Some Results on Rectifiable Spaces

Jing Zhang\textsuperscript{a}, Jiewen Chen\textsuperscript{a}, Hanfeng Wang\textsuperscript{b}

\textsuperscript{a} School of Mathematics and Statistics, Minnan Normal University, Zhangzhou 363000, P. R. China
\textsuperscript{b}Department of Mathematics, Shandong Agricultural University, Taian 271018, China

Abstract. In this paper, it is mainly proved that (1) if $G$ is a rectifiable space, then $\ell\Pi(G) \leq \ell(G)$; (2) if $G$ is a rectifiable space and $A$ is a discrete rectifiable subspace of $G$, then $|A| \leq \ell(G)$; (3) every locally compact NSS rectifiable space $G$ is first-countable. The above results improve the corresponding results in topological groups.

1. Introduction

A topological space $G$ is said to be a rectifiable space provided that there exists a homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. If $G$ is a rectifiable space, then $\varphi$ is called a rectification on $G$. M.M. Choban \cite{6} proved the next theorem.

**Theorem 1.1.** A topological space $G$ is a rectifiable space if and only if there exist $e \in G$ and two continuous maps $p : G \times G \to G$, $q : G \times G \to G$ such that for any $x \in G$, $y \in G$ the next identities hold:

$$p(x, q(x, y)) = q(x, p(x, y)) = y \quad \text{and} \quad q(x, x) = e.$$ 

In fact, we can assume that $p = \pi_2 \circ q^{-1}$ and $q = \pi_2 \circ \varphi$ in Theorem 1.1. Fixing a point $x \in G$, we get that the maps $f_x, g_x : G \to G$ defined by $f_x(y) = p(x, y)$ and $g_x(y) = q(x, y)$ for each $y \in G$, are homeomorphisms.

The above map $p : G \times G \to G$ will be called multiplication on $G$. Let $G$ be a rectifiable space, and let $p$ be the multiplication on $G$. Therefore, $q(x, y)$ is an element such that $p(x, q(x, y)) = y$. Since $p(x, e) = p(x, q(x, x)) = x$ and $p(x, q(x, e)) = e$, it follows that $e$ is a right neutral element for $G$ and $q(x, e)$ is a right inverse for $x$.

Recall that a topological group $G$ is a group $G$ with a topology such that the product maps of $G \times G$ into $G$ is jointly continuous and the inverse map of $G$ onto itself associating $x^{-1}$ with arbitrary $x \in G$ is continuous. A paratopological group $G$ is a group $G$ with a topology such that the product maps of $G \times G$ into $G$ is jointly continuous. It is well known that rectifiable spaces are good generalization of topological groups. In fact, for a topological group with the neutral element $e$, as it is easy to see, the map $q(x, y) = (x, x^{-1}y)$ is a rectification.
on $G$. The 7-dimensional sphere $S_7$ is a rectifiable space but not a topological group [20]. Further, it is easy to see that paratopological groups and rectifiable spaces are all homogeneous. W. Atiponrat [3] introduced the concept of topological gyrogroups as a generalization of topological groups. In [5], the authors proved that each topological gyrogroup is a rectifiable space.

Cardinal functions are an interesting topic in general topology (see [9, 10]). Many topologists have investigated cardinal invariants in topological groups and paratopological groups extensively ([2, 16, 17]). In 1996, A.S. Gul’ko [8] proves that if $G$ is a rectifiable space, then (1) $\pi\chi(G) = \chi(G)$; (2) $\omega(G) \leq k(G)\chi(G)$; (3) $\omega(G) = \pi\omega(G) = d(G)\chi(G)$. F. Lin, C. Liu and R. Shen in [11–13] also study cardinal functions in rectifiable spaces.

The notations $\omega(G)$, $\pi\omega(G)$, $\chi(G)$ and $\ell(G)$ denote the first infinite ordinal, the weight of a space $G$, the $\pi$-weight of a space $G$, the character of a space $G$ and the Lindelöf number of a space $G$, respectively. The symbol $\kappa$ denotes an infinite cardinal. The letter $e$ denotes the neutral element of a group and the right neutral element of a rectifiable space, respectively. The readers may consult [2, 7] for notation and terminology not explicitly given here. All spaces are assumed to be $T_2$.

2. The Index of Narrowness of Rectifiable Spaces

The next two problems about the index of narrowness was posed by A.V. Arhangel’skii, A. Bella [1] and F. Lin [11], respectively.

**Problem 2.1.** Let $G$ be a paratopological (semitopological) (Hausdorff, regular) group of countable extent. Must $G$ be $\omega$-narrow?

**Problem 2.2.** Is every Souslin rectifiable space $G$ left $\omega$-narrow?

In [19], we prove that if $G$ is a Hausdorff quasitopological group of countable extent, then $G$ is $\omega$-narrow. Recall that a rectifiable space $G$ is said to have the property $*$ if for each open neighborhood $U$ of $e$ in $G$, there exists an open neighborhood $V$ of $e$ in $G$ such that $g(p(x, V), x) \subset U$ for every $x \in G$. In [18], we show that if $G$ is a rectifiable space with the property $*$, then $\mathcal{I}_G \leq \ell(G)$ (The definition of $\mathcal{I}_G$ see below before Theorem 2.11.1). In this section, further, we show that if $G$ is a rectifiable space, then $\mathcal{I}_G \leq \ell(G)$, which improves the related results in [18]. First of all, we give some concepts and technical lemmas. Recall that if $U$ is an open neighborhood of the right neutral element $e$ of a rectifiable space $G$, a subset $A$ of $G$ is called $U$-discrete if $b \notin p(a, U)$, for any distinct $a, b \in A$ [18].

The next Lemma 2.3 and Lemma 2.4 can be found in [18]. For completeness of our proof, the specific proof of the next two lemmas are given again.

**Lemma 2.3.** Let $G$ be a rectifiable space, $A$ be a subset of $G$ and $U$ be an open neighborhood of $e$ in $G$. Then $A$ is $U$-discrete if and only if $q(A, A) \cap U = \{e\}$.

**Proof.** Firstly, we shall verify that if $A$ is $U$-discrete, then we have $q(A, A) \cap U = \{e\}$. Suppose to the contrary that, there are two distinct elements $a, b \in A$ such that $q(a, b) \in U$. Then $b \in p(a, U)$, thus contradicting the assumption that the set $A$ is $U$-discrete.

If $q(A, A) \cap U = \{e\}$, the following we shall verify that $A$ is $U$-discrete. If there exist $a, b \in G$ and $a \neq b$ such that $b \in p(a, U)$, then $e \notin q(a, b) \in U \cap q(A, A)$, which is a contradiction.

**Lemma 2.4.** Let $G$ be a rectifiable space and $A \subset G$. Then for each open neighborhood $U$ of the neutral element $e$ in $G$, $A$ is $U$-discrete if and only if $\overline{A}$ is $U$-discrete.

**Proof.** It is only need to verify that if $A$ is $U$-discrete, then $\overline{A}$ is $U$-discrete. It follows from Lemma 2.3 that $q(A, A) \cap U = \{e\}$. We only need to prove that $q(\overline{A}, \overline{A}) \cap U = \{e\}$. The following we shall prove that for arbitrary distinct elements $a, b \in \overline{A}$, we can conclude $q(a, b) \notin U$.

Case 1: If $a, b \in A \subset \overline{A}$, since $A$ is $U$-discrete, it is obvious that $q(a, b) \notin U$. 

Case 2: If \( a \in \overline{A} \setminus A \), \( b \in A \) and \( q(a, b) = u \in U \), then there exists an open neighborhood \( V \) of \( e \) in \( G \) such that \( q(p(a, V), b) \subset U \). Since \( a \in \overline{A} \), there is \( a' \in p(a, V) \cap A \) where \( a' \neq b \). We have \( e \neq q(a', b) \in q(p(a, V), b) \subset U \), contradicting the assumption that \( A \) is \( U \)-discrete.

Case 3: If \( a, b \in A \) and \( q(a, b) \in U \), then there is an open neighborhood \( V \) of \( e \) in \( G \) such that \( q(p(a, V), b) \subset U \). Since \( b \in A \), there is \( b' \in p(b, V) \cap A \) where \( b' \neq a \). It is easy to see that \( e \neq q(a, b') \in q(p(a, V), b) \subset U \), contradicting the assumption that \( A \) is \( U \)-discrete.

Case 4: If \( a, b \in A \) and \( q(a, b) = u \in U \), then there is an open neighborhood \( W \) of \( e \) in \( G \) such that \( q(p(a, W), p(b, W)) \subset U \). Since \( a, b \in A \), there are \( a' \in p(a, W) \cap A \) and \( b' \in p(b, W) \cap A \) where \( a' \neq b' \). Thus \( e \neq q(a', b') \in U \), contradicting the assumption that \( A \) is \( U \)-discrete. \( \square \)

Recall that if \( A \) is a subspace of a rectifiable space \( G \), then \( A \) is called a rectifiable subspace of \( G \) \([12]\) if we have \( p(A, A) \subset A \) and \( q(A, A) \subset A \).

**Lemma 2.5.** \([12]\) Let \( G \) be a rectifiable space and \( A \) be a rectifiable subspace of \( G \). Then \( \overline{A} \) is also a rectifiable subspace of \( G \).

It is quite easy to verify the next theorem.

**Theorem 2.6.** Let \( G \) be a rectifiable space and \( A \) be a rectifiable subspace of \( G \). Then \( A \) is a discrete rectifiable subspace if and only if \( \overline{A} \) is a discrete rectifiable subspace.

**Proof.** If \( A \) is discrete, then there is an open neighborhood \( U \) of the right neutral element \( e \) in \( G \) such that \( U \cap A = \{e\} \). Since \( A \) is a rectifiable subspace, \( U \cap q(A, A) = \{e\} \). It follows from Lemma 2.4 and Lemma 2.5 that \( \overline{A} \) is \( U \)-discrete rectifiable subspace.

It is easy to see that the following corollaries are true. \( \square \)

**Corollary 2.7.** Every discrete rectifiable subspace \( A \) of a countably compact rectifiable space \( G \) is finite.

**Corollary 2.8.** Every discrete subgroup \( A \) of a countably compact topological group \( G \) is finite.

**Corollary 2.9.** Let \( G \) be a rectifiable space and \( A \) be a discrete rectifiable subspace of \( G \). Then \( |A| \leq \ell(G) \).

**Proof.** By Theorem 2.6, \( \overline{A} \) is a discrete rectifiable subspace of \( H \). Since \( \ell(\overline{A}) \leq \ell(G) \) and \( \overline{A} \) is discrete, \( |A| \leq |\overline{A}| \leq \ell(\overline{A}) \leq \ell(G) \). \( \square \)

**Corollary 2.10.** Let \( G \) be a topological group and \( A \) be a discrete subgroup of \( G \). Then \( |A| \leq \ell(G) \).

Let \( G \) be a rectifiable space and \( N(e) \) the family of open neighborhoods of the right neutral element \( e \) in \( G \). The left index of narrowness \( \text{In}_l(G) \) and the right index of narrowness \( \text{In}_r(G) \) of \( G \) are defined, respectively, as follows:

\[
\text{In}_l(G) = \min\{\kappa \geq \omega : (\forall U \in N(e))(\exists F \subset G)(p(F, U) = G \wedge |F| \leq \kappa)\},
\]

\[
\text{In}_r(G) = \min\{\kappa \geq \omega : (\forall U \in N(e))(\exists F \subset G)(p(U, F) = G \wedge |F| \leq \kappa)\}.
\]

If \( G \) satisfies \( \text{In}_l(G) \leq \kappa(\text{In}_r(G) \leq \kappa) \), then \( G \) is left (right) \( \kappa \)-narrow \([11]\). We also define the index of narrowness of \( G \) by

\[
\text{In}(G) = \text{In}_l(G) \cdot \text{In}_r(G).
\]

Given a space \( X \), we denote by \( e(X) \) the supremum of cardinality of closed discrete subsets of \( X \).

It is proved that the inequality \( \text{In}(H) \leq e(H) \) hold in each topological group \( H \) \([2]\). The next theorem generalizes the above result.

**Theorem 2.11.** Let \( H \) be a rectifiable space. Then \( \text{In}_l(H) \leq e(H) \).
Proof. Let \( \kappa = e(H) \). It suffices to show that \( \text{In}_{1}(H) \leq \kappa \). If \( \text{In}_{1}(H) > \kappa \), then there is an open neighborhood \( U \) of \( e \) in \( H \) such that for an arbitrary subset \( A \subset H \) with \( |A| \leq \kappa \), we have \( G \setminus p(A, U) \neq \emptyset \). The family \( \mathcal{E} \) of all \( U \)-discrete subset of \( H \) is (partially) ordered by inclusion, and the union of any chain of \( U \)-discrete sets is also a \( U \)-discrete set. Therefore, according to Zorn’s Lemma, there exists a maximal element \( A \) of the family \( \mathcal{E} \). It follows from Lemma 2.4 and \( \kappa = e(H) \) that \( A \) is closed and \( |A| \leq \kappa \). Then there is \( x_{1} \in G \setminus p(A, U) \). Since \( x_{1} \notin A \), there is an open neighborhood \( V_{1} \) of \( e \) in \( H \) such that \( V_{1} \subset U \) and \( p(x_{1}, V) \cap A = \emptyset \). Put \( A_{1} = A \cup \{x_{1}\} \). It is clear that \( A_{1} \) is \( V_{1} \)-discrete and \( |A_{1}| \leq \kappa \).

Assume that we have defined open neighborhood \( V_{a} \) of \( e \) and \( V_{q} \)-discrete subset \( A_{a} \) with \( |A_{a}| \leq \kappa \) for some infinite cardinal \( a \). Thus \( G \setminus p(A_{a}, U) \neq \emptyset \). By Lemma 2.4, it is clear that \( A_{a} \) is \( V_{a} \)-discrete, so \( |A_{a}| \leq \kappa \). Then there are \( x_{a+1} \in G \setminus p(A_{a}, U) \) and an open neighborhood \( V_{a+1} \) of \( e \) such that \( p(x_{a+1}, V_{a+1}) \cap A_{a} = \emptyset \), where \( V_{a+1} \subset U \). Put \( A_{a+1} = A_{a} \cup \{x_{a+1}\} \), then \( A_{a+1} \) is \( V_{a+1} \)-discrete. By induction, we shall have a \( V \)-discrete subset \( B \) such that \( |B| > \kappa \) for some open neighborhood \( V \) of \( e \). It follows from Lemma 2.4 that \( B \) is \( V \)-discrete. Since \( |B| > |B| > \kappa \), this contradicts the definition of \( \kappa \). Hence \( \text{In}_{1}(H) \leq \kappa = e(H) \). \( \square \)

At the end of this section, we give two-element properties of \( U \)-discrete subsets preserved by homomorphism maps.

Let \( G, H \) be rectifiable spaces and \( f : G \to H \) be a map from \( G \) to \( H \). The map is called a homomorphism if for arbitrary \( x, y \in G \) we have \( f(p_{G}(x, y)) = p_{H}(f(x), f(y)) \). Moreover, if \( f \) is a one-to-one homomorphism map from \( G \) onto \( H \), then \( f \) is called an isomorphism [11].

Lemma 2.12. ([11]) Let \( G, H \) be rectifiable spaces and \( f : G \to H \) be a homomorphism from \( G \) to \( H \). Then \( f(e_{G}) = e_{H} \) and \( f(q_{G}(x, y)) = q_{H}(f(x), f(y)) \) for arbitrary \( x, y \in G \).

Making use of Lemma 2.3 and the definition of homomorphism, it is easy to deduce the propositions below.

Proposition 2.13. Let \( G, H \) be rectifiable spaces and \( f : G \to H \) be a isomorphism from \( G \) to \( H \). If \( A \) is a \( U \)-discrete subset of \( H \), then \( f^{-1}(A) \) is a \( W \)-discrete subset of \( G \) for each open neighborhood \( W \) of \( e_{G} \) in \( G \) with \( f(W) \subset U \).

Proof. According to Lemma 2.3, it is only need to prove that the equation \( W \cap q_{G}(f^{-1}(A), f^{-1}(A)) = \{e_{G} \} \) hold, where \( e_{G} \) denotes the right neutral element of \( G \). If not, then there exist \( a_{1}, a_{2} \in f^{-1}(A), b_{1}, b_{2} \in A \) and \( w \in W \) where \( w \neq e_{G}, a_{1} \neq a_{2} \) such that \( f(a_{1}) = b_{1}, f(a_{2}) = b_{2} \) and \( q_{G}(a_{1}, a_{2}) = w \). Thus \( f(q_{G}(a_{1}, a_{2})) = f(w) \). Since \( f \) is a homomorphism, \( f(w) = f(q_{G}(a_{1}, a_{2})) = q_{H}(f(a_{1}), f(a_{2})) = q_{H}(b_{1}, b_{2}) \). Since \( f \) is a isomorphism, \( e_{H} \neq f(w) \in f(W) \cap q_{H}(A, A) = U \cap q_{H}(A, A) \), which is a contradiction. \( \square \)

Proposition 2.14. Let \( G, H \) be rectifiable spaces and \( f : G \to H \) be a homomorphism from \( G \) to \( H \). If \( A \) is a \( U \)-discrete subset of \( G \), then \( f(A) \) is a \( V \)-discrete subset of \( H \) for each open neighborhood \( V \) of \( e_{H} \) with \( f^{-1}(V) \subset U \).

Proof. According to Lemma 2.3, it is only need to prove that the equation \( V \cap q_{H}(f(A), f(A)) = \{e_{H} \} \) hold. If not, then there exist \( a, b \in A \) and \( v \in V \) where \( v \neq e_{H}, f(a) \neq f(b) \) such that \( q_{H}(f(a), f(b)) = v \). Since \( f \) is a homomorphism, \( v = q_{H}(f(a), f(b)) = f(q_{G}(a, b)) \). Then \( q_{G}(a, b) \in f^{-1}(V) \cap q_{G}(A, A) \subset U \cap q_{G}(A, A) \), which is a contradiction. \( \square \)

It is well known that every first-countable rectifiable space is metrizable [8]. It is easy to deduce that the following result is true.

Theorem 2.15. Suppose that \( f \) is an open continuous homomorphism of a metrizable rectifiable space \( G \) onto a rectifiable space \( H \), then \( H \) is also a metrizable rectifiable space.

Proof. Since \( f \) is open and continuous, and the space \( G \) is first-countable, the space \( H \) is also first-countable. Thus \( H \) is a metrizable rectifiable space. \( \square \)
3. Locally Compact NSS-Rectifiable Spaces

It is known that in every locally compact totally disconnected topological group $G$, there exists a local base $\mathcal{B}$ of $G$ at $e$ such that every element of $\mathcal{B}$ is an open compact subgroup of $G$.

In [14], the authors proved that every locally $\sigma$-compact rectifiable space with a bc-base is locally compact or zero-dimensional and posed the next question:

**Problem 3.1.** Does each totally disconnected locally compact rectifiable space have an open compact rectifiable subspace?

In this section, we prove that each totally disconnected locally compact rectifiable space have a closed compact rectifiable subspace. We also show that every locally compact NSS rectifiable space $G$ is first-countable.

First of all, we give a simple lemma which will be used in our proof.

**Lemma 3.2.** ([15]) Let $C$ be a compact subset and $F$ be a closed subset of a rectifiable space $G$ such that $C \cap F = \emptyset$. Then there exists an open neighborhood $V$ of $e$ in $G$ such that $p(C,V) \cap F = \emptyset$.

We can conclude the next propositions easily by Lemma 3.2.

**Proposition 3.3.** Let $G$ be a rectifiable space and $C$ be a $\sigma$-compact subset of $G$ and $F$ be a closed subset of $G$ such that $C \cap F = \emptyset$. Then there exists a $G_\sigma$-subset of every open neighborhood $V$ of $e$ such that $p(C,V) \cap F = \emptyset$.

**Proof.** Assume that $C = \bigcup_{n \in \mathbb{N}} C_n$, where each $C_n$ is compact, there exists an open neighborhood $V_a$ of $e$ in $G$ such that $p(C_n,V_a) \cap F = \emptyset$ according to Lemma 3.2. Put $V = \bigcap_{n \in \mathbb{N}} V_n$, then $V$ is a $G_\sigma$-subset of $G$ and $p(C,V) \cap F = \emptyset$, which completes our proof. \qed

**Proposition 3.4.** Let $G$ be a rectifiable space, and let $C$ be a non-empty compact $G_\sigma$-set in $G$. Then there exists a $G_\sigma$-set $V$ in $G$ such that $e \in V$ and $p(C,V) \subset C$.

**Proof.** Let $C = \bigcap \gamma$, where $\gamma$ is a family of open subsets of $G$ and $|\gamma| \leq \kappa$. Take any $U \in \gamma$. By Lemma 3.2, there exists an open neighborhood $V_U$ of $e$ such that $p(C,V_U) \subset U$. Put $V = \bigcap \{V_U : U \in \gamma\}$. Then $V$ is a $G_\sigma$-subset of $G$ such that $e \in V$ and $p(C,V) \subset C$. \qed

The following theorem implies that each totally disconnected locally compact rectifiable space have a closed compact rectifiable subspace.

**Theorem 3.5.** Suppose that $G$ is a rectifiable space and $C$ is a compact neighborhood of $e$ in $G$. Then there exists a closed compact rectifiable subspace $H$ of $G$ such that $H \subset C$.

**Proof.** Since $C$ is a compact neighborhood of $e$ in $G$, we can choose an open neighborhood $V_1$ of $e$ in $G$ such that $V_1 \subset \overline{V_1} \subset C$. Assume that open neighborhoods $V_i$ of $e$ in $G$ are defined for each $i = 1, 2, \cdots, n$. Then there is an open neighborhood $V_{n+1}$ of $e$ in $G$ such that $p(V_{n+1},V_{n+1}) \subset V_n, q(V_{n+1},V_{n+1}) \subset V_n$ and $\overline{V_{n+1}} \subset V_n$. Put $H = \bigcap_{n \in \mathbb{N}} V_n$. Then $H = \bigcap_{n \in \mathbb{N}} V_n$. It is easy to check that $H$ is a closed rectifiable subspace and $H \subset C$. Indeed, we only need to verify that $H$ is a rectifiable subspace. For arbitrary $x, y \in H$ and for each $n \in \mathbb{N}$, we have $x, y \in V_n$. Therefore $p(x,y) \in p(V_{n+1},V_{n+1}) \subset V_n$ and $q(x,y) \in q(V_{n+1},V_{n+1}) \subset V_n$, which implies that $p(H,H) \subset H$ and $q(H,H) \subset H$. Since $C$ is compact and $H$ is closed contained in $C$, $H$ is a compact rectifiable subspace contained in $C$. \qed

It is clear that the next corollary gives a partial answer to Problem 3.1. We call a space $X$ a $P$-space if every $G_\sigma$-set in $X$ is open.

**Corollary 3.6.** Suppose that $G$ is a totally disconnected locally compact rectifiable space and $G$ is a $P$-space. Then there exists a local base $\mathcal{B}$ of $G$ at the right neutral element $e$ such that each element of $\mathcal{B}$ is an open compact rectifiable subspace.
Proof. Since $G$ is a totally disconnected locally compact space, there exists a base $\mathcal{P}$ of $G$ at $e$ consisting of open compact subsets of $G$. It follows from Theorem 3.5 that for every $V \in \mathcal{P}$, there is a compact rectifiable subspace $H_V$ which is also a $G_\delta$-subset in $G$ such that $H_V \subset V$. Let $\mathcal{B} = \{H_V : V \in \mathcal{P}\}$. Since $G$ is a $P$-space, $\mathcal{B}$ is a base of $G$ at $e$. □

Let $G$ be a rectifiable space. We say that $G$ is a rectifiable space with no small rectifiable subspace or, for brevity, an NSS-rectifiable space if there exists a neighborhood $U$ of $e$ such that every rectifiable subspace $H$ of $G$ contained in $U$ is trivial, that is, $H = \{e\}$.

**Theorem 3.7.** If $G$ is a NSS-rectifiable space, then the following two conditions are equivalent:

1. there exists a non-empty compact $G_\delta$-set $C$ in $G$;
2. the right neutral element $e$ of $G$ is a $G_\delta$-point in $G$.

**Proof.** It is only need to show that the first condition implies the second one. Since the space $G$ is homogeneous, we can assume that $e \in C = \bigcap_{n \in \mathbb{N}} U_n$. Since $G$ is a NSS-rectifiable space, there exists a neighborhood $U$ of $e$ such that every rectifiable subspace $H$ of $G$ contained in $U$ is trivial. Let $C' = \bigcap_{n \in \mathbb{N}} (U_n \cap U)$. It is clear that $C' \subset C \cap U$. Buy the regularity of $G$, there exists an open neighborhood $V$ of $e$ in $G$ such that $\overline{V_n} \subset U_n \cap U$, for each $n \in \mathbb{N}$. Let $C'' = \bigcap_{n \in \mathbb{N}} \overline{V_n}$. It is easy to see that $C'' = \bigcap_{n \in \mathbb{N}} \overline{V_n} \subset C \cap U$. Thus $C''$ is a non-empty compact $G_\delta$-set contained in $U$. There is an open neighborhood $W_1' \subset V_1$. Assume that open neighborhoods $W_i'$ of $e$ in $G$ are defined for each $i = 1, 2, \ldots, n$. Then there is an open neighborhood $W_n'\subset G$ such that $p(W_n', W_n') \subset W_n'$, $q(W_n', W_n') \subset W_n$ and $\overline{W_n} \subset W_n'$. Put $H_1 = \bigcap_{n \in \mathbb{N}} W_n'$. It is clear that $H_1$ is a closed rectifiable subspace contained in $V_1$. Similarly, we can define closed rectifiable subspace $H_n$ contained in $V_n$ for each $n = 2, 3, \ldots$. Put $H = \bigcap_{n \in \mathbb{N}} H_n$. It is clear that $H$ is a $G_\delta$-set and is a rectifiable subspace contained in $C''$. Since $C'' \subset U$ and $G$ is NSS, $H = \{e\}$. This completes the proof. □

**Theorem 3.8.** Every locally compact NSS rectifiable space $G$ is first-countable.

**Proof.** Clearly, each locally compact regular space contains a non-empty compact $G_\delta$-set. Therefore, according to Theorem 3.7, the right neutral element $e$ of $G$ is a $G_\delta$-point in $G$. Since $G$ is locally compact and Hausdorff, $G$ is first-countable at $e$. Hence, by homogeneity, the space $G$ is first-countable. □

In [5], the authors proved that each topological group is a topological gyrogroup and each topological gyrogroup is a rectifiable space. It is easy to see that the following corollary is true.

**Corollary 3.9.** ([4]) Every locally compact NSS-gyrogroup $(G, \tau, \oplus)$ is first-countable.

---

**References**


