P-Hirano Inverses in Rings

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Abstract. We introduce and study a new class of generalized inverses in rings. An element \( a \) in a ring \( R \) has p-Hirano inverse if there exists \( b \in R \) such that \( bab = b, b \in \text{comm}^2(a), (a^2 - ab)^k \in J(R) \) for some \( k \in \mathbb{N} \). We prove that \( a \in R \) has p-Hirano inverse if and only if there exists \( p = p^2 \in \text{comm}^2(a) \) such that \( (a^2 - p)^k \in J(R) \) for some \( k \in \mathbb{N} \). Multiplicative and additive properties for such generalized inverses are thereby obtained. We then completely determine when a \( 2 \times 2 \) matrix over local rings has p-Hirano inverse.

1. Introduction

Let \( R \) be an associative ring with an identity. The commutant of \( a \in R \) is defined by \( \text{comm}(a) = \{ x \in R \mid xa = ax \} \). The double commutant of \( a \in R \) is defined by \( \text{comm}^2(a) = \{ x \in R \mid xy = yx \text{ for all } y \in \text{comm}(a) \} \). Following Wang and Chen [8], an element \( a \) in \( R \) has p-Drazin inverse (that is, pseudo Drazin inverse) if there exists \( b \in R \) such that

\[
b = bab, b \in \text{comm}^2(a), a^k - a^{k+1}b \in J(R)
\]

for some \( k \in \mathbb{N} \). Here, \( J(R) \) denotes the Jacobson radical of the ring \( R \). The preceding \( b \) is unique, if such element exists, and called the p-Drazin inverse of \( a \) and denote \( b \) by \( a^{pD} \). Pseudo Drazin inverses in a ring are extensively studied in both matrix theory and Banach algebra (see [2, 3, 8, 11] and [12]). Recently, Mosić [6], has introduced and studied new classes of generalized Drazin inverses and pseudo n-strong Drazin inverses.

We shall see that \( a \in R \) has p-Drazin inverse if and only if there exists \( b \in R \) such that

\[
b = bab, b \in \text{comm}^2(a), (a - a^2b)^k \in J(R)
\]

for some \( k \in \mathbb{N} \) (see Lemma 2.1). The motivation of this paper is to discuss the dual of pseudo Drazin inverses in a ring. We introduce and study a new class of generalized inverses in a ring. An element \( a \in R \) has pseudo Hirano inverse (p-Hirano inverse as an abbreviation) if there exists \( b \in R \) such that

\[
b = bab, b \in \text{comm}^2(a), (a^2 - ab)^k \in J(R)
\]

for some \( k \in \mathbb{N} \). We shall prove that the preceding \( b \) is unique, if such element exists, and call \( b \) the p-Hirano inverse of \( a \), and denote \( b \) by \( a^{ph} \).

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In Section 2, the relations of p-Hirano and p-Drazin inverses are obtained, and we prove that \( a \in R \) has p-Hirano inverse if and only if there exists \( p = p^2 \in \text{comm}^2(a) \) such that \( (a^2 - p)^k \in J(R) \) for some \( k \in \mathbb{N} \).

Let \( a, b \in R \). Then \( ab \) has p-Drazin inverse if and only if \( ba \) has p-Drazin inverse and \( (ba)^{pd} = b((ab)^{pd})a \). This was known as Cline’s formula for p-Drazin inverses (see [5, 6, 8]). In Section 3, we establish Cline’s formula for p-Hirano inverses. Further, we explore multiplicative property of p-Hirano inverses for elements in a Banach algebra.

In [7, Theorem 2.6], the authors proved that for any elements \( a, b \in R, 1 - ab \) has p-Drazin inverse if and only if \( 1 - ba \) has p-Drazin inverse. This is the extension of Jacobson’s lemma for Drazin and generalized Drazin inverses in a ring (see [6, 7, 10]). In Section 4, we investigate Jacobson’s lemma for p-Hirano inverses and prove that \( 1 - ab \in R \) has p-Hirano inverse if and only if \( 1 - ba \in R \) has p-Hirano inverse.

A ring \( R \) is local if \( R \) has only one maximal right ideal. Finally, in the last section, we completely determine when a \( 2 \times 2 \) matrix over local rings has p-Hirano inverse. This provides many \( 2 \times 2 \) matrices over a local ring additionally generated by tripotent and nilpotent matrices.

Throughout the paper, all rings are associative with an identity and all Banach algebras are complex. \( R^{nil} = \{ a \in R \mid 1 + ax \in U(R) \text{ for every } x \in \text{comm}(a) \} \). We use \( N(R) \) to denote the set of all nilpotent elements in \( R \). \( \mathbb{N} \) stands for the set of all natural numbers and \( GL_2(R) \) is the group of \( 2 \times 2 \) invertible matrices over \( R \).

2. Pseudo Drazin inverses

The goal of this section is to investigate elementary properties of p-Hirano inverses and explore the relations between p-Drazin and p-Hirano inverses. We begin with

**Lemma 2.1.** Let \( R \) be a ring and \( a \in R \). Then \( a \) has p-Drazin inverse if and only if there exists \( b \in R \) such that

\[
b = bab, b \in \text{comm}^2(a), (a - a^2b)^k \in J(R)
\]

for some \( k \in \mathbb{N} \).

**Proof.** \( \Rightarrow \) Since \( a \) has p-Drazin inverse, we can find \( b \in \text{comm}^2(a) \) such that \( b = bab \) and \( a^k - a^{k+1} b \in J(R) \) for some \( k \in \mathbb{N} \). Hence, \( a^k(a - a^2b) \in J(R) \). This implies that \( (a - a^2b)^k = a^{k-1}(a - a^2b)(1 - ab) \in J(R) \), as desired.

\( \Leftarrow \) By hypothesis, there exists \( b \in R \) such that \( b = b^2a, b \in \text{comm}^2(a), (a - a^2b)^k \in J(R) \) for some \( k \in \mathbb{N} \). Then we have \( c \in R \) such that \( (a^k - a^{k+1}b) - (a - a^2b)^k = a^k(a - a^2b) - (a - a^2b)^k = (a - a^2b)(a^{k-2} - (a - a^2b)^{k-1}) = (a - a^{2}b)a^2bc = (ab - a^2b^2)a^2c = 0 \). Thus \( a^k - a^{k+1}b = (a - a^2b)^k \in J(R) \), as required. \( \square \)

**Lemma 2.2.** Let \( R \) be a ring and \( a \in R \). If \( a \) has p-Hirano inverse, then it has p-Drazin inverse.

**Proof.** Let \( a \in R \). Then there exists some \( b \in \text{comm}^2(a) \) such that \( bab = b \) and \( (a^2 - ab)^n \in J(R) \). Hence \( (a^2 - a^2b^2)^n = (a^2 - a(bab))^n = (a^2 - ab)^n \in J(R) \). So we have

\[
(a^2(1 - a^2b^2)^n) = (a^2 - a^2b^2)^n(1 - a^2b^2)^n \in J(R).
\]

It follows that

\[
(a(a - a^2b))^n = (a^2(1 - a^2b^2)^n) \in J(R).
\]

We conclude that

\[
(a - a^2b)^{2n} = (a(a - a^2b)(1 - ab))^n \in J(R).
\]

Since \( b \in \text{comm}^2(a) \) and \( bab = b \) we deduce that \( a \) has p-Drazin inverse by Lemma 2.1. \( \square \)

Let \( R \) be a ring and \( a \in R \). Since the p-Drazin inverse of an element is unique, we see that \( a \) has at most one p-Hirano inverse in \( R \), and if the p-Hirano inverse of \( a \) exists, it is exactly its p-Drazin inverse and we denote it by \( a^{ph} \). Now we give the relations of p-Hirano and p-Drazin inverses.

**Theorem 2.3.** Let \( R \) be a ring and \( a \in R \). Then \( a \) has p-Hirano inverse if and only if
(1) $a$ and $-a$ have $p$-Drazin inverses;
(2) $(a - a^3)^k \in J(R)$ for some $k \in \mathbb{N}$.

Proof. $\implies$ Clearly, $a$ has $p$-Hirano inverse if and only if so does $-a$. Thus, $a$ and $-a$ have $p$-Drazin inverses by Lemma 2.2.

$\Leftarrow$ By [8, Theorem 1.4], there exist idempotents $e, f \in \text{comm}^2(a)$ such that $a - e, a + f \in U(R)$. Clearly, $a, e, f$ commute with each other. Since $(ae)^k, (af)^k \in J(R)$, we have

$$(ae - af)^{2k+1} = \sum_{i=0}^{2k+1} \binom{2k+1}{i} (ae)^i (af)^{2k+1-i} (-af)^i \in J(R).$$

As $a + e, a + f \in U(R)$, we see that,

$$[(a + f)^{-1}(a - e)^{-1}(ae - af)]^{2k+1} \in J(R);$$

hence,

$$1 - [(a + f)^{-1}(a - e)^{-1}(ae - af)]^{2k+1} \in U(R).$$

This implies that

$$1 - (a + f)^{-1}(a - e)^{-1}(ae - af) \in U(R).$$

Hence,

$$u = a^2 - ef = (a - e)(a + f) - af + ae = (a - e)(a + f)[1 - (a + f)^{-1}(a - e)^{-1}(ae - af)] \in U(R).$$

By hypothesis, $a - a^3 \in N(R/J(R))$, and so $a^2 - a^4 \in N(R/J(R))$. In view of [9, Lemma 3.5], we can find $g \in \mathbb{Z}[a]$ such that $(a^2 - g)^m, g - g^2 \in J(R)$ for some $m \in \mathbb{N}$. Hence, $(u + ef - g)^m \in J(R)$. This shows that $ef - g \in U(R)$. Clearly, $(ef - g)^3 - (ef - g) \in J(R)$, we see that $(ef - g)((ef - g)^2 - 1) \in J(R)$. Hence $g - (1 - 2ef)(1 - ef) \in J(R)$. This implies that $(a^2 - (1 - ef))^m \in J(R)$. We complete the proof by Lemma 2.2.

A ring $R$ is strongly 2-nil-clean if every element in $R$ is the sum of a tripotent and a nilpotent that commute (see [1]). It follows by Theorem 2.3 that a ring $R$ is strongly 2-nil-clean if and only if every element in $R$ has $p$-Hirano inverse and $J(R)$ is nil.

Lemma 2.4. Let $R$ be a ring and $a \in R$. Then the following are equivalent:

(1) $a$ has $p$-Hirano inverse.
(2) There exists $b \in R$ such that

$$b = ba^2b, b \in \text{comm}^2(a), (a^2 - a^2b)^k \in J(R)$$

for some $k \in \mathbb{N}$.

Proof. It follows from [6, Theorem 3.2].

We are now ready to prove the following.

Theorem 2.5. Let $R$ be a ring and $a \in R$. Then the following are equivalent:

(1) $a$ has $p$-Hirano inverse.
(2) There exists $p^2 = p \in \text{comm}^2(a)$ such that $(a^2 - p)^k \in J(R)$ for some $k \in \mathbb{N}$.

Proof. It is obvious by [6, Theorem 3.1].

Corollary 2.6. Let $R$ be a ring and $a \in R$. Then the following are equivalent:

(1) $a$ has $p$-Hirano inverse.
(2) There exists $b \in R$ such that

$$ab = (ab)^2, b \in \text{comm}^2(a), (a^2 - ab)^k \in J(R)$$

for some $k \in \mathbb{N}$. 

3. Multiplicative Property

The aim of this section is to generalize Cline’s formula from p-Drazin inverses to p-Hirano inverses. We record [8, Theorem 2.1].

**Lemma 3.1.** Let $R$ be a ring, and let $a, b \in R$. Then $ab$ has p-Drazin inverse if and only if $ba$ has p-Drazin inverse.

**Theorem 3.2.** Let $R$ be a ring, and let $a, b \in R$. Then $ab$ has p-Hirano inverse if and only if $ba$ has p-Hirano inverse, and

$$(ab)^p = b((ab)^p)^2a.$$  

**Proof.** It is a consequence of [6, Corollary 3.2].

**Corollary 3.3.** Let $R$ be a ring, and let $a, b \in R$. If $(ab)^k$ has p-Hirano inverse, then so is $(ba)^k$.

In a Banach algebra, it shall be suffice to require that the p-Hirano inverse of an element $a$ merely commutes with $a$. That is, we have

**Lemma 3.4.** Let $A$ be a Banach algebra and $a \in A$. Then the following are equivalent:

1. $a$ has p-Hirano inverse.
2. There exists $p^2 = p \in \text{comm}(a)$ such that $(a^2 - p)^k \in J(A)$ for some $k \in \mathbb{N}$.
3. There exists $b \in \text{comm}(a)$ such that
   $$b = bab, (a^2 - ab)^k \in J(A)$$
   for some $k \in \mathbb{N}$.

**Proof.** (1) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (2) Set $p = ab$. Then $p^2 = p \in \text{comm}(a)$ with $(a^2 - p)^k \in J(A)$, as required.

(2) $\Rightarrow$ (1) Set $w = a^2 - p$. Then $w^k \in J(A)$. Let $\lambda \in \mathbb{C}$. Then $(\lambda w)^k = \lambda^k w^k \in J(A)$. Hence, $1 - (\lambda w)^k \in U(A)$. That is,

$$(1 - \lambda w)[1 + \lambda w + (\lambda w)^2 + \cdots + (\lambda w)^{k-1}] \in U(A).$$

This implies that $1 - \lambda w \in U(A)$. By the definition of quasinilpotent element, we get $w \in \text{A}^{qnil}$.

We see that

$$a^2 + (1 - p) = 1 + w \in U(R), a^2(1 - p) = w(1 - p) \in \text{A}^{qnil}, (1 - p)^2 = 1 - p \in \text{comm}(a^2).$$

That is, $1 - p$ is the spectral idempotent of $a^2$. In light of [4, Theorem 7.5.3], we see that $1 - p \in \text{comm}^2(a^2) \subseteq \text{comm}^2(a)$. This implies that $p \in \text{comm}^2(a)$, as desired.

**Theorem 3.5.** Let $A$ be a Banach algebra, and let $a, b \in A$. If $a, b$ have p-Hirano inverses and $ab = ba$, then $ab$ has p-Hirano inverse.

**Proof.** Since $a, b$ have p-Hirano inverses, we can find $x \in \text{comm}^2(a), y \in \text{comm}^2(b)$ such that

$$x = x^2 a, y = y^2 b, (a^2 - ax)^k, (b^2 - by)^k \in J(A)$$

for some $k \in \mathbb{N}$. Obviously, $x, y, a, b$ commute with each other. We easily check that

$$(ab)^2 - (ab)(yx) = -(b^2 - by)(a^2 - ax) + a^2(b^2 - by) + (a^2 - ax)b^2.$$ 

Then we have some $m \in \mathbb{N}$ such that

$$(ab)^2 - (ab)(yx))^m \in J(A).$$

Moreover, we see that $yx = (yx)^2(ab)$ and $yx \in \text{comm}(ab)$. Therefore $ab$ has p-Hirano inverse, by Lemma 3.4.

**Corollary 3.6.** Let $A$ be a Banach algebra. If $a \in A$ has p-Hirano inverse, then $a^n$ has p-Hirano inverse for all $n \in \mathbb{N}$. 

4. Jacobson’s Lemma for p-Hirano inverses

Jacobson’s Lemma states that for any $a, b \in R$, $1 - ab \in R$ is invertible if and only if $1 - ba \in R$ is invertible. An element $a \in R$ has generalized Drazin inverse in case there exists $b \in R$ such that

$$b = bab, b \in \text{comm}^2(a), a - a^2b \in R^\text{nil}.$$

In [10, Theorem 2.1], the authors generalized Jacobson’s Lemma to generalized Drazin invertibility. Motivated by this known theorem, we consider Jacobson’s Lemma for p-Hirano inverses. The main theorem in this section is the following.

**Theorem 4.1.** Let $R$ be a ring, and let $a, b \in R$. Then $1 - ab$ has p-Hirano inverse if and only if $1 - ba$ has p-Hirano inverse, and

$$\left(1 - ba\right)^p = 1 + b((1 - ab)^p - (1 - ab)^q(1 - (1 - ab)^p(1 - ab))^{-1})a,$$

where $(1 - ab)^q = 1 - (1 - ab)^q(1 - ab)$.

**Proof.** $\implies$ Let $a = (1 - ab)^2$. Then $\alpha = 1 - cd$, where $c = a$ and $d = (2 - ba)b$. In view of Lemma 2.4, there exists $x \in R$ such that

$$x = xax, x \in \text{comm}^2(1 - ab), (a - ax)^k \in J(R)$$

for some $k \in \mathbb{N}$. One easily checks that $a(1 - ax) = (a - ax)(1 - ax)$. Hence, $(a - ax)^k = (a - ax)^k(1 - ax) \in J(R)$. Let $\beta = (1 - ba)^2$. Then $\beta = 1 - dc$. Denote $p = 1 - ax$. Since $(a - ax)^k \in J(R)$, we see that $1 - p\alpha = 1 - (a - ax)^k \in U(R)$. Set $q = dp(1 - p\alpha)^{-1}c$. Then

$$q^2 = dp(1 - p\alpha)^{-1}(cd)p(1 - p\alpha)^{-1}c = d(1 - p\alpha)^{-1}p(1 - p\alpha)^{-1}pc = q.$$

Clearly, $\beta d = (1 - dc)d = d(1 - cd) = dx$ and $\beta\beta = a(1 - dc) = a(1 - da) = 1 - d\alpha = 1 - cd\alpha = ac$. Further, we check that $\beta q = \beta dp(1 - p\alpha)^{-1}c = q\beta$. We claim that $\beta \in \text{comm}^2(\beta)$. Let $y \in R$ be such that $y\beta = \beta y$. Then $y(1 - dc) = (1 - dc)y$, and so $ydc = dcy$. This implies that $(cyd)(1 - a) = (1 - a)(cyd)$, and then $(cyd)a = a(cyd)$. As $p \in \text{comm}^2(\alpha)$, we get $(cyd)p = p(cyd)$.

Thus, we get $(cyd)p(1 - p\alpha)^{-1} = p(cyd)(1 - p\alpha)^{-1} = p(1 - p\alpha)^{-1}(cyd)$. Hence,

$$\begin{align*}
(yqd)q &= dcy(dp(1 - p\alpha)^{-1}c) \\
&= q(ydc),
\end{align*}$$

and so $(1 - \beta)yq = qy(1 - \beta).$ Therefore

$$\begin{align*}
yq(1 - \beta q) &= yq(1 - \beta) \\
y(1 - \beta)q &= yq(1 - \beta).
\end{align*}$$

Multiplying the above by $q$ on the right side yields $yq(1 - \beta q) = qyq(1 - \beta q)$. As

$$1 - c\beta dp(1 - p\alpha)^{-1} = 1 - c(1 - dc)dp(1 - p\alpha)^{-1} = 1 - p\alpha \in U(R),$$

we see that $1 - \beta q = 1 - \beta dp(1 - p\alpha)^{-1}c \in U(R)$. This implies that $yq = qyq$. As $(1 - \beta)yq = (1 - \beta)yq$, we deduce that $(1 - \beta)yq = (1 - \beta q)yq$, and so $qy = qyq$. Therefore $yq = qyq = qy$, and so $q \in \text{comm}^2(\beta)$.

Write $r = (p(1 - p\alpha)^{-1} - 1)c$. Then $rd = (p(1 - p\alpha)^{-1} - 1)cd = p - 1 + a$, and so $(1 - p\alpha)(1 + rd) = (1 - p\alpha)(\alpha + p)$. As $1 - p\alpha \in U(R)$, we see that $1 + rd = \alpha + p$, and so $\alpha - ax = \alpha - 1 + p = rd$. This shows that $(rd)^k \in J(R), (dr)^k \in J(R)$. On the other hand, $\beta + q = 1 + dr$, and so $(\beta - (1 - q))^k = (dr)^k \in J(R)$. Here $(1 - q)^2 = 1 - q \in \text{comm}^2(\beta)$, and so $1 - q \in \text{comm}^2(1 - ba)$. Therefore $1 - ba \in R$ has p-Hirano inverse. The formula is then obtained by Lemma 2.2 and [7, Theorem 2.6].

$\iff$ This is symmetric. $\square$
Corollary 4.2. Let $R$ be a ring, let $n \in \mathbb{N}$, and let $a, b \in R$. Then $(1 - ab)^n$ has $p$-Hirano inverse if and only if $(1 - ab)^n$ has $p$-Hirano inverse.

Proof. $\implies$ Since $b(1 - ab) = (1 - ba)b$, we have $b(1 - ab)^2 = [b(1 - ab)](1 - ab) = [(1 - ba)b](1 - ab) = (1 - ba)[b(1 - ab)] = (1 - ba)^2b$. By induction, we have

$$b(1 - ab)^m = (1 - ba)^mb$$

for any $m \in \mathbb{N}$. Then we have

$$1 - (1 - ab)^n = [1 - (1 - ab)][1 + (1 - ab) + (1 - ab)^2 + \cdots + (1 - ab)^{n-1}] = ab[1 + (1 - ab) + (1 - ab)^2 + \cdots + (1 - ab)^{n-1}] = a[b + b(1 - ab) + b(1 - ab)^2 + \cdots + b(1 - ab)^{n-1}] = a[b + (1 - ba)b + (1 - ba)^2b + \cdots + (1 - ab)^{n-1}b] = a[1 + (1 - ba) + (1 - ba)^2 + \cdots + (1 - ba)^{n-1}].$$

Then we have

$$(1 - ab)^n = 1 - a[1 + (1 - ba) + (1 - ba)^2 + \cdots + (1 - ba)^{n-1}]b.$$ Likewise, we have

$$(1 - ba)^n = 1 - b[1 + (1 - ab) + (1 - ab)^2 + \cdots + (1 - ab)^{n-1}]a.$$ Therefore

$$(1 - ba)^n = 1 - [b + b(1 - ab) + b(1 - ab)^2 + \cdots + b(1 - ab)^{n-1}]a = 1 - [b + (1 - ba)b + (1 - ba)^2b + \cdots + (1 - ba)^{n-1}b]a = 1 - [1 + (1 - ba) + (1 - ba)^2 + \cdots + (1 - ba)^{n-1}]ba.$$

$\iff$ This is symmetric. $\blacksquare$

For rectangular matrices over a ring, we derive

Proposition 4.3. Let $R$ be a ring, and let $A \in M_{m \times n}(R), B \in M_{n \times m}(R)$. Then $I_m + AB \in M_m(R)$ has $p$-Hirano inverse if and only if $I_n + BA \in M_n(R)$ has $p$-Hirano inverse.

Proof. Let $k = m + n$. Set

$$C = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in M_{k \times k}(R).$$

Then we observe that

$$I_k + CD = \begin{pmatrix} I_n & 0 \\ 0 & I_m + AB \end{pmatrix}, I_k + DC = \begin{pmatrix} I_n + BA & 0 \\ 0 & I_m \end{pmatrix}.$$ Theorem 4.1 is applied to give the following result: $I_k + CD \in M_{k \times k}(R)$ has $p$-Hirano inverse if and only if so has $I_k + DC \in M_{k \times k}(R)$. Therefore we easily obtain the result. $\blacksquare$

Corollary 4.4. Let $A$ be a Banach algebra, $a, b \in A$ and $ab = 0$. If $a, b$ have $p$-Hirano inverse. Then $a + b$ has $p$-Hirano inverse.

Proof. Let $C = \begin{pmatrix} 1 \\ a \end{pmatrix}$ and $B = (b, 1)$. Clearly, $CB = \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}$. By virtue of Theorem 2.3 and [8, Theorem 5.3], $CB \in M_2(A)$ has $p$-Hirano inverse. In light of Proposition 4.3, $BC = a + b \in A$ has $p$-Hirano inverse, as asserted. $\blacksquare$
5. Matrices over local rings

The goal of this section is to completely determine when a $2 \times 2$ matrix over a local ring has p-Hirano inverse. Recall that a local ring $R$ is called co-bleached if for any $j \in J(R)$ and $u \in U(R), l_u - r_j$ and $l_j - r_u$ are injective. Where $l_u$ and $r_j$ will denote the abelian group endomorphisms of $R$ given by left or right multiplication by $u$ or $j$. The following lemma is crucial.

Lemma 5.1. ([3, Theorem 3.5]) Let $R$ be a local ring and $A \in M_2(R)$. Then $A$ has p-Drazin inverse if and only if

1. $A \in GL_2(R)$; or
2. $A^2 \in M_2(J(R));$ or
3. $A$ is similar to \[
\begin{pmatrix}
\alpha & 0 \\
0 & \beta
\end{pmatrix},
\] where $l_u - r_\beta, l_\beta - r_u$ are injective and $\alpha, \beta \in J(R)$.

We come now to characterize $2 \times 2$ matrices over a local ring by means of the similarity.

Theorem 5.2. Let $R$ be a local ring, and let $A \in M_2(R)$. Then $A$ has p-Hirano inverse if and only if

1. $A^2 \in M_2(J(R))$, or ($l_2 - A^2)^2 \in M_2(J(R))$, or
2. $A$ is similar to \[
\begin{pmatrix}
\lambda & 0 \\
0 & \mu
\end{pmatrix},
\] where $l_\lambda - r_\mu, l_\mu - r_\lambda$ are injective and $\lambda, \mu \in J(R)$.

Proof. $\implies$ In light of Theorem 2.5, we may write $(A^2 - E)^k \in M_2(J(R))$ for some $k \in \mathbb{N}$, where $E^2 = E \in \text{comm}^2(A)$. By virtue of Theorem 2.3, $A$ has p-Drazin inverse, i.e., it is pseudopolar. Then we have three cases.

Case 1. $A \in GL_2(R)$. Hence $E \in GL_2(R)$; and so $E = I_2$. This shows that $(l_2 - A^2)^2 \in M_2(J(R))$.

Case 2. $A^2 \in M_2(J(R))$.

Case 3. $A$ is similar to \[
\begin{pmatrix}
\lambda & 0 \\
0 & \mu
\end{pmatrix},
\] where $l_\lambda - r_\mu, l_\mu - r_\lambda$ are injective and $\lambda \in U(R), \mu \in J(R)$. In light of Theorem 2.3,

$$(\begin{pmatrix}
\lambda & 0 \\
0 & \mu
\end{pmatrix})^k \in M_2(J(R))$$

for some $k \in \mathbb{N}$, and so $(\lambda - \lambda^3)^k \in J(R)$. This shows that $\lambda \in \pm 1 + J(R)$.

$\implies$ Case 1. $A^2 \in M_2(J(R))$. Then $A$ has p-Hirano inverse.

Case 2. $(l_2 - A^2)^2 \in M_2(J(R))$. In light of Theorem 2.5, $A$ has p-Hirano inverse.

Case 3. $A$ is similar to \[
\begin{pmatrix}
\lambda & 0 \\
0 & \mu
\end{pmatrix},
\] where $l_\lambda - r_\mu, l_\mu - r_\lambda$ are injective and $\lambda \in \pm 1 + J(R), \mu \in J(R)$.

We see that

$$(\begin{pmatrix}
\lambda & 0 \\
0 & \mu
\end{pmatrix})^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \lambda^2 - 1 & 0 \\ 0 & \mu^2 \end{pmatrix},$$

where $\begin{pmatrix} \lambda^2 - 1 & 0 \\ 0 & \mu^2 \end{pmatrix} \in M_2(J(R))$. Let $\begin{pmatrix} x & s \\ t & y \end{pmatrix} \in \text{comm} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. Then $\lambda s = s \mu$ and $\mu t = t \lambda$; hence, $s = t = 0$.

This implies that

$$(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$ 

Therefore $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \text{comm}^2 \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, as needed. $\square$

We have at our disposal all the information necessary to characterize p-Hirano inverses of matrices over a cotobleached local ring by means of the solvability of quadratic equations.

Theorem 5.3. Let $R$ be a cotobleached local ring, and let $A \in M_2(R)$. Then $A$ has p-Hirano inverse if and only if
Proof. \(\Rightarrow\) As in the proof of Theorem 5.2, we may assume
\[
U^{-1} \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} U = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}
\]
for some \(U \in GL_2(R)\), where \(\alpha, \mu \in \pm 1 + J(R)\). Write \(U^{-1} = \begin{pmatrix} z & y \\ s & t \end{pmatrix}\). It follows from
\[
\begin{pmatrix} z & y \\ s & t \end{pmatrix} \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} z & y \\ s & t \end{pmatrix}
\]
that
\[
y = az; \\
z\lambda + y\mu = ay; \\
t = bs; \\
s\lambda + t\mu = \beta t.
\]

Clearly, \(t = bs \in J(R)\). If \(y \) or \(s \) in \(J(R)\), then \(U\) is not invertible, a contradiction. Since \(R\) is local, we see that \(y, s \in U(R)\). If \(z \in J(R)\), then \(y = az \in J(R)\), a contradiction. This implies that \(z \in U(R)\). Let \(\delta = y^{-1}ay \) and \(\gamma = s^{-1}bs\). Then \(\delta \in \pm 1 + J(R)\), \(\gamma \in J(R)\). We compute that
\[
\delta^2 - \delta \mu = y^{-1}a^2y - y^{-1}ay\mu = (y^{-1}a)(ay - y\mu) = (y^{-1}a)z\lambda = y^{-1}(az)\lambda = \lambda;
\]
hence, \(\delta^2 - \delta \mu - \lambda = 0\). Moreover we check that
\[
\gamma^2 - \gamma \mu = (s^{-1}b)(bs - s\mu) = s^{-1}(bs - t\mu) = s^{-1}(s\lambda) = \lambda.
\]

Therefore the equation \(x^2 - x\mu - \lambda = 0\) has a root \(\delta \in \pm 1 + J(R)\) and a root \(\gamma \in J(R)\), as desired.

\(\Leftarrow\) Suppose that the equation \(x^2 - x\mu - \lambda = 0\) has a root \(\alpha \in \pm 1 + J(R)\) and a root \(\beta \in J(R)\). Then \(a^2 = a\mu + \lambda; \beta^2 = \beta\mu + \lambda\). Hence,
\[
\begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix} \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix},
\]
where
\[
\begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & \beta - \alpha \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in GL_2(R).
\]

Therefore \(\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}\) is similar to \(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\), where \(\alpha \in \pm 1 + J(R)\) and \(\beta \in J(R)\). By virtue of Theorem 5.2, we complete the proof. \(\square\)
Corollary 5.4. Let \( R \) be a commutative local ring, and let \( A \in M_2(R) \). Then \( A \) has \( p \)-Hirano inverse if and only if

1. \( A^2 \in M_2(J(R)), \) or \((I_2 - A^2)^2 \in M_2(J(R)), \) or
2. \( x^2 - \text{tr}(A)x + \det(A) \) has a root \( \alpha \in \pm 1 + J(R) \) and a root \( \beta \in J(R). \)

Proof. \( \implies \) By virtue of Theorem 5.3, we may assume that \( A \) is similar to \( \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} \), where \( \lambda \in J(R), \mu \in U(R) \) and the equation \( x^2 - \mu x - \lambda = 0 \) has a root in \( \pm 1 + J(R) \) and a root in \( J(R). \) Hence \( \lambda = -\det(A) \) and \( \mu = \text{tr}(A) \), as desired.

\( \Leftarrow \) Case 1. \( A \) has \( p \)-Hirano inverse.

Case 2. Since \( \det(A) = \alpha \beta \in J(R), \) we see that \( A \notin GL_2(R). \) As \( \text{tr}(A) = \alpha + \beta \in \pm 1 + J(R), \) we have \( \det(I_2 - A) = \pm 1 - \text{tr}(A) + \det(A) \in J(R); \) hence, \( I_2 - A \notin GL_2(R). \) In view of [5, Lemma 2.4], \( A \) is similar to \( \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} \), where \( \lambda \in J(R), \mu \in U(R). \) Thus \( \lambda = -\det(A) \) and \( \text{tr}(A) = u, \) and so the equation \( x^2 - \mu x - \lambda = 0 \) has a root in \( \pm 1 + J(R) \) and a root in \( J(R). \) Therefore \( A \) has \( p \)-Hirano inverse by Theorem 5.3. \( \Box \)

We are ready to prove:

Theorem 5.5. Let \( R \) be a commutative local ring, and let \( A \in M_2(R). \) If \( J(R) \) is nil, then the following are equivalent:

1. \( A \) has \( p \)-Hirano inverse.
2. \( A \) is the sum of a tripotent and a nilpotent that commute.
3. \( A \) or \( I_2 - A^2 \) is nilpotent, or \( x^2 - \text{tr}(A)x + \det(A) \) has a root \( \alpha \in \pm 1 + N(R) \) and a root \( \beta \in N(R). \)

Proof. \( (1) \Leftrightarrow (3) \) This is obvious by Corollary 5.4, as \( J(R) = N(R). \)

\( (1) \implies (2) \) In light of Theorem 2.3, \((A - A^3)^2 \in M_2(J(R)),\) and so \( A - A^3 \in N(M_2(R)).\)

Case 1. \( 2 \in J(R). \) Then \( A - A^2 \in N(M_2(R)). \) In view of [9, Lemma 3.5], there exists \( E^2 = E \in Z[A] \) such that \( A - E \in N(M_2(R)), \) as desired.

Case 2. \( 2 \notin J(R). \) Then \( 2 \in R \) is invertible. Let \( B = \frac{A^2 + A}{2}, \) \( C = \frac{A^2 - A}{2}. \) Then \( A = B - C. \) We have

\[
B - B^2 = \frac{A^2 + A}{2} - \left(\frac{A^2 + A}{2}\right)^2 = \frac{A^2 + 2A - A^4 - 2A^3}{4} = \frac{(A - A^3)(A + 2I_2)}{4},
\]

\[
C - C^2 = \frac{A^2 - A}{2} - \left(\frac{A^2 - A}{2}\right)^2 = \frac{-A^4 + 2A^3 + A^2 - 2A}{4} = \frac{(A - A^3)(A + 2I_2)}{4}.
\]

Since \( A - A^3 \in N(M_2(R)), \) we see that \( B - B^2, C - C^2 \in N(M_2(R)). \) In light of [9, Lemma 3.5], there exists idempotents \( E, F \in Z[A] \) such that \( B - E, C - F \in N(M_2(R)). \) Therefore \( A = E - F + (B - E) - (C - F), \) where \((E - F)^3 = E - F \in Z[A] \subseteq \text{comm}(A), \) \((B - E) - (C - F) \in N(M_2(R)). \) Therefore \((A - (E - F))^2 = 0 \in M_2(J(R))\) for some \( k \in F, \) as desired.

\( (2) \implies (1) \) Write \( A = E + W, E^3 = E \in \text{comm}(A), W \in N(M_2(R)). \) Then \( A^2 - A^4 = ((2E + W) - 2E^2(2E + W) - (2E + W^2)W)W \in N(M_2(R)). \) According to [9, Lemma 3.5], there exists \( F^2 = F \in Z[A] \) such that \( A^2 - F \in N(M_2(R)). \) As \( F \in \text{comm}(A), \) it follows by Theorem 2.5 that \( A \) has \( p \)-Hirano inverse, as asserted. \( \Box \)

Corollary 5.6. Let \( F \) be a field, and let \( A \in M_2(F). \) Then the following are equivalent:

1. \( A \) has \( p \)-Hirano inverse.
2. \( A \) is the sum of a tripotent and a nilpotent that commute.
3. \( A^2 = 0, \) or \((I_2 - A^2)^2 = 0, \) or \( A^2 = \pm A. \)

Proof. \( (1) \Leftrightarrow (2) \) This is obvious, by Theorem 5.4.

\( (1) \implies (3) \) This is clear by Theorem 5.5 and [3, Lemma 3.2], as \( J(F) = 0. \)

\( (3) \implies (1) \) This is obtained by Theorem 5.5. \( \Box \)
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