, (5) holds. This shows that $(I - T)^{-1}$ is continuous on C . The theorem is proved. □

Theorem 4. *Let (E, K, Γ) be sequentially complete in the Cauchy sense and (X, p, τ) be sequentially complete in the Weierstrass sense. Assume that $T, S : X \longrightarrow X$ are operators such that*

(1) T is uniformly continuous, S is continuous, $S(C) \subset C$ and $S(C)$ is relatively compact.

(2) There is a sequence of positive continuous operators $\{Q_n : E \to E\}_{n=1,2,...}$ satisfies the conditions (2a), (2b) and (2c) of Theorem 3.

Then the operator $T + S$ has a fixed point.

Proof. By the Theorem 3 the operator $(I - T)^{-1} : X \longrightarrow X$ is well defined and continuous. The operator $(I - T)^{-1} \circ S : C \to X$ is continuous, it shows that $\overline{(I - T)^{-1} \circ S(C)}$ is compact. By the Tychonoff theorem there exists $x \in X$ such that $x = (I - T)^{-1} \circ S(x)$ or equivalently $x = T(x) + S(x)$. □

4. Application.

Let $(F_s, \|.\|_s)_{s\in(0,1]}$ be a family of Banach spaces such that $1 \geq r > s$ implies that $F_r \subset F_s$ and $\|.\|_s \leq \|.\|_r$. Set $F = \cap_{s\in(0,1]} F_s$ and we denote $\Omega = [0, \infty)$. Let $f, g : \Omega \times (F, \|.\|_r) \to (F, \|.\|_s)$ be continuous functions for every (r, s) satisfying $0 < s < r \leq 1$. In this section we study the existence of solution of a Cauchy problem with perturbation, in the scale of Banach spaces $(F_s, \|.\|_s)_{s\in(0,1]}$, of the form

$$\begin{cases} x'(t) = f[t, x(t)] + g[t, x(t)], \ t \in \Omega, \\ x(0) = x_0 \in F_1, \end{cases} \tag{21}$$

where f and g satisfy the following conditions:

(A1) For any $(r, s) \in (0, 1) \times (0, 1)$ satisfying $r > s$, then

$$\left\|f(t, x) - f(t, y)\right\|_s \leq k(r, s)\left\|x - y\right\|_r \text{ for all } t \in \Omega, \tag{22}$$

where $k(r, s) = c(r - s)^{-\mu}$, $(c, \mu) \in (0, \infty) \times (0, 1)$ and

(A2) for every $s \in (0, 1)$ there exist $\xi(s) > 0$ and $l(s) > 0$ such that

$$\left\|x - y\right\|_s < \xi(s) \text{ implies that } \left\|f(t, x) - f(t, y)\right\|_s \leq l(s)\left\|x - y\right\|_s \ (t \in \Omega, x, y \in F), \tag{23}$$

(A3) for every $s \in (0, 1)$ the set $g(I \times F)$ is relatively compact in $(F_s, \|.\|_s)$, where I is an any bounded segment of $[0, \infty)$.

The problem (21) is equivalent to the following integral equation

$$x(t) = x_0 + \int_0^t f(v, x(v))\,dv + \int_0^t g(v, x(v))\,dv := Tx(t) + Sx(t). \tag{24}$$

(where $Tx(t) = x_0 + \int_0^t f(v, x(v))\,dv$, $Sx(t) = \int_0^t g(v, x(v))\,dv$).

Let us denote by (E, K, Γ) the ordered locally convex space, where

$E = \left\{x = \left(x^{(1)}, x^{(2)}, ...\right) : x^{(i)} \in \mathbb{R}\right\}$ equipped with the normal algebraic operations;

$K = \left\{x = \left(x^{(1)}, x^{(2)}, ...\right) : x^{(j)} \geq 0\ \forall j \in \mathbb{N}^*\right\}$ and the topology on E defined by a family of seminorms $\Gamma = \{\varphi_n : E \to \mathbb{R}\}_{n=1,2,...}$, where $\varphi_n(x) = \left|x^{(n)}\right|$. We can verify that (E, K, Γ) is sequentially complete and the condition (2) holds.

Let $(s_n) \subset (0, 1)$ be an sequence such that $s_1 < s_2 < ... < s_n < ...$ and $\lim s_n = 1$. Set

$$X = \{x : \Omega \to (F, \|.\|_s) \mid x \text{ is continuous } \forall s \in (0, 1)\}.$$

For $n \in \mathbb{N}^*$ we write $\Omega_n = [0, n]$ and define $q_n(x) = \sup_{t \in \Omega_n} \|x(t)\|_{s_n}$ and $p(x) = \{q_n(x)\}_{n=1,2,...}$. We will denote by X_n the set of continuous functions from Ω_n to the Banach space F_{s_n}. We can verify that (X_n, q_n) is a Banach space, $q_n = \varphi_n p$, and (X, p, τ) is a cone normed space, where τ is a topology on X defined by family of seminorms $\{\varphi \circ p : \varphi \in \Gamma\}$.

Lemma 5. *1. Let $x : [0, \infty) \to F$ be a function such that $x|_{\Omega_n} \in X_n$ for all $n \in \mathbb{N}^*$, then $x \in X$.*

2. (X, p, τ) is sequentially complete in the Weierstrass sense.

Proof. Fix $s \in (0, 1)$ and $t_0 \in \Omega$. We can choose $n \in \mathbb{N}^*$ so that $s_n > s$ and $t_0 + 1 \in \Omega_n$. By the continuity of $x|_{\Omega_n} : \Omega_n \to F_{s_n}$ and inequality $\|.\|_s \leq \|.\|_{s_n}$ we can prove that x is continuous at t_0. Hence, the first assertion holds. Let $\{x_n\}$ be a sequence in X, and we assume that $\sum_{n=1}^{\infty} p(x_{n+1} - x_n)$ converges in E. From this we see that the sequence $\{S_n\}$, $S_n = \sum_{k=1}^{n} p(x_{k+1} - x_k)$, is convergent in E. For every $a \in \mathbb{N}^*$ we have

$$\varphi_a[S_{n+k} - S_n] = \sum_{j=n+1}^{n+k} q_a\left(x_{j+1} - x_j\right). \tag{25}$$

We conclude from (25) that $\sum_{n=1}^{\infty} q_a(x_{n+1} - x_n)$ converges in $\mathbb{R}$. We have

$$\begin{aligned} q_a(x_n|_{\Omega_a} - x_{n+k}|_{\Omega_a}) &= q_a(x_n - x_{n+k}) \\ &= \varphi_a p(x_n - x_{n+k}) \leq \varphi_a\left(\sum_{j=n}^{n+k-1} p\left(x_j - x_{j+1}\right)\right) \\ &\leq \sum_{j=n}^{n+k-1} \varphi_a p\left(x_j - x_{j+1}\right) = \sum_{j=n}^{n+k-1} q_a\left(x_j - x_{j+1}\right) \end{aligned} \tag{26}$$

It follows from (26) that $\{x_n|_{\Omega_a}\}_n$ is Cauchy in (X_a, q_a), hence that $\exists y_a \in X_a$ such that $x_n|_{\Omega_a} \xrightarrow{\|\cdot\|_{s_a}} y_a$. Now, assume that $a, a' \in \mathbb{N}^*$, $a' > a$, $x_n|_{\Omega_a} \xrightarrow{q_a} y_a$ and $x_n|_{\Omega_{a'}} \xrightarrow{q_{a'}} y_{a'}$. We will prove that $y_{a'}|_{\Omega_a} = y_a$. Indeed, we have

$$\begin{aligned} q_a(y_a - y_{a'}|_{\Omega_a}) &\leq q_a(y_a - x_n|_{\Omega_a}) + q_a(x_n|_{\Omega_a} - y_{a'}|_{\Omega_a}) \\ &\leq q_a(y_a - x_n|_{\Omega_a}) + q_{a'}\left(x_n|_{\Omega_{a'}} - y_{a'}|_{\Omega_{a'}}\right). \end{aligned} \tag{27}$$

Letting $n \to \infty$ in (27) we obtain $y_{a'}|_{\Omega_a} = y_a$. We define $x : \Omega \to F$, $x(t) := y_a(t)$ if $t \in \Omega_a$. Then $x|_{\Omega_n} = y_n \in X_n$ for all $n \in \mathbb{N}^*$. That $x \in X$ follows from the first assertion. Since

$$\begin{aligned} q_a(x_n - x) &= q_a(x_n|_{\Omega_a} - x|_{\Omega_a}) \\ &= q_a(x_n|_{\Omega_a} - y_a) \to 0 \text{ for all } a \in \mathbb{N}^*, \end{aligned}$$

we conclude that $x_n \xrightarrow{\tau} x$. □

Lemma 6. *Let T be defined by (24). Assume that the condition (A2) holds. Then for every $(\varphi_a, n) \in \Gamma \times \mathbb{N}^*$ and for every $z \in X$, there exists $\delta(a,n) > 0$ such that*

$$\left\|(T_z^n x - T_z^n y)(t)\right\|_{s_a} \leq \frac{(l_a t)^n}{n!} q_a(x-y) \text{ if } \varphi_a p(x-y) < \delta(a,n), \tag{28}$$

where $l_a = l(s_a)$, $x, y \in X$, $t \in \Omega$.

Proof. We will prove this lemma by induction. For $n = 1$. Since condition (A2), $\exists \xi_a > 0$ such that

$$\left\|x(v) - y(v)\right\|_{s_a} < \xi_a \text{ implies that } \left\|f(v, x(v)) - f(v, y(v))\right\|_{s_a} \leq l_a \left\|x(v) - y(v)\right\|_{s_a}. \tag{29}$$

Choose $\delta(a,1) \in (0, \xi_a)$ and if $\varphi_a p(x-y) < \delta(a,1)$, then (29) holds. Hence that

$$\begin{aligned} \left\|(T_z x - T_z y)(t)\right\|_{s_a} &\leq \int_0^t \left\|f(v, x(v)) - f(v, y(v))\right\|_{s_a} dv \\ &\leq l_a \int_0^t \left\|x(v) - y(v)\right\|_{s_a} dv \leq l_a t q_a(x-y). \end{aligned}$$

It follows that (28) holds for $n = 1$.

Suppose that, by induction, there exists $\{\delta(a,i)\}_{i=1,2,\ldots,j} \subset (0,\infty)$ such that

$$\varphi_a p(x-y) < \delta(a,j) \Rightarrow \left\|\left(T_z^j x - T_z^j y\right)(t)\right\|_{s_a} \leq \frac{(l_a t)^j}{j!} q_a(x-y). \tag{30}$$

Then for any $(t,s) \in \Omega \times [0,1)$, $z, x, y \in X$ we have

$$\left\|\left(T_z^{j+1} x - T_z^{j+1} y\right)(t)\right\|_s \leq \int_0^t \left\|f\left(v, T_z^j x(v)\right) - f\left(v, T_z^j y(v)\right)\right\|_s dv. \tag{31}$$

Choose $\delta(a, j+1) \in \left(0, \min\left\{\frac{\xi_a j!}{(l_a a)^j}, \delta(a,j)\right\}\right)$, then, if

$$\varphi_a p(x-y) < \delta(a, j+1),$$

by (30), then

$$\begin{aligned} \left\|\left(T_z^j x - T_z^j y\right)(t)\right\|_{s_a} &\leq \frac{(l_a t)^j}{j!} q_a(x-y) \\ &\leq \frac{(l_a a)^j}{j!} q_a(x-y) < \xi_a \ \forall t \in \Omega. \end{aligned}$$

This shows that

$$\varphi_a p\left(T_z^j x - T_z^j y\right) < \xi_a.$$

From (31) and (29) we have

$$\begin{aligned}\left\|\left(T_z^{j+1}x - T_z^{j+1}y\right)(t)\right\|_{s_a} &\leq l_a \int_0^t \left\|T_z^j x(v) - T_z^j y(v)\right\|_{s_a} dv \\ &\leq l_a \int_0^t \frac{(k_a t)^j}{j!} q_a(x-y)\, dv \\ &= \frac{(l_a t)^{j+1}}{(j+1)!} q_a(x-y),\end{aligned}$$

which finish the induction process. □

Lemma 7. *Let T be defined by (24). Assume that the condition (A1) holds. Then for every $a, n \in \mathbb{N}^*$ and for every $z \in X$ we have*

$$q_a\left(T_z^n(x) - T_z^n(y)\right) \leq \frac{(k_a a)^n}{n!} q_{a+1}(x-y) \ \ (\forall x, y \in X), \tag{32}$$

where $k_a = k(s_{a+1}, s_a)$.

Proof. By a similar argument as that of Lemma 6 we have

$$\left\|\left(T_z^n x - T_z^n y\right)(t)\right\|_{s_a} \leq \frac{(k_a t)^n}{n!} q_{a+1}(x-y) \ \ \forall x, y \in X \text{ and } t \in \Omega, \tag{33}$$

where $a, n \in \mathbb{N}^*$, $z \in X$. This assertion (33) completes the proof. □

Lemma 8. *Assume that the condition (A1) holds. Then T is uniformly continuous from (X, p, τ) to (X, p, τ).*

Proof. Fix $(\varphi_a, \varepsilon) \in \Gamma \times (0, \infty)$ $(a \in \mathbb{N}^*)$. By the Lemma 7 we see that

$$\varphi_a p(Tx - Ty) \leq k_a a \varphi_{a+1} p(x-y) \ \text{ for all } x, y \in X.$$

Choose $\delta \in \left(0, \varepsilon (k_a a)^{-1}\right)$ and $\phi = \varphi_{a+1}$, then $\phi p(x-y) < \delta$ implies $\varphi_a p(Tx - Ty) < \varepsilon$. It follows that T is uniformly continuous. □

Lemma 9. *Assume that g satisfies the condition (A3) and the operator $S : X \to X$ defined by (24). Then*
1. *S is continuous,*
2. *$S(X)$ is relatively compact in (X, p, τ).*

Proof. 1. Let $\{x_n\}_n \subset X$, and assume that $x_n \xrightarrow{\tau} x$. We will prove that $\varphi_a p(S(x_n) - S(x)) \to 0$ for all $a \in \mathbb{N}^*$. Indeed, set $A = \{x_n(t) : n \in \mathbb{N},\ t \in \Omega_a\}$. Let $\{x_{n_k}(t_k)\}$ be a sequence in A. We can assume that $\{t_k\}_k$ converges to $t \in \Omega_a$. We have

$$\begin{aligned}\left\|x_{n_k}(t_k) - x(t)\right\|_{s_{a+1}} &\leq \left\|x_{n_k}(t_k) - x(t_k)\right\|_{s_{a+1}} + \|x(t_k) - x(t)\|_{s_{a+1}} \\ &\leq \varphi_{a+1} p(x_{n_k} - x) + \|x(t_k) - x(t)\|_{s_{a+1}}.\end{aligned} \tag{34}$$

Since $x : \Omega \to \left(F_{s_{a+1}}, \|.\|_{s_{a+1}}\right)$ is continuous, from (34) it follows that $x_{n_k}(t_k) \overset{\|.\|_{s_{a+1}}}{\longrightarrow} x(t)$ (as $k \to \infty$). From this we see that A is relatively compact in $F_{s_{a+1}}$. Therefore, $B := \Omega_a \times \overline{A}$ is compact in $\Omega \times \left(F_{s_{a+1}}, \|.\|_{s_{a+1}}\right)$. For any $\varepsilon > 0$, since g is uniformly continuous on B, $\exists \delta > 0$ such that

$$\left\|g(v, z) - g(v, z')\right\|_{s_a} < \frac{\varepsilon}{2a} \text{ for all } v \in \Omega_a \text{ if } \|z - z'\|_{s_{a+1}} < \delta. \tag{35}$$

Since $x_n \xrightarrow{\tau} x$, it follows that $\varphi_{a+1}p(x_n - x) \longrightarrow 0$, hence that there exists $N_0 \in \mathbb{N}$ such that $\varphi_{a+1}p(x_n - x) < \delta$ for $n \geq N_0$. Therefore,

$$\|x_n(t) - x(t)\|_{s_{a+1}} < \delta \text{ for all } t \in \Omega,\ n \geq N_0. \tag{36}$$

From (36) we obtain

$$\|S(x_n)(t) - S(x)(t)\|_{s_a} \leq \int_0^t \left\|g(v, x_n(v)) - g(v, x(v))\right\|_{s_a} dv \leq \int_0^t \frac{\varepsilon}{2M} dv < \frac{\varepsilon}{2}. \tag{37}$$

We conclude from (37) that $\varphi_a p(S(x_n) - S(x)) < \varepsilon$.

2. For every $a \in \mathbb{N}^*$, we set $S(X)|_{\Omega_a} = \{S(x)|_{\Omega_a} : x \in X\}$. We first prove two following assertions:
(i) $S(X)|_{\Omega_a}$ is equicontinous on Ω_a,
(ii) for every $t \in \Omega_a$, the set $\{S(x)|_{\Omega_a}(t) : x \in X\}$ is relatively compact in F_{s_a}.

From the assumption (A3), we see that $g(\Omega_a \times F)$ is relatively compact in F_{s_a}. Therefore, $\exists \beta > 0$ such that

$$\left\|g(v, x)\right\|_{s_a} \leq \beta \text{ for all } (v, x) \in \Omega_a \times F. \tag{38}$$

Fix $\delta \in \left(0, \frac{\varepsilon}{\beta}\right)$, for every $x \in X$ and $(t, t') \in \Omega_a \times \Omega_a$ satisfying $|t - t'| < \delta$, then, we have

$$\left\|S(x)|_{\Omega_a}(t) - S(x)|_{\Omega_a}(t')\right\|_{s_a} \leq \int_{\min\{t,t'\}}^{\max\{t,t'\}} \left\|g(v, x(v))\right\|_{s_a} dv \leq |t - t'|\beta < \varepsilon.$$

Hence, $S(X)|_{\Omega_a}$ is equicontinous on Ω_a.

Assume that $n \in \mathbb{N}$ satisfying $n > a$. Let

$$G_a = \overline{co}^{F_{s_a}}[g(\Omega_a \times F)] \cup \left\{\theta_{F_{s_a}}\right\},$$

then G_a is compact in F_{s_a}. We have

$$\{S(x)|_{\Omega_a}(t) : x \in X\} \subset \left\{\int_0^t g(v, x(v))\, dv : x \in X\right\} \subset tG_a.$$

It follows that $\{S(x)|_{\Omega_a}(t) : x \in X\}$ is relatively compact in F_{s_a}. By the results just proved, Theorem Ascoli now shows that $S(X)|_{\Omega_a}$ is relatively compact in (X_a, q_a).

Finally, we shall prove that $S(X)$ is relatively compact in (X, p, τ). Indeed, given any $\{y_n\}_n \subset S(X)$, we can assume that the set $\{y_n : n \in \mathbb{N}\}$ is infinite. For $a = 1$, since $S(X)|_{\Omega_1}$ relatively compact in (X_1, q_1), there exits a subsequence $\left\{y_n^1\right\}_{n=1,2,...}$ of $\{y_n\}_n$ such that $y_n^1|_{\Omega_1} \xrightarrow{q_1} x_1$ $(x_1 \in X_1)$. By induction, we can assume that there exist $\left\{y_n^1\right\}_n, \left\{y_n^2\right\}_n, ..., \left\{y_n^j\right\}_n, ...$ satisfying the following conditions:

(a) $\left\{y_n^k\right\}_n$ is a subsequence of $\left\{y_n^{k-1}\right\}_n$ and

(b) $\left\{y_n^k|_{\Omega_k}\right\}_n \xrightarrow{q_k} x_k$ $(x_k \in X_k)$, $k \in \mathbb{N}^*$.

Assume that $i, j \in \mathbb{N}^*$ satisfying $j \geq i$. Then we have

$$\begin{aligned} q_i\left(x_i - x_j\right) &\leq q_i\left(x_i - y_n^j\right) + q_i\left(y_n^j - x_j\right) \\ &\leq q_i\left(x_i - y_n^j\right) + q_j\left(y_n^j - x_j\right) \overset{n\to\infty}{\to} 0. \end{aligned}$$

This give $x_i = x_j|_{\Omega_i}$. The function $x : \Omega \to F$ defined by $x(t) = x_i(t)$ if $t \in \Omega_i$. We now consider the sequence $\{y_n^n\}_n$. For every $a \in \mathbb{N}^*$, then, $\{y_n^n\}_{n\geq a}$ is subsequence of $\{y_n^a\}_n$. Since $q_a(y_n^a - x) \to 0$, it shows that $q_a(y_n^n - x) \to 0$. Hence, $\{y_n^n\} \xrightarrow{\tau} x$. □

We are now in position to prove the following theorem.

Theorem 10. *Assume that the conditions (A1)-(A3) hold, then the equation (21) has a positive solution.*

Proof. We will prove that $T + S$ has a fixed point in X by using Theorem 4. Let $Q_n : K \to K$ be defined by

$$[Q_n] = dig\left[\left(\frac{(k_a a)^n}{n!}\right)_{a=1,2,\ldots}\right] (n \in \mathbb{N}^*). \tag{39}$$

We will verify the conditions of Theorem 4. For any $(\varphi_a, \xi) \in \Gamma \times K$, we have

$$\sum_{n=1}^{\infty} Q_n\varphi_a(\xi) = \sum_{n=1}^{\infty} \frac{(k_a a)^n}{n!}\xi < \infty,$$

hence that the condition (2a) holds. By Lemma 7 we see that the condition (2c) holds.

For any given $(\varphi_a, \varepsilon) \in \Gamma \times (0, \infty)$ $(a \in \mathbb{N}^*)$ and $x, y, z \in X$, by the Lemma 6, there exists a sequence $\{\delta(a, n)\}_n \subset (0, \infty)$ such that

$$\left\|(T_z^n x - T_z^n y)(t)\right\|_{s_a} \leq \frac{(l_a t)^n}{n!} q_a(x - y) \ \forall t \in \Omega \text{ whenever } \varphi_a p(x - y) < \delta(a, n). \tag{40}$$

Since $\lim_{n\to\infty} \frac{(l_a a)^n}{n!} = 0$, there exists $r \in \mathbb{N}^*$ such that $\frac{(l_a a)^r}{r!} < \frac{1}{3}$. Let $\delta = \min\left\{\frac{\varepsilon}{2}, \delta(a, r)\right\}$. Since (40) we deduce that

$$\varphi_a p(T_z^r x - T_z^r y) < \varepsilon.$$

whenever $\varphi_a p(x - y) < \delta + \varepsilon$.This shows that the condition (2b) holds. The others assumptions of Theorem 4 are verified easy by Lemma 5, Lemma 8 and Lemma 9. □

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