The Absolute Center of $p$-Groups of Maximal Class

R. Orfi$^a$, S. Fouladi$^a$

$^a$Department of Mathematics, Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran

Abstract. The purpose of this paper is to determine $L(G)$, the absolute center of the group $G$, when $G$ is a $p$-group of maximal class. Particularly we find $L(G)$ for metabelian $p$-groups of maximal class, all $p$-groups of maximal class of order less than $p^3$ and $p$-groups of maximal class for $p = 2, 3$.

1. Introduction

In 1994, Hegarty [6] introduced $L(G)$, the absolute center of a group $G$ as follows: $L(G) = \{ g \in G \mid g^a = g \}$ for all $a \in \text{Aut}(G)$. As we see there is an analogue between $L(G)$ and $Z(G)$, on the other hand we may define $Z(G) = \{ g \in G \mid g^a = g \}$ for all $a \in \text{Inn}(G)$. Obviously $L(G) \leq Z(G)$. Hegarty [6] proved an analogue of Schur’s theorem for the absolute center, that is, if $G$ is a group such that $G/L(G)$ is finite, then $\langle g^{-1}g^a \mid g \in G, a \in \text{Aut}(G) \rangle$ is also finite. Moreover Meng and Guo [12] explore the relationship between $L(G)$ and the Frattini subgroup $\Phi(G)$ for a finite group $G$, they also determine the structure of the absolute center of all finite minimal non-abelian $p$-groups.

In this paper we study $L(G)$ for $p$-groups of maximal class. As the definition of $L(G)$ shows, studying $L(G)$ directly depends on the structure of $\text{Aut}(G)$. Therefore we use a structure of the Sylow $p$-subgroup of $\text{Aut}(G)$ for metabelian $p$-groups of maximal class from our paper [5] and also the structure of $p'$-automorphism of $p$-groups of maximal class from [13] to prove our main theorem. Moreover we need the concept of the degree of commutativity of $p$-groups of maximal class. Specially we prove that $|L(G)| = 2$ for all 2-groups of maximal class, $|L(G)| = 1$ for all 3-groups of maximal class and also $|L(G)| = 1$ for $p$-groups of maximal class of order $p^3$. Moreover we show that there is only one group of maximal class of order $p^3$ with $|L(G)| = p$ and all other groups of maximal class of order $p^3$ have trivial absolute center (See Theorem 2.12). Furthermore we determine the absolute center for all metabelian $p$-groups of maximal class (See corollaries 2.5, 2.6 and Theorem 2.8).

Throughout this paper the following notation is used. The terms of the lower and the upper central series of $G$ are denoted by $\gamma_i(G)$ and $Z_i(G)$, respectively. The centre of $G$ is denoted by $Z = Z(G)$. If $a$ is an automorphism of $G$ and $x$ is an element of $G$, we write $x^a$ for the image of $x$ under $a$. For a normal subgroup $N$ of $G$, we let $\text{Aut}^N(G)$ denote the group of all automorphisms of $G$ centralizing $G/N$. Let $H \leq G$ and $A \leq \text{Aut}(G)$, we note that $C_A(H) = \{ a \in A \mid h^a = h, \forall h \in H \}$ and $C_A^H(A) = \{ h \in H \mid h^a = h, \forall a \in A \}$. The Frattini subgroup of $G$ is denoted by $\Phi = \Phi(G)$ and $\text{Aut}_p(G)$ for the Sylow $p$-subgroup of $\text{Aut}(G)$. Also we use the notation $x \equiv y \pmod{H}$ to indicate that $Hx = Hy$, where $H$ is a subgroup of a group $G$ and $x, y \in G$. Let $(a, p) = 1$, we note that $\text{ord}_p(a)$ is the smallest positive integer $t$ such that $a^t \equiv 1 \pmod{p}$. All unexplained notation is standard and follows that of [9].
2. Main results

Let $G$ be a $p$-group of maximal class of order $p^n$ ($n \geq 3$), where $p$ is a prime. We note that if $n = 3$, then $L(G) = 1$ for $p > 2$ and $L(G) = Z(G)$ for $p = 2$. Therefore in the rest of the paper we assume that $n \geq 4$. Following [9], we define the 2-step centralizer $K_i$ in $G$ to be the centralizer in $G$ of $\gamma_i(G)/\gamma_{i+2}(G)$ for $2 \leq i \leq n-2$ and define $P_i = P_i(G)$ by $P_0 = G$, $P_1 = K_2$, $P_i = \gamma_i(G)$ for $2 \leq i \leq n$. The degree of commutativity $l = l(G)$ of $G$ is defined to be the maximum integer such that $[P_i, P_j] \leq P_{i+j+1}$ for all $i, j \geq 1$ if $P_1$ is not abelian and $l = n - 2$ if $P_1$ is abelian.

Take $s \in G - \bigcup_{i=2}^{n-2} K_i$, $s_1 \in P_1 - P_2$ and $s_i = [s_{i-1}, s]$ for $2 \leq i \leq n-1$. It is easily seen that $\{s, s_1\}$ is a generating set for $G$ and $P_i(G) = \langle s, \ldots, s_{i-1} \rangle$ for $1 \leq i \leq n-1$ and so $Z(G) = P_{n-1}(G) = \langle s_{n-1} \rangle$. For the rest of the paper we fix the above notation.

By [9, Corollary 3.2.7] and [2, Corollary p.59] we have the following result.

**Lemma 2.1.** Let $G$ be a $p$-group of maximal class of order $p^n$.

(i) The degree of commutativity of $G$ is positive if and only if the 2-step centralizers of $G$ are all equal.

(ii) If $G$ is metabelian then $G$ has positive degree of commutativity.

**Lemma 2.2.** [7, Hillfassitz III. 14.13] If $G$ is a $p$-group of maximal class of order $p^n$ and $s \notin K_i$ for $2 \leq i \leq n-2$, then $C_G(s) = \langle s \rangle P_{n-1}(G)$ and $s' \in P_{n-1}(G)$.

**Theorem 2.3.** [3, Theorem 3.2] Let $G = \langle a, b \rangle$ be a two-generated metabelian group. Then the following are equivalent:

(i) For all $u, v \in G'$, there is an automorphism of $G$ that maps $a$ to $au$ and $b$ to $bv$;

(ii) $G$ is nilpotent.

By the above theorem we see that if $G$ is a metabelian $p$-group of maximal class of order $p^n$, then for any elements $x, y \in G'$ there is an automorphism that maps $s$ to $sx$ and $s_1$ to $s_1 y$ hence $|\text{Aut}(\Phi(G))| = p^{2n-4}$. Moreover $\frac{\text{Aut}(G)}{\text{Aut}(\Phi(G))} \hookrightarrow \text{Aut}(G)$ and so $|\text{Aut}(G) : \text{Aut}(\Phi(G))|$ divides $p$, since $\frac{G}{\Phi(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

**Lemma 2.4.** If $G$ is a $p$-group of maximal class of order $p^n$, then $\text{Aut}(G)$ fix $Z(G)$ elementwise.

**Proof.** Consider the action of $\text{Aut}(G)$ on $Z(G)$. It is obvious that $C_{Z(G)}(\text{Aut}(G)) \neq 1$ since $\text{Aut}(G)$ and $Z(G)$ are $p$-groups. As $|Z(G)| = p$, we have $C_{Z(G)}(\text{Aut}(G)) = Z(G)$, which completes the proof. □

**Corollary 2.5.** If $G$ is a $p$-group of maximal class of order $p^n$ and $\text{Aut}(G)$ is also a $p$-group, then $L(G) = Z(G)$.

**Proof.** This is obvious by the fact that $L(G) \leq Z(G) \cong \mathbb{Z}_p$ and Lemma 2.4. □

**Corollary 2.6.** Let $G$ be a 2-group of maximal class of order $2^n$, then $L(G) = Z(G)$.

**Proof.** By [5, Theorem 5.9], we see that $\text{Aut}(G)$ is also a 2-group which completes the proof by using Corollary 2.5. □

**Lemma 2.7.** Let $G$ be a $p$-group of maximal class of order $p^n$. If $\delta \in \text{Aut}(G)$ with $s^\delta = s^\varphi x$ and $s_1^\delta = s_1^\varphi y$, where $x, y \in \Phi(G)$ and $0 < a, c < p$. Then $s_i^r = s_i^r \mod p$.

**Proof.** By induction on $m$ we have $[s_i^m, s_i^r] \equiv s_i^{m-r} \mod p$ and so $[s_i^m, s_i^r] \equiv s_i^{m-r} \mod p$ for all $i, j \geq 1$. Therefore by induction on $i$ we see that $s_i^j = s_i^{r+c} \mod p$, as required. □
Now for the rest of the paper by using corollaries 2.5 and 2.6 we may assume that $G$ is a metabelian $p$-group of maximal class of order $p^n(p > 2)$ and Aut$(G)$ is not a $p$-group. It is straightforward to see that when $p$ is odd, Aut$(G)$ is supersolvable and is a split extension of Aut$_p(G)$ by a subgroup of the direct product of two cyclic groups of order $p - 1$. On the other hand, if $H$ be a $p'$-subgroup of Aut$_p(G)$, then we have Aut$_p(G) = Aut_p(G) \ltimes H$ and $H$ is embedded in $\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$ (see [1] Section 1). Since $P_1(G)$ and $\Phi(G)$ are characteristic subgroups of $G$, $G/\Phi(G)$ and $P_1/\Phi(G)$ are invariant under $H$. So by Maschke's Theorem there exists $s \in G - P_1$ such that $G/\Phi(G) = P_1/\Phi(G) \ltimes (\Phi(G), s)/\Phi(G)$ and $(\Phi(G), s)/\Phi(G)$ is invariant under $H$. In the rest of the paper $s$ will be as above. Therefore if $\delta \in H$ then $s^\delta = s^x$ and $s^\delta_1 = s^y_1$, where $x, y \in \Phi(G)$ and $0 \leq a, c < p$. We recall that if $G$ is metabelian $p$-group of maximal class, then $G$ has positive degree of commutativity and $|s|$ divides $p^2$ by Lemma 2.2. In the next theorem we find the absolute center for finite metabelian $p$-group of maximal class when $H \neq 1$.

**Theorem 2.8.** Let $G$ be a metabelian $p$-group of maximal class of order $p^n(p > 2)$ and $H \neq 1$. If $H$ is not cyclic, then $L(G) = 1$. Let $H$ be cyclic such that $H = \langle \delta \rangle$ with $s^\delta = s^x$, $s^\delta_1 = s^y_1$, where $1 \leq a, c < p$ and $x, y \in \Phi(G)$.

(i) If $|s| = p^2$, then $L(G) = 1$.

(ii) If all elements out of $P_1$ have order $p$, then

(a) if $\text{ord}_p(c) \nmid \text{ord}_p(a)$, then $L(G) = 1$.

(b) if $\text{ord}_p(c) \mid \text{ord}_p(a)$, then there exists $0 \leq r < \text{ord}_p(a)$ such that $c \equiv a^r$ (mod $p$). On setting $\text{ord}_p(a) = t$ we have $L(G) = Z(G)$ when $t \mid n - 2 + r$ and $L(G) = 1$ when $t \nmid n - 2 + r$.

**Proof.** By [13, Theorem A], we have $C_H(Z(G))$ is cyclic. Hence there exists $a \in H$ such that $C_{Z(G)}(a) \neq Z(G)$. As $Z(G) = p$ we deduce that $C_{Z(G)}(a) = 1$, which completes the proof, since $L(G) \leq C_{Z(G)}(a)$.

(i) Since $\delta \notin \text{Aut}^\Phi(G)$, we have $(a, c) \neq (1, 1)$. By Lemma 2.7, if $a = 1$ then $s^\delta_{n-1} = s^y_{n-1} \neq s_{n-1}$, as desired.

If $a > 1$, then by Theorem 2.3, the map $\beta$ defined by $s^\delta = su^r$ and $s^\delta_1 = s^1 w^r$, where $u^\delta = x$ and $w^0 = y$, is an automorphism of $G$ lying in Aut$_p^\Phi(G)$. On setting $a = \beta s^\delta$, we see that $s^a = s^\delta$ and $s^a_1 = s^\delta_1$ and so $(s^\delta)^1 = s^\delta_1 \neq s^\delta$. Moreover by Lemma 2.2, $Z(G) = (s^\delta)$, which completes the proof.

(ii) The map $\beta$ defined by $s^\delta = su^r$ and $s^\delta_1 = s^1 w^r$, where $u^\delta = x$ and $w^0 = y$, is an automorphism of $G$ lying in Aut$_p^\Phi(G)$. On setting $a = \beta s^\delta$ and $\text{ord}_p(a) = t$, we have $s^a = s$ and $s^a_1 = s^\delta_1$ and so by Lemma 2.7, $s^a_1 = s^\delta_{n-1} \neq s_{n-1}$ since $\text{ord}_p(c) \nmid t$.

(ii) First we see that $1, a, \ldots, a^{n-1}$ are all distinct roots of the equation $x^t \equiv 1$ (mod $p$). Therefore there exists $0 \leq r < |s|$ such that $c \equiv a^r$ (mod $p$). Now by Lemma 2.7, $s^a_1 = s^\delta^{an/r}$, which completes the proof. □

In what follows first we find the absolute center for all finite 3-groups of maximal class and finally we obtain the absolute center for all $p$-groups of maximal class of order $p^n$, where $4 \leq n \leq 5$.

**Lemma 2.9.** Let $G$ be a $p$-group of maximal class of order $p^n(p > 2)$ and $H \neq 1$. If $P_1$ is abelian, then $L(G) = 1$.

**Proof.** First we may assume that $|s| = p$ by Theorem 2.8. Now we see that any element of $G$ is uniquely determined by $s^t u$, where $0 \leq t < p$ and $u \in P_1$. Assume that $1 < b < p$, we define $\beta : G \to G$ by $(s^t u)^\beta = s^t u^b$, and we show that $\beta$ is an automorphism. Let $g_1 = s^t u$ and $g_2 = s^{t'} u'$, where $0 \leq t, t' < p$ and $u, u' \in P_1$. We may write $g_1 g_2 = s^{t+t'} [s^{t}, u^{-1}][u t']$. If $t + t' \equiv r$ (mod $p$), then $s^{t+t'} = s^r$ since $|s| = p$ and so $(g_1 g_2)^\beta = s^{t+t'} [s^r, u^{-1}] u t'$. Moreover $s^r g_1^\beta = s^{r+t'} [s^r, u^{-1}] u t'$. We have $[s^r, u^{-1}] = [s^{t'}, u^{-1}]$ since $P_1$ is abelian and so $\beta$ is a homomorphism. Also $\beta$ is onto since $G = \langle s, s^1 \rangle$. Thus $\beta$ is an automorphism. Furthermore $s^\delta_{n-1} = s^\delta_{n-1}$, which completes the proof since $L(G) \leq Z(G) = \langle s_{n-1} \rangle$. □

**Lemma 2.10.** Let $G$ be a 3-group of maximal class of order $3^n$ ($n \geq 4$), then $L(G) = 1$. 


Proof. First we see that for $n = 4$, $G$ is metabelian; and for $n = 5$, $G$ has degree of commutativity $n - 4$ by [2, Theorem 3.13] and so is metabelian. Moreover by [5, Theorem 5.8], we have $H \neq 1$. Now if $P_1$ is abelian, then by Lemma 2.9, $L(G) = 1$. Furthermore if $P_1$ is not abelian, then by observing the proof of [5, Theorem 5.6 (i)], we have $H = \langle \beta_2 \rangle$ when $n$ is odd and $H = \langle \beta_3 \rangle$ when $n$ is even, where $\beta_2^n = s^{-1}, s_2^2 = s_1$ and $\beta_3^n = s^{-1}, s_1^5 = s_1^{-1}$. Note that $s^{-1} = s^2s^{-3}$ and $s^{-3} \in \Phi(G)$. Therefore Lemma 2.7 completes the proof. □

Lemma 2.11. Let $G$ be a $p$-group of maximal class of order $p^4$ ($p > 2$). Then $L(G) = 1$.

Proof. First we see that $H \neq 1$ by [11, Lemma 9]. Since $P_1 = C_G(\gamma_2(G))$, we have $\gamma_2(G) \leq Z(P_1) \leq P_1$ which implies that $P_1/Z(P_1)$ is cyclic and so $P_1$ is abelian, as desired. □

Now for $p > 3$, Curran [4, Corollary 5] shown that there is only one group of order $p^5$ whose automorphism group is also a $p$-group in which $(p - 1,3) = 1$. The presentation of this group is as follows:

$$G_0 = \langle a_1, a \mid a^p = [a_1, a]^p = [a_1, a, a, a]^p = [a_1, a, a, a] = 1, a^2Z = [a_1, a, a, a]^{-1} \rangle.$$ 

We note that $G_0$ is of maximal class. By this observation we state the following theorem.

Theorem 2.12. Let $G$ be a $p$-group of maximal class of order $p^5$ with $p > 3$. If $G = G_0$ then $L(G) = Z(G)$, for otherwise $L(G) = 1$.

Proof. First we claim that $G$ is metabelian. To prove this we have $[\gamma_2(G), Z_2(G)] = 1$ and so $\gamma_3(G) = Z_2(G) \leq Z(\gamma_2(G)) \leq \gamma_2(G)$, which implies that $\gamma_2(G)$ is abelian. If $G = G_0$ then Corollary 2.5 completes the proof. Therefore for the rest of the proof we may assume that $H \neq 1$. Since $p \geq 5$, by using [9, Proposition 3.3.2] we have $\exp(G/Z(G)) = \exp(G') = p$ which yields that $\Omega_1(G) \leq Z(G) = Z_p$. Moreover by [9, Lemma 1.2.11] $G$ is regular. Now if $\Omega_1(G) = Z(G)$, then $[\Omega_1(G)] = p^4$. Hence $\Omega_1(G)$ is a maximal subgroup of $G$ and $\Omega_1(G) = \{ x \in G | x^p = 1 \}$ since $G$ is regular. On setting $s \in G - (P_1 \cup \Omega_1(G))$, we have $|s| = p$ and so $L(G) = 1$ by Theorem 2.8. If $\Omega_1(G) = 1$, then $\exp(G) = p$. Now from Jamie’s list [8], there are only two families $\Phi_9$ and $\Phi_{10}$ of groups of maximal class of order $p^5$. By observing the presentation of these groups, we see that only $\Phi_9(1^5)$ and $\Phi_{10}(1^5)$ are of exponent $p$. Now if $G = \Phi_9(1^5)$ with the following presentation:

$$\langle s, s_1, \ldots, s_4 \mid [s, s] = s_1, s^p = s_1^2 = 1 \ (1 \leq i \leq 4) \rangle,$$

then obviously $P_1$ is abelian and so $L(G) = 1$ by Lemma 2.9. Furthermore if $G = \Phi_{10}(1^5)$ with the presentation

$$\langle s, s_1, \ldots, s_4 \mid [s, s] = s_1, [s_1, s_2] = s_4, s^p = s_1^p = 1 \ (1 \leq i \leq 4) \rangle,$$

then the map $\alpha$ defined by $s^\alpha = s^{-1}, s_1^\alpha = s_1$ is an automorphism of order 2 and it is easily seen that $s_4^\alpha = s_4^{-1}$, completing the proof. □

Acknowledgments. The authors are grateful to the referee for useful comments. The paper was revised accordingly and also second author would like to thank the Kharazmi University(Iran) for financial support.

References